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HARMONIC ANALYSIS OF THE TWO-DIMENSIONAL FLOW OF AN INCOMPRESSIBLE VISCOUS FLUID*

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Introduction. In this paper, we consider the two dimensional flow of an incompressible viscous fluid

a) in a *bounded domain D*, the velocity vanishing at the boundary B of the domain; B is assumed to be a regular curve;[1]

b) in the entire plane, the velocity vanishing at infinity, this case being regarded as a limiting case of a).

Since the fluid is assumed to be *incompressible*, there exists a *stream function* $\psi(x, y, t)$ such that the *two components u* and v of the velocity are given by

$$u(x, y, t) = \frac{\partial\psi}{\partial y}, \tag{1}$$

$$v(x, y, t) = -\frac{\partial\psi}{\partial x}. \tag{2}$$

According to the *boundary conditions*, we have

$$u(x, y, t) = 0, \qquad v(x, y, t) = 0 \qquad \text{on } B \tag{3}$$

or

$$\psi(x, y, t) = 0, \qquad \frac{\partial\psi}{\partial n} = 0 \qquad \text{on } B. \tag{4}$$

The *vorticity* is given by[2]

$$\zeta(x, y, t) = -\tfrac{1}{2}\Delta\psi. \tag{5}$$

According to the Navier-Stokes equations the vorticity must satisfy the equation

$$\frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} = \nu\Delta\zeta \tag{6}$$

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[1] What is meant here by the term *regular curve* is only that B fulfills the conditions required for the use of the Green formula.

[2] $\Delta\psi$ indicating the Laplacian: $\Delta\psi = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2$.

in which ν is a constant, the *kinematic viscosity* of the fluid. Substituting the values (1), (2), and (5) into (6), we obtain a partial (non-linear) differential equation for $\psi(x, y, t)$, which is characteristic for the two dimensional flow of an incompressible

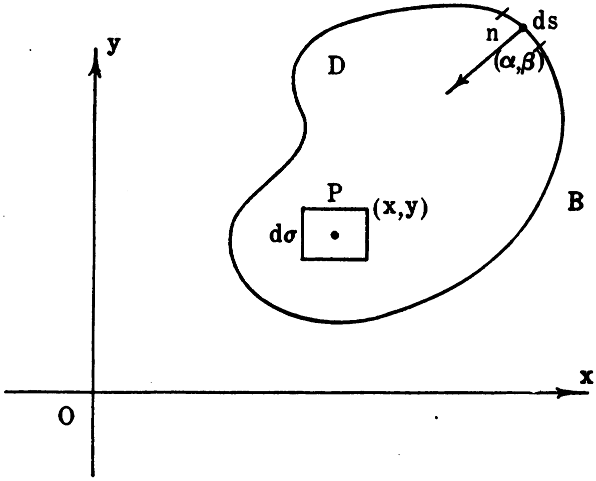

FIG. 1

viscous fluid. We shall only deal with *regular flows*, that is with stream functions such that *the functions*

$$\psi,\ u,\ v,\ \zeta,\ \frac{\partial\zeta}{\partial x},\quad \frac{\partial\zeta}{\partial y},\quad \Delta\zeta,\ \frac{\partial\psi}{\partial t},\quad \frac{\partial u}{\partial t},\quad \frac{\partial v}{\partial t},\quad \frac{\partial\zeta}{\partial t}$$

are continuous in x, y, t for (x, y) in $D+B$ and $t_1 \leqq t \leqq t_2$.

This paper is divided in three parts. In the first part, the Fourier transforms $\Psi(\omega_1, \omega_2, t)$, $U(\omega_1, \omega_2, t)$, $V(\omega_1, \omega_2, t)$, $Z(\omega_1, \omega_2, t)$ of the stream function ψ, the velocity components u, v, and the vorticity ζ are introduced and it is proved that Ψ, U and V are obtained from Z by multiplication by simple rational functions of ω_1 and ω_2. Next the kinetic energy of the flow

$$E = \frac{1}{2}\int_D (u^2 + v^2)d\sigma, \qquad (d\sigma = dxdy) \tag{7}$$

and its *spectral decomposition*

$$E = \int_\Omega \gamma(\omega_1, \omega_2, t)d\omega, \qquad (d\omega = d\omega_1 d\omega_2) \tag{8}$$

are considered and it is shown that the *spectral function* $\gamma(\omega_1, \omega_2, t)$ has also a very simple expression in Z. In other words, the entire harmonic analysis of the flow can be based on the Fourier transform of the vorticity.

This part of the paper uses only the definitions of ψ, u, v, ζ and the boundary condition; the results are valid whether the fluid is viscous or not.

