$$x_{mn} = 0, \qquad z_{mn} = 0, \qquad \bar{z}_{0n} = r_0^{2n}[W_{0n} - w_{0n} + (n-2)r_0^{2-2n}\bar{x}_{0,n-2}],$$

$$\bar{z}_{0n} = r_1^{2n}[Y_{1n} - M_1 w_{1n} - (1+N_1)w_{0n} + (n-2)r_1^{2-2n}\bar{x}_{0,n-2}],$$

$$\bar{z}_{jn} = r_j^{2n}[W_{jn} - (1+M_j)w_{jn} - N_j w_{j-1,n} + (n-2)r_j^{2-2n}\bar{x}_{j,n-2}],$$

$$y_{0,n-2} = r_0^{4-2n}[\overline{X}_{0,n-2} - nr_0^{2n-2}w_{0n} - \bar{x}_{0,n-2}],$$

$$y_{0,n-2} = r_1^{4-2n}[\overline{Z}_{1,n-2} - M_1\bar{x}_{1,n-2} - (1+N_1)\bar{x}_{0,n-2} - nr_1^{2n-2}w_{0n}],$$

$$y_{j,n-2} = r_j^{4-2n}[\overline{X}_{j,n-2} - (1+M_j)\bar{x}_{j,n-2} - N_j\bar{x}_{j-1,n-2} - nr_j^{2n-2}w_{jn}].$$

For the case of the temperature a function of the radius the functions become very simple with the series in Eqs. (8) and (9) reducing to one term. With the temperature as $T_j(r)$ in S_j, the particular integral of $\nabla^2 V = kT$ yields

$$z\,\overline{Y}_m = k_m \int_0^r rT_m(r)\,dr,$$

$$z\,\overline{Y}_j = k_m \int_0^{r_m} rT_m(r)\,dr + k_{m-1}\int_{r_m}^{r_{m-1}} rT_{m-1}(r)\,dr + \cdots + k_j \int_{r_{j+1}}^{r} rT_j(r)\,dr,$$

whence, using Eq. (9), $[Y]_j = L_j t = A_{j1}t = [Y]_{j-1} = E_{j1}t$ with $r_j^2 L_j = [z\overline{Y}_j]_{r=r_j}$. Further, $\phi_j(z) = a_{j1}z$ and $\psi_j(z) = g_{j1}/z$, where a_{j1} and g_{j1} are given by Eq. (10). The function U is $U_j = a_{j1}r^2 + g_{j1}\log r$ and the radial and tangential stresses in polar coordinates are

$$\sigma_{rj} = 2a_{j1} + (g_{j1} - z\overline{Y}_j)/r^2, \qquad \tau_{r\theta j} = 0,$$

$$\sigma_{\theta j} = 2a_{j1} - (g_{j1} - z\overline{Y}_j)/r^2 - k_j T_j\,,$$

$$\sigma_{zj} = E_j e_z - k_j T_j + 4a_{j1}/q_j\,.$$

ON THIRD-ORDER CORRELATION AND VORTICITY IN ISOTROPIC TURBULENCE*

By F. N. FRENKIEL (*Cornell University*)

The fundamental equation of the propagation of the correlation function obtained by Th. von Kármán and L. Howarth[1] for isotropic turbulence in an incompressible viscous fluid, is written

$$\frac{\partial(\overline{u'^2}f)}{\partial t} + 2(\overline{u'^2})^{3/2}\left(\frac{\partial h}{\partial r} + \frac{4}{r}h\right) = 2\nu\overline{u'^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right). \tag{1}$$

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[1]Th. von Kármán and L. Howarth. *On the statistical theory of isotropic turbulence*, Proc. Roy. Soc. London (A) **164**, 192 (1938).

For small distances r, the longitudinal correlation function f may be represented by the relation†

$$f = 1 + \frac{1}{2} r^2 f''(0) + \frac{1}{24} r^4 f^{IV}(0)$$

and using the two relations

$$\lambda^{-2} = -f''(0) \qquad \text{and} \qquad \lambda_\omega^{-2} = \frac{7}{15} \lambda^2 f^{IV}(0)$$

where λ is Taylor's dimension of the smallest eddies, there will be obtained for small r the following expression:

$$f = 1 - \frac{r^2}{2\lambda^2} + \frac{5}{56} \frac{r^4}{\lambda^2 \lambda_\omega^2} \tag{2}$$

Applying now the well-known relation[1]

$$\frac{1}{\overline{u'^2}} \frac{d\overline{u'^2}}{dt} = -10 \frac{\nu}{\lambda^2} \tag{3}$$

and multiplying by λ^3/r^2, one has, for small r,

$$\frac{d\lambda}{dt} + 5 \frac{\nu}{\lambda} \left(1 - \frac{\lambda^2}{\lambda_\omega^2}\right) + 2 \frac{\lambda^3 (\overline{u'^2})^{1/2}}{r^2} \left(\frac{\partial h}{\partial r} + \frac{4}{r} h\right) = 0 \tag{4}$$

