After $(n-1)$ modes have been found, and the reduction of the matrix a_{rs} carried out for each of them in the manner described, we must be left with the matrix

$$a_{rs} - \sum_{j \neq i} a_{rs}^{(j)} = a_{rs}^{(i)},$$

where i is the number of the mode left to be found. In this matrix the terms of any row are proportional to $\sum_e a_{se}x_e^{(i)}$ so the remaining i-th mode can be found by solving the set of linear equations

$$\sum_s a_{rs}x_s = a_{1r}^{(i)}.$$

The reductions can often be carried out with advantage in terms of the matrix b_{rs} ; this is done by normalizing the modes so that instead of satisfying (10) they satisfy

$$\sum_{rs} b_{rs}y_r^{(i)}y_s^{(i)} = 1,$$

and calculating $a_{rs}^{(i)}$ from the formula

$$a_{rs}^{(i)} = \lambda^{(i)} \sum_k b_{rk}y_k^{(i)} \sum_e b_{se}y_e^{(i)},$$

which is easily deduced from (8). This is particularly convenient in the most usual type of problem in which b_{rs} is a diagonal matrix—in mechanical problems, those in which the kinetic energy can be expressed as a sum of squares. If $| b_{rs} | = | m_r\delta_{rs} |$ the modes must be normalized by

$$\sum_r m_r(y_r^{(i)})^2 = 1$$

and then

$$a_{rs}^{(i)} = \lambda^{(i)}m_rm_sx_r^{(i)}x_s^{(i)}.$$

After reductions corresponding to all but the i-th mode have been made on the matrix a_{rs} , the remaining matrix is $a_{rs}^{(i)}$ from whose rows the i-th mode can be found immediately.

A NORM CRITERION FOR NON-OSCILLATORY DIFFERENTIAL EQUATIONS*

By AUREL WINTNER (*The Johns Hopkins University*)

Let $f(t)$, $x(t)$, $\lambda(t)$, $\cdots$ denote real-valued, continuous functions on an unspecified half-line, $t_0 \leqq t < \infty$. If $\lambda(t)$ is positive on this half-line, put

$$\lambda^* = \lambda^*(t) = \lambda(t) \int_t^\infty (du)/\lambda^2(u), \tag{1}$$

provided that the second factor on the right of (1) is a convergent integral. Under this proviso, a direct substitution of (1) shows that, if $\lambda(t)$ is a solution of the differential equation $D_f(\lambda) = 0$, where

*Received Nov. 17, 1947.

$$D_f(\lambda) = D_f(\lambda(t)) = \lambda''(t) + f(t)\lambda(t), \qquad (' = d/dt) \tag{2}$$

then $D_f(\lambda^*) = 0$, i.e., that $\lambda^*(t)$ represents another (linearly independent) solution.

Following A. Kneser[1], let the differential equation $D_f(x) = 0$ be called oscillatory or non-oscillatory according as each or none of its solutions $x(t) \not\equiv 0$ has zeros clustering at $t = \infty$. This alternative is complete, since, in view of Sturm's separation theorem, either every or no solution $x(t) \not\equiv 0$ of $D_f(x) = 0$ has an infinity of zeros on the half-line $t \geqq t_0$. The decision of the alternative (for a *given* coefficient function, $f = f(t)$, of D_f) is fundamental in certain questions of stability and related applications[2].

It seems to be of both theoretical and practical interest that the decision can always be based on a criterion similar to the "norm" conditions in the theory of linear functionals and operators (Lebesgue-Toeplitz). It is a criterion the applicability of which does not involve, in principle, the knowledge of a *solution* $x(t) \not\equiv 0$ of $D_f(x) = 0$, since it depends on the consideration of *arbitrary functions.* It can be formulated as follows:

The differential equation $D_f(x) = 0$ is of non-oscillatory type if and only if there exists some positive function, say $\lambda(t)$, corresponding to which the assignments (1), (2) *define two continuous functions the product of which is absolutely integrable, i.e.,*

$$\int^{\infty} \lambda^* \mid D_f(\lambda) \mid dt < \infty. \tag{3}$$

As an illustration of how to apply this criterion, choose the arbitrary function $\lambda(t)$ to be t. Then (1) and (2) reduce to $\lambda^* = 1$ and $D_f(\lambda) = f(t)t$, respectively, and so (3) will be satisfied if $|f(t)| \, t$ has a finite integral over the half-line. It follows that the absolute integrability of $f(t)t$ (which, incidentally, is compatible with $\limsup f(t)t = \infty$ *and* $\liminf f(t) = -\infty$, where $t \to \infty$) is sufficient in order that the differential equation $D_f(x) = 0$ be of non-oscillatory type.

Actually, this particular sufficient condition is contained in an asymptotic result of Bôcher[3]. But this is not a necessary condition. In fact, other sufficient conditions result if the choice $\lambda = t$ is replaced by other choices of the arbitrary function $\lambda(t)$. Such