In the second part Eq. (6) is used and some inequalities are given which must be satisfied by the Fourier transforms when the variation of the time t is considered. Some of these inequalities concern the variation of the spectral function with time.

All inequalities are based on the fact that *the kinetic energy of the flow is decreasing more rapidly than an exponential function of t.*

Next, an integrodifferential equation for the Fourier transform Z of the vorticity is derived from (6). This equation might be the starting point for the study of the variation of the spectral function with the time t.

Finally, in the third part, the case that the flow fills the entire plane

$$X \qquad -\infty < x < +\infty, \qquad -\infty < y < +\infty$$

is considered as the limit of the flow in a bounded domain, the fluid being at rest at infinity. The integrodifferential equation simplifies considerably in this case.

If $f(x, y)$ is a real function of the real variables x and y defined[3] and continuous in $D+B$, its *Fourier transform* is given by

$$F(\omega_1, \omega_2) = \frac{1}{4\pi^2}\int_D f(x, y)e^{-i(\omega_1 x+\omega_2 y)}d\sigma, \tag{9}$$

where the *frequencies* ω_1, ω_2 are real. We shall use the following well-known properties of the Fourier transforms.[4]

a) $F(\omega_1, \omega_2)$ is a complex function of the two real variables ω_1, ω_2 defined in the entire plane

$$\Omega \qquad -\infty < \omega_1 < +\infty, \qquad -\infty < \omega_2 < +\infty.$$

b)
$$F(-\omega_1, -\omega_2) = \overline{F}(\omega_1, \omega_2),$$

where $\overline{F}$ denotes the conjugate of F.

c) $F(\omega_1, \omega_2)$ is a continuous function of the two variables ω_1, ω_2 in every point of Ω.

d) $F(\omega_1, \omega_2)$ is bounded in Ω:

$$|F(\omega_1, \omega_2)| \leqq \frac{1}{4\pi^2}\int_D |f(x, y)|\, d\sigma. \tag{10}$$

We shall also use the bound

$$|F(\omega_1, \omega_2)| \leqq \frac{\sqrt{S}}{4\pi^2}\left[\int_D f(x, y)^2 d\sigma\right]^{1/2}, \tag{11}$$

where $S=\int_D d\sigma$. Equation (11) is deduced from (10) by means applying Schwarz' inequality.

e) If $|\omega_1|+|\omega_2|\to+\infty$, then $F(\omega_1, \omega_2)\to 0$ (Riemann's theorem).

f) If $\phi(x, y)=af(x, y)+bg(x, y)$, where a, b are constants, and $f(x, y)$, $g(x, y)$ continuous functions in $D+B$ having the Fourier transforms $F(\omega_1, \omega_2)$, $G(\omega_1, \omega_2)$, then $\phi(x, y)$ has the Fourier transform $\Phi(\omega_1, \omega_2)=aF(\omega_1, \omega_2)+bG(\omega_1, \omega_2)$.

g) If $f(x, y)$ and $g(x, y)$ are continuous functions in $D+B$ having the Fourier transforms $F(\omega_1, \omega_2)$ and $G(\omega_1, \omega_2)$, respectively, and if

$$F(\omega_1, \omega_2) = G(\omega_1, \omega_2),$$

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[3] In order to apply directly most of the known results of the theory of the Fourier transforms, it is often useful to consider $f(x, y)$ as defined in the entire plane X with $f(x, y)=0$ at every point outside $D+B$.

[4] S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932, pp. 183–197.

then

$$f(x, y) = g(x, y)$$

in every point of $D+B$.

h) If $F(\omega_1, \omega_2)$ is *absolutely integrable,*

$$\int_\Omega |F(\omega_1, \omega_2)| d\omega < +\infty,$$

the integral

$$\int_\Omega F(\omega_1, \omega_2) e^{i(\omega_1 x+\omega_2 y)} d\omega$$

defines a continuous function of x, y for all values of x, y; in $D+B$:

$$f(x, y) = \int_\Omega F(\omega_1, \omega_2) e^{i(\omega_1 x+\omega_2 y)} d\omega,$$

while

$$0 = \int_\Omega F(\omega_1, \omega_2) e^{i(\omega_1 x+\omega_2 y)} d\omega$$

outside $D+B$. Since the integral is continuous for all values of x, y, the function $f(x, y)$ must vanish on B. Thus, if the given function $f(x, y)$ does not vanish on B, its Fourier transform cannot be absolutely integrable.[5]

i) If $f(x, y)$ has continuous derivatives $\partial f/\partial x$, $\partial f/\partial y$ in $D+B$ (which is always the case for the functions considered here), one has

$$f(x, y) = \int_\Omega F(\omega_1, \omega_2) e^{i(\omega_1 x+\omega_2 y)} d\omega \tag{12}$$

in every point of D, the integral being now an *improper integral* defined as the limit

$$\lim_{\lambda\to+\infty} \int_{C_\lambda} F(\omega_1, \omega_2) e^{i(\omega_1 x+\omega_2 y)} d\omega, \tag{13}$$