This equation gives the law of the propagation of the dimension λ. It may be also written in the following form

$$\frac{\partial h}{\partial r} + \frac{4}{r} h = \left[\frac{5}{2}\left(\frac{\lambda^2}{\lambda_\omega^2} - 1\right) - \frac{1}{4\nu} \frac{d\lambda^2}{dt}\right] \frac{\nu r^2}{\lambda^4 (u'^2)^{1/2}} \tag{4a}$$

and after integrating this differential equation we find

$$h = \left[\frac{5}{14}\left(\frac{\lambda^2}{\lambda_\omega^2} - 1\right) - \frac{1}{28} \frac{1}{\nu} \frac{d\lambda^2}{dt}\right] \frac{1}{N_\lambda} \left(\frac{r}{\lambda}\right)^3 \tag{5}$$

where

$$N_\lambda = \frac{\lambda (\overline{u'^2})^{1/2}}{\nu}$$

is the number of Reynolds of microturbulence.

Let us consider the decay of isotropic turbulence and assume with von Kármán the case of "large" Reynolds number of turbulence, i.e. the case when the correlation functions $f(r, t)$ and $h(r, t)$ are independent of the viscosity ν, except for small r. In addition, von Kármán assumes that the correlation curves preserve their shape and that only a scale L changes. As a result the relation[1]

$$\frac{(\overline{u'^2})^{1/2} \lambda^2}{L\nu} = A \tag{6}$$

is found, where A is a numerical constant and L a dimension representing the scale of turbulence.

†To avoid confusion between values at an initial moment t_0 which appears later and values at the origin, we shall use in this paper $f''(0)$ for f'' at $r = 0$ instead of f_0'', $h'''(0)$ instead of h_0''', and so on.

L. G. Loitsianskii[2] shows, that when the rate of decrease of f, h and $\partial f/\partial r$ exceeds the rate of increase of r^4, there exists the following theorem of the conservation of the disturbance moment:

$$\overline{u'^2} \int_0^\infty s^4 f(s)\, ds = \overline{u'^2}\, L^{*5} = \text{const.} \tag{7}$$

As the disturbance length L^* is given as a function of the fourth moment of f and as the shape of this curve is assumed to be preserved for large r, we may use L^* in place of L, assuming that L^*/L is constant. Thus the equation (6) becomes

$$\frac{(\overline{u'^2})^{1/2}\lambda^2}{L^*\nu} = A_1 \tag{6a}$$

where A_1 is a new numerical constant.

After eliminating L^* between (6a) and (7) it is found that

$$\frac{(\overline{u'^2})^{7/10}\lambda^2}{\nu} = \frac{1}{C} \tag{8}$$

with C a numerical constant. Eliminating now λ^2/ν between the last relation and the equation (3) and integrating, we find

$$(\overline{u'^2})^{-7/10} - (\overline{u_0'^2})^{-7/10} = 7C(t - t_0) \tag{9}$$

where $\overline{u'^2} = \overline{u_0'^2}$ at a chosen initial moment $t = t_0$. C is found as a function of the initial values $\overline{u_0'^2}$ and λ_0 using equation (8), and as a result the following equation of decay is obtained:

$$\overline{u'^2} = \overline{u'^2_0}\left[7\frac{\nu}{\lambda_0^2}(t - t_0) + 1\right]^{-10/7} \tag{10}$$

It will also be found that

$$L^* = L_0^*\left[7\frac{\nu}{\lambda_0^2}(t - t_0) + 1\right]^{2/7} \tag{11}$$

and

$$\lambda^2 = \lambda_0^2\left[7\frac{\nu}{\lambda_0^2}(t - t_0) + 1\right] \tag{12}$$

or

$$\lambda^2 = \lambda_0^2 + 7\nu(t - t_0) \tag{12a}$$

In another paper[3] these results are compared with experimental measurements of the decay of turbulence, and the agreement with equations (10) and (12) seems quite satisfactory.

The variation of the Reynolds Number of microturbulence is easily found using equations (10) and (12) and is written

$$N_\lambda = N_{\lambda,0}\left[7\frac{\nu}{\lambda_0^2}(t - t_0) + 1\right]^{-3/14} \tag{13}$$

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[2]L. G. Loitsianskii. *Some basic laws of isotropic turbulent flow*, N. A. C. A., T. M. 1079, 1945 (Translated from Rep. No. 440 of Central Aero-Hydrodynamical Institute, Moscow, 1939).

[3]F. N. Frenkiel. *The decay of isotropic turbulence*, (to be published).

For the decay of vorticity it will be found

$$\overline{\omega^2} = \overline{\omega_0^2}\left[7\frac{\nu}{\lambda_0^2}(t - t_0) + 1\right]^{-17/7} \tag{14}$$

with $\overline{\omega^2} = 15\overline{u'^2}/\lambda^2$, the value of the mean square of the vorticity, $\overline{\omega_0^2}$ being its value at the initial moment t_0.

Differentiating equation (12a) it will be found

$$\frac{1}{\nu}\frac{d\lambda^2}{dt} = 7 \tag{15}$$

which combined with equation (5) gives for the third order correlation coefficient the expression

$$h = \left[\frac{5}{14}\frac{\lambda^2}{\lambda_\omega^2} - \frac{17}{28}\right]\frac{1}{N_\lambda}\left(\frac{r}{\lambda}\right)^3 \tag{16}$$

In this relation the third order correlation coefficient h at the moment t is given as a function of λ, λ_ω and N_λ at the same moment t.