C_λ being the circle $\omega_1^2+\omega_2^2 \leqq \lambda^2$ (Cauchy's principal value). As a rule, (12) does not hold on the boundary B.

j) $f(x, y)$ and $g(x, y)$ being continuous functions with continuous derivatives in $D+B$ and $F(\omega_1, \omega_2)$ and $G(\omega_1, \omega_2)$ their Fourier transforms, we have

$$\frac{1}{4\pi^2}\int_D f(x, y) g(x, y) e^{-i(\theta_1 x+\theta_2 y)} d\sigma = \int_\Omega F(\omega_1, \omega_2)\overline{G}(\omega_1+\theta_1, \omega_2+\theta_2) d\omega, \tag{14}$$

the meaning of the integral being the same as in (12). Here, θ_1 and θ_2 are arbitrary real variables; in particular for $\theta_1=\theta_2=0$ one has Parseval's formula:

[5] For instance, taking for $D+B$ the square $-1\leqq x\leqq +1$, $-1\leqq y\leqq +1$ and assuming that $f(x, y)=1$ in $D+B$ we obtain the Fourier transform $F(\omega_1, \omega_2)=(1/\pi^2\omega_1\omega_2)\sin\omega_1 \sin\omega_2$ which is not absolutely integrable. Eq. (12) holds, however, with the definition (13).

$$\frac{1}{4\pi^2}\int_D f(x, y)g(x, y)d\sigma = \int_\Omega F(\omega_1, \omega_2)\overline{G}(\omega_1, \omega_2)d\omega. \tag{15}$$

1. Let us introduce the Fourier transforms of the stream function, the velocity components and the vorticity:

$$\Psi(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D \psi(x, y, t)e^{-i(\omega_1 x+\omega_2 y)}d\sigma, \tag{16}$$

$$U(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D u(x, y, t)e^{-i(\omega_1 x+\omega_2 y)}d\sigma, \tag{17}$$

$$V(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D v(x, y, t)e^{-i(\omega_1 x+\omega_2 y)}d\sigma, \tag{18}$$

$$Z(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D \zeta(x, y, t)e^{-i(\omega_1 x+\omega_2 y)}d\sigma. \tag{19}$$

The first part of the present paper deals with the purely kinematical significance of ψ, u, v, ζ, all computations being supposed to be made at a given time t. Thus, there is no particular need for stressing the particular value of t. For the sake of brevity we shall therefore write $\psi(x, y)$, $\Psi(\omega_1, \omega_2)$, etc. for $\psi(x, y, t)$, $\Psi(\omega_1, \omega_2, t)$, etc.

THEOREM I—The Fourier transforms of the stream function and of the velocity components are expressed in terms of the Fourier transform of the vorticity by means of

$$\Psi(\omega_1, \omega_2) = \frac{2}{\omega_1^2 + \omega_2^2}Z(\omega_1, \omega_2), \tag{20}$$

$$U(\omega_1, \omega_2) = \frac{2i\omega_2}{\omega_1^2 + \omega_2^2}Z(\omega_1, \omega_2), \tag{21}$$

$$V(\omega_1, \omega_2) = \frac{-2i\omega_1}{\omega_1^2 + \omega_2^2}Z(\omega_1, \omega_2). \tag{22}$$

Substituting (1) into (17) we obtain

$$U = \frac{1}{4\pi^2}\int_D \frac{\partial\psi}{\partial y}e^{-i(\omega_1 x+\omega_2 y)}d\sigma$$
$$= \frac{1}{4\pi^2}\int_D \frac{\partial}{\partial y}\left[\psi e^{-i(\omega_1 x+\omega_2 y)}\right]d\sigma + \frac{i\omega_2}{4\pi^2}\int_D \psi e^{-i(\omega_1 x+\omega_2 y)}d\sigma.$$

Application of Green's formula to the first integral yields

$$U = -\frac{1}{4\pi^2}\int_B \beta\psi e^{-i(\omega_1 x+\omega_2 y)}ds + i\omega_2\Psi.$$

On account of the boundary condition (4) the first integral equal zero. Thus

$$U = i\omega_2\Psi. \tag{23}$$

Similarly

$$V = -i\omega_1\Psi. \tag{24}$$

Next, let us apply Green's formula

$$\int_D (f\Delta g - g\Delta f)d\sigma = -\int_B \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)ds$$

setting $f=\psi$, $g=e^{-i(\omega_1 x+\omega_2 y)}$. On account of the boundary condition (4) we have:

$$\int_D [\psi\Delta e^{-i(\omega_1 x+\omega_2 y)} - \Delta\psi e^{-i(\omega_1 x+\omega_2 y)}]d\sigma = 0.$$

If we replace $\Delta\psi$ by -2ζ in accordance with (5), and remark that

$$\Delta e^{-i(\omega_1 x+\omega_2 y)} = -(\omega_1^2+\omega_2^2)e^{-i(\omega_1 x+\omega_2 y)}, \tag{25}$$

we obtain

$$-(\omega_1^2+\omega_2^2)\int_D \psi e^{-i(\omega_1 x+\omega_2 y)}d\sigma + 2\int_D \zeta e^{-i(\omega_1 x+\omega_2 y)}d\sigma = 0$$

which is Eq. (20). Inserting the value (20) of Ψ into (23) and (24) we immediately obtain the expression (21) and (22) and our theorem is proved.[6]

Combining (21) and (22) we obtain the following relation

$$2iZ = \omega_2 U - \omega_1 V. \tag{26}$$

THEOREM II—The spectral function $\gamma(\omega_1, \omega_2)$ of the kinetic energy E which is defined by (8), *is expressed in terms of the Fourier transform of the vorticity by means of*

$$\gamma(\omega_1, \omega_2) = 8\pi^2\frac{|Z(\omega_1, \omega_2)|^2}{\omega_1^2+\omega_2^2}\cdot \tag{27}$$

To prove this, we start from Green's formula

$$\int_D \psi\Delta\psi d\sigma + \int_D \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right]d\sigma = -\int_B \psi\frac{\partial\psi}{\partial n}ds.$$

The integral over B equals zero on account of the boundary condition (4). If we replace $\Delta\psi$ by its value (5) and $\partial\psi/\partial x$, $\partial\psi/\partial y$ by (1) and (2), we obtain

$$E = \int_D \psi\zeta d\sigma. \tag{28}$$

Applying Parseval's formula (15), we find

$$E = 4\pi^2\int_\Omega \Psi\overline{Z}d\omega, \tag{29}$$

and hence

$$\gamma(\omega_1, \omega_2) = 4\pi^2\Psi(\omega_1, \omega_2)\overline{Z}(\omega_1, \omega_2). \tag{30}$$

[6] It is perhaps worthwhile to stress that the boundary condition (4) plays an essential part in this proof. As a rule, the relation $F_1 = i\omega_1 F$ does not hold between the Fourier transforms $F(\omega_1, \omega_2)$ and $F_1(\omega_1, \omega_2)$ of $f(x, y)$ and $\partial f/\partial x$. For instance for the function $f=1$ considered in remark 5, $F_1=0$, while $i\omega_1 F = (i\pi^2/\omega_2)\sin\omega_1 \sin\omega_2$.

From this we deduce (27) by inserting the value (20) for Ψ.

The theorems I and II reduce the harmonic analysis of the flow to the study of the Fourier transform Z of the vorticity. We shall now establish some properties of this function. According to (12).

$$\zeta(x, y) = \int_\Omega Z(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 y)} d\omega$$

$$= \lim_{\lambda \to +\infty} \int_{C_\lambda} Z(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 y)} d\omega. \tag{31}$$

We can then interpret $Z(\omega_1, \omega_2)d\omega_1 d\omega_2$ as the contribution of vortices, the frequencies of which are between ω_1 and $\omega_1 + d\omega_1$, and ω_2 and $\omega_2 + d\omega_2$.

$Z(\omega_1, \omega_2)$ is continuous and bounded in the whole plane Ω. Noting that

$$\Psi(0, 0) = \frac{1}{4\pi^2} \int_D \psi d\sigma$$

is finite, we obtain from (20),

$$Z(0, 0) = 0. \tag{32}$$

Equation (26) then yields

$$2|Z| \leqq |\omega_2||U| + |\omega_1||V|. \tag{33}$$

On account of (11) we have

$$|U| \leqq \frac{\sqrt{S}}{4\pi^2}\left[\int_D u^2 d\sigma\right]^{1/2} |V| \leqq \frac{\sqrt{S}}{4\pi^2}\left[\int_D v^2 d\sigma\right]^{1/2},$$

and consequently

$$|U|^2 + |V|^2 \leqq \frac{S}{8\pi^2} E. \tag{34}$$

Equation (33) gives then the following upper bound for $|Z|$:

$$|Z| \leqq \frac{\sqrt{2S}}{8\pi^2}(|\omega_1| + |\omega_2|)\sqrt{E}. \tag{35}$$