Equation (16) represents a parabola of the third order to which the triple correlation curve $h(r)$ is tangent at its origin.

The last equation gives

$$h'''(0) = \left[\frac{5}{14}\frac{\lambda^2}{\lambda_\omega^2} - \frac{17}{28}\right]\frac{6}{\lambda^3 N_\lambda} \tag{17}$$

It should be recalled that

$$\frac{\lambda^2}{\lambda_\omega^2} = \frac{7}{15}\frac{f^{IV}(0)}{[f''(0)]^2}$$

depends on the shape of the correlation curve. Using the notations of Batchelor and Townsend[4] $S = -\lambda^3 k'''(0)$; $G = \lambda^4 f^{IV}(0) = f^{IV}(0)/[f''(0)]^2$ and remarking that for isotropic turbulence[1] $k = -2h$ and $k'''(0) = -2h'''(0)$ it will be found from (17) that

$$G = \frac{51}{14} + \frac{1}{2}N_\lambda S \tag{18}$$

giving a relation between G, S and N_λ at the same moment t.

Consider the case of decay of isotropic turbulence and assume that $G = G_0$ is constant during the decay. This assumption agrees with the results of the experiments of Batchelor and Townsend.[4] The constant value of G_0 depends upon the experimental conditions and may be different in each experiment, depending on the shape of the correlation curve. Thus during the decay

$$S = \frac{2}{N_\lambda}\left(G_0 - \frac{51}{14}\right) \tag{19}$$

and at the initial moment t_0

$$S = S_0\left[7\frac{\nu}{\lambda_0^2}(t - t_0) + 1\right]^{3/14} \tag{20}$$

[4]G. K. Batchelor and A. A. Townsend. *Decay of vorticity in isotropic turbulence*, Proc. Roy. Soc. London (A), **190**, 534 (1947).

with
$$S_0 = \frac{2}{N_{\lambda,0}}\left[G_0 - \frac{51}{14}\right] \tag{21}$$

The decay of the vorticity is given by the relation[1]

$$\frac{d\overline{\omega^2}}{dt} = 70h'''(0)\cdot(\overline{u'^2})^{3/2} - 10\nu\frac{\overline{\omega^2}}{\lambda_\omega^2} \tag{22}$$

and it will easily be found, using the relations given before, that

$$\frac{d\overline{\omega^2}}{dt} = \frac{7}{3(15)^{1/2}}(\overline{\omega_0^2})^{3/34}\left[S_0 - 2\frac{G_0}{N_{\lambda,0}}\right](\overline{\omega^2})^{24/17} \tag{23}$$

In the decay of vorticity, the rate of production of vorticity by diffusive stretching of the vortex tubes and the rate of dissipation of vorticity by viscosity are each proportional to $(\overline{\omega^2})^{24/17}$. The ratio of the production of vorticity to its dissipation is constant and equal to

$$\frac{SN_\lambda}{2G} = 1 - \frac{51}{14}\frac{1}{G_0} \tag{24}$$

The equations (23) and (21) give

$$\frac{d\overline{\omega^2}}{dt} = -\frac{17}{(15)^{1/2}}\frac{(\overline{\omega_0^2})^{3/34}}{N_{\lambda,0}}(\overline{\omega^2})^{24/17} \tag{25}$$

which could be found directly by differentiating equation (14).

There exist some differences between the conclusions drawn here for isotropic turbulence at "large" Reynolds number of turbulence and the conclusions of Batchelor and Townsend. In particular, in the present paper it is found that the decay of vorticity is proportional to $(\overline{\omega^2})^{24/17}$ and not to $(\overline{\omega^2})^{3/2}$. The factor G is considered to be constant as in the Batchelor-Townsend paper but N_λ and S are variable. For the case studied here, G is given as function of S by the equation (18), which is of the same form as that found by Batchelor and Townsend but instead of 30/7 the constant is equal to 51/14.

The experimental results of Batchelor and Townsend[4] seem to agree only very roughly with these relations. However, if account is taken of probable inaccuracies in the experimental data, such as the imperfect isotropy behind grids and errors due to finite lengths of hot-wires (especially as this concerns measurements of λ), it appears that the agreement may be satisfactory after all.

BOOK REVIEWS

Relaxation methods in theoretical physics. By R. V. Southwell. Oxford, at the Clarendon Press, 1946. vii + 248 pp. $7.00.

The relaxation method of approximate numerical solution of systems of equations was first introduced in 1936 by R. V. Southwell in connection with the solution of engineering problems arising in the field of structural design. In questions of the loading and deflection of complex structural frames, one is struck by the inherent precision of the conventional analytical methods of attack and the inherent lack of precision in the given data defining a problem. A number of people devised numerical schemes of calculation by which the precision of the answer could be made as good as desired and thus made comparable with the given data. Several of these methods involved successive relaxation of constraints. To Southwell