At this point it seems interesting to note a property of the Fourier transform Z. *If $\phi(x, y)$ is an arbitrary harmonic function in D, and $\Phi(\omega_1, \omega_2)$ is its Fourier transform, Z is orthogonal to Φ*, i.e.,

$$\int_\Omega Z(\omega_1, \omega_2) \overline{\Phi}(\omega_1, \omega_2) d\omega = 0.$$

In fact, the vorticity ζ is orthogonal to any harmonic function ϕ in D.[7] To prove this we need only use Green's formula

[7] J. Kampé de Fériet, *On a property of the Laplacian of a function in a two dimensional bounded domain, when the first derivatives of the function vanish at the boundary,* Mathematics Magazine No. 2, 1947.

$$\int_D (\phi\Delta\psi - \psi\Delta\phi)d\sigma = -\int_B \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds.$$

On account of the boundary condition (4) the second integral vanishes. Setting

$$\Delta\psi = -2\zeta, \qquad \Delta\phi = 0$$

in the first integral we find

$$\int_D \phi\zeta d\sigma = 0$$

which, on account of (15), is fully equivalent to our statement.

Let us now consider the spectral function $\gamma(\omega_1, \omega_2)$. We can interpret $\gamma(\omega_1,\omega_2)d\omega_1 d\omega_2$ as the amount of kinetic energy coming from the vortices with frequencies between ω_1 and $\omega_1+d\omega_1$, and ω_2 and $\omega_2+d\omega_2$.

The Fourier transforms Ψ and Z being continuous and bounded in the entire plane Ω, we see from (30) that $\gamma(\omega_1, \omega_2)$ is also continuous and bounded in the entire plane. Moreover, $\gamma(\omega_1, \omega_2) \geqq 0$ for all values of ω_1 and ω_2; on account of the same formula

$$\gamma(0, 0) = 4\pi^2\Psi(0, 0)\overline{Z}(0, 0),$$

and hence

$$\gamma(0, 0) = 0. \tag{36}$$

More precisely, if we write Eq. (30) in the following manner

$$\gamma(\omega_1, \omega_2) = 2\pi^2(\omega_1^2 + \omega_2^2) \mid \Psi(\omega_1, \omega_2) \mid^2,$$

we see that

$$\lim_{\omega_1\to 0, \omega_2\to 0} \frac{\gamma(\omega_1, \omega_2)}{\omega_1^2 + \omega_2^2} = 2\pi^2 \mid \Psi(0, 0) \mid^2. \tag{37}$$

Since Ψ and Z tend towards zero when $|\omega_1| + |\omega_2| \to +\infty$, we have

$$\lim_{|\omega_1|+|\omega_2|\to+\infty} \gamma(\omega_1, \omega_2) = 0. \tag{38}$$

From (27) it follows then that $\gamma(\omega_1, \omega_2)$ is decreasing more rapidly than $1/(\omega_1^2+\omega_2^2)$. Combining (27) and the inequality (35) we obtain an upper bound for $\gamma(\omega_1, \omega_2)$:

$$\gamma(\omega_1, \omega_2) \leqq \frac{S}{4\pi^2} \frac{(\mid \omega_1 \mid + \mid \omega_2 \mid)^2}{\omega_1^2 + \omega_2^2} E,$$

and hence

$$\gamma(\omega_1, \omega_2) \leqq \frac{S}{2\pi^2} E. \tag{39}$$

2. Let us now consider the variation of the flow with the time t, in accordance with Eq. (6) for a viscous incompressible fluid.

We shall use the following proposition:[8] *The kinetic energy is always decreasing*

[8] J. Kampé de Fériet, *Sur la décroissance de l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine plan borné*, C. R. Acad. Sci. Paris **223**, 1096–1098 (1946).

$(dE/dt<0)$; *more precisely, it is decreasing more rapidly than an exponential function of the time*:

$$E \leqq E_0 \exp\left(-\frac{8\nu}{\sqrt{K_D}} t\right). \tag{40}$$

Here, E_0 is the value of E at the time $t=0$, and K_D is a positive constant depending only on the domain D.

Applying this result to (35) and (39), we have the following

THEOREM III—The upper bounds of $|Z|$ *and* γ *given by the inequalities* (35) *and* (39) *are always decreasing, more precisely,*

$$|Z(\omega_1, \omega_2, t)| \leqq \frac{\sqrt{2SE_0}}{8\pi^2}(|\omega_1| + |\omega_2|) \exp\left(-\frac{4\nu}{\sqrt{K_D}} t\right) \tag{41}$$

$$|\gamma(\omega_1, \omega_2, t)| \leqq \frac{SE_0}{2\pi^2} \exp\left(-\frac{8\nu}{\sqrt{K_D}} t\right). \tag{42}$$

A more concrete picture is furnished by the surface $\gamma(\omega_1, \omega_2, t)$. This surface passes through the origin O and is asymptotic to the plane $\gamma=0$ for large values of ω_1, ω_2; its possible maxima are all below a given plane $\gamma=M$; with increasing time this plane approaches the plane $\gamma=0$ in an exponential manner.

In order to study the variations of the spectral function $\gamma(\omega_1, \omega_2, +)$ with the time t, we shall establish the equivalent of Eq. (6) in terms of Fourier transforms. To this end, we must compute the Fourier transforms of $\partial\zeta/\partial t$, of $u\partial\zeta/\partial x + v\partial\zeta/\partial y$ and of $\Delta\zeta$.

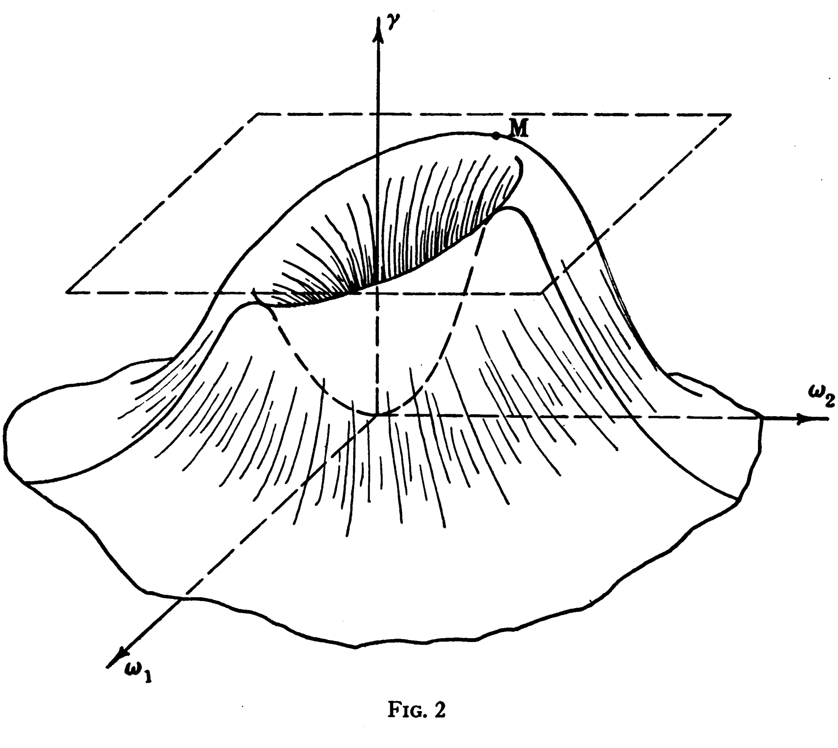

FIG. 2

For the first function the result is obvious. Since ζ and $\partial\zeta/\partial t$ are continuous in t, the functions Z and $\partial Z/\partial t$ are also continuous in t. Thus,

$$\frac{\partial}{\partial t} Z(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D \frac{\partial \zeta}{\partial t} e^{-i(\omega_1 x+\omega_2 y)} d\sigma. \tag{43}$$

The Fourier transform of $\partial\zeta/\partial t$ is therefore given by $\partial Z/\partial t$.

To compute the second Fourier transform, we start from the equation

$$\left(u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y}\right)e^{-i(\omega_1 x+\omega_2 y)} = \frac{\partial}{\partial x}\left[u\zeta e^{-i(\omega_1 x+\omega_2 y)}\right] + \frac{\partial}{\partial y}\left[v\zeta e^{-i(\omega_1 x+\omega_2 y)}\right] + i(\omega_1\zeta u + \omega_2\zeta v)e^{-i(\omega_1 x+\omega_2 y)}.$$

We integrate this equation over D and note that by Green's formula,

$$\int_D \frac{\partial}{\partial x}\left[u\zeta e^{-i(\omega_1 x+\omega_2 y)}\right]d\sigma = -\int_B \alpha u\zeta e^{-i(\omega_1 x+\omega_2 y)} ds = 0, \tag{44}$$

$$\int_D \frac{\partial}{\partial y}\left[v\zeta e^{-i(\omega_1 x+\omega_2 y)}\right]d\sigma = -\int_B \beta v\zeta e^{-i(\omega_1 x+\omega_2 y)} ds = 0, \tag{45}$$

on account of the boundary condition (3). Accordingly,

$$\int_D\left(u\frac{\partial\zeta}{\partial y} + v\frac{\partial\zeta}{\partial y}\right)e^{-i(\omega_1 x+\omega_2 y)}d\sigma = i\omega_1\int_D u\zeta e^{-i(\omega_1 x+\omega_2 y)}d\sigma + i\omega_2\int_D v\zeta e^{-i(\omega_1 x+\omega_2 y)}d\sigma.$$

Using formula (14), we obtain

$$\frac{1}{4\pi^2}\int_D u\zeta e^{-i(\omega_1 x+\omega_2 y)}d\sigma = \int_\Theta U(\theta_1, \theta_2)\overline{Z}(\theta_1+\omega_1, \theta_2+\omega_2)d\theta,$$

$$\frac{1}{4\pi^2}\int_D v\zeta e^{-i(\omega_1 x+\omega_2 y)}d\sigma = \int_\Theta V(\theta_2, \theta_2)\overline{Z}(\theta_1+\omega_1, \theta_2+\omega_2)d\theta,$$

where the variables of integration are now called θ_1, θ_2, and $d\theta$ denotes $d\theta_1 d\theta_2$, the integral being extended over the entire plane Θ. We thus have proved that the Fourier transform of $u\partial\zeta/\partial x + v\partial\zeta/\partial y$ equals

$$i\int_\Theta [\omega_1 U(\theta_1, \theta_2) + \omega_2 V(\theta_1, \theta_2)]\overline{Z}(\theta_1+\omega_1, \theta_2+\omega_2)d\theta$$

or, what is the same by (21) and (22),

$$2\int_\Theta \frac{\theta_1\omega_2 - \theta_2\omega_1}{\theta_1^2+\theta_2^2} Z(\theta_1, \theta_2)\overline{Z}(\theta_1+\omega_1, \theta_2+\omega_2)d\theta. \tag{46}$$

Let us now consider the Fourier transform of the Laplacian of the vorticity:

$$\mathfrak{E}(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D \Delta\zeta e^{-i(\omega_1 x+\omega_2 y)}d\sigma. \tag{47}$$

Applying Green's formula

$$\int_D [\Delta\zeta e^{-i(\omega_1 x+\omega_2 y)} - \zeta\Delta(e^{-i(\omega_1 x+\omega_2 y)})]d\sigma$$

$$= -\int_B \left[\frac{\partial\zeta}{\partial n} e^{-i(\omega_1 x+\omega_2 y)} - \zeta \frac{\partial}{\partial n}(e^{-i(\omega_1 x+\omega_2 y)})\right] ds$$

and setting.

$$\Phi(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_B \left[\frac{\partial\zeta}{\partial n} + i(\omega_1\alpha + \omega_2\beta)\zeta\right] e^{-i(\omega_1 x+\omega_2 y)}ds, \tag{48}$$

we obtain

$$C(\omega_1, \omega_2, t) = -(\omega_1^2 + \omega_2^2)Z(\omega_1, \omega_2, t) - \Phi(\omega_1, \omega_2, t). \tag{49}$$

Reviewing the expressions of the Fourier transforms of $\partial\zeta/\partial t$, $u\partial\zeta/\partial x+v\partial\zeta/\partial y$, and $\Delta\zeta$ we may state the following

THEOREM IV—Equation (6) *is equivalent to the following integrodifferential equation for the Fourier transforms*:

$$\frac{\partial}{\partial t} Z(\omega_1, \omega_2, t) + 2\int_\Theta \frac{\theta_1\omega_2 - \theta_2\omega_1}{\theta_1^2 + \theta_2^2} Z(\theta_1, \theta_2, t)\overline{Z}(\theta_1 + \omega_1, \theta_2 + \omega_2, t)d\theta$$

$$= -\nu(\omega_1^2 + \omega_2^2)Z(\omega_1, \omega_2, t) - \nu\Phi(\omega_1, \omega_2, t). \tag{50}$$

This equation, being fully equivalent to the Navier-Stokes equations, is the rational starting point for any rigorous study of the Fourier transform of the vorticity and hence, of the spectral function. This study seems to be very difficult; perhaps it will be only possible to check, in some approximate way, some of the working hypotheses which were assumed in recent papers, e.g., the assumption that the big eddies have a tendency to degenerate into smaller ones. This means that when t is increasing the peak on the spectral surface $\gamma(\omega_1, \omega_2, t)$ is gradually shifted towards the high frequencies ω_1, ω_2.

3. We can get interesting general results, if we consider the case where the flow extends over the entire plane.

$$X \qquad -\infty < x < +\infty, \qquad -\infty < y < +\infty.$$

This case may be regarded as a limiting case of previous problem. The boundary condition (3) means now that the fluid is at rest at infinity:

$$\lim_{r\to+\infty} u(x, y, t) = 0 \qquad \lim_{r\to+\infty} v(x, y, t) = 0. \tag{51}$$

$$r = +\sqrt{x^2 + y^2}.$$

If the kinetic energy is to remain finite,

$$E = \frac{1}{2}\int_X (u^2 + v^2)d\sigma < +\infty, \tag{52}$$

the decrease of u and v must be sufficiently rapid. A sufficient condition is

$$u = O(r^{-3/2}), \qquad v = O(r^{-3/2}) \quad \text{for large values of } r. \tag{53}$$

To insure the existence of the Fourier transform

$$Z(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_X \zeta(x, y, t)e^{-i(\omega_1 x+\omega_2 y)}d\sigma, \tag{54}$$

it is sufficient to assume that ζ is absolutely integrable

$$\int_X |\zeta| \, d\sigma < +\infty. \tag{55}$$

In this case, all the properties a) to j) enumerated for the Fourier transform (9) still hold for (54). The condition is obviously satisfied if:

$$\zeta = O(r^{-3}) \quad \text{for large values of } r. \tag{56}$$

To compute the Fourier transforms of $\partial\zeta/\partial t$, $u\partial\zeta/\partial x+v\partial\zeta/\partial y$, and $\Delta\zeta$, we can use the same process as before, taking for D the circle $x^2+y^2<R^2$ and then letting R tend towards $+\infty$. We must observe, however, that we no longer have $u=0$, $v=0$ on the boundary B; we must therefore carefully examine some terms in our equations.

In the evaluation of the Fourier transform of $u\partial\zeta/\partial x+v\partial\zeta/\partial y$, the two terms (44) and (45) do no longer vanish. According to Schwarz' inequality we have, however,

$$\left|\int_B \alpha u\zeta e^{-i(\omega_1 x+\omega_2 y)}ds\right| \leqq \int_B |u|\,|\zeta|\,ds \leqq \left[\int_B u^2 ds\right]^{1/2}\left[\int_B \zeta^2 ds\right]^{1/2}.$$

From our assumption concerning the decrease of u and ζ, it is obvious that

$$\lim_{R\to+\infty}\int_B u^2 ds = 0, \qquad \lim_{R\to+\infty}\int_B \zeta^2 ds = 0.$$

Similar considerations apply to (45). Thus, at the limit, the expression (46) for the Fourier transform still holds.

In the evaluation of the Fourier transform of $\Delta\zeta$ we obtain a very fortunate simplification: from (48) ,we have

$$|\Phi| \leqq \frac{1}{4\pi^2}\int_B \left[\left|\frac{\partial\zeta}{\partial n}\right| + (|\omega_1| + |\omega_2|)\,|\zeta|\right]ds.$$

The assumption of decreasing of ζ yields

$$\lim_{R\to+\infty}\int_B \left|\frac{\partial\zeta}{\partial n}\right|ds = 0, \qquad \lim_{R\to+\infty}\int_B |\zeta|\,ds = 0;$$

thus

$$\lim_{R\to+\infty} \Phi = 0.$$

The Fourier transforms C and Z being extended to the whole plane, we have thus now:

$$C = -(\omega_1^2 + \omega_2^2)Z.$$

THEOREM V—In the case of a flow extending over the entire plane X, where the fluid is at rest at infinity, and where u, v, ζ decrease so rapidly that

$$E = \frac{1}{2}\int_X (u^2 + v^2)d\sigma < +\infty$$

and

$$\int_X |\zeta|\, d\sigma < +\infty,$$

the Fourier transform of the vorticity satisfies the integrodifferential equation:

$$\frac{\partial}{\partial t} Z(\omega_1, \omega_2, t) + 2\int_\Theta \frac{\theta_1\omega_2 - \theta_2\omega_1}{\theta_1^2 + \theta_2^2} Z(\theta_1, \theta_2, t)\cdot \overline{Z}(\theta_1 + \omega_1, \theta_2 + \omega_2, t)d\theta \qquad (57)$$
$$= -\nu(\omega_1^2 + \omega_2^2)Z(\omega_1, \omega_2, t).$$

This equation seems to me the most simple that can be obtained in this manner.

To give at least an application, we shall establish the general solution of (57) for the particular, but interesting, class of flows, where the vorticity is constant along every stream line.

The flows characterized by $\zeta = F(\psi)$ have been the subject of interesting researches.[9] It is in this case and only in this case that non-linear terms disappear in the Navier-Stokes equations; their Fourier transform vanishes, of course. Equation (57) is then reduced to

$$\frac{\partial Z}{\partial t} = -\nu(\omega_1^2 + \omega_2^2)Z,$$

and the general solution is

$$Z(\omega_1, \omega_2, t) = Z_0(\omega_1, \omega_2)e^{-\nu(\omega_1^2+\omega_2^2)t}.$$

Here, $Z_0(\omega_1, \omega_2)$ denotes the arbitrary value of Z for $t = 0$. It is seen that, in this case, not only an upper bound of Z, but Z itself is decreasing exponentially for every value of ω_1, ω_2.

[9] Ratib Berker, *Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible,* Thesis, Lille, 1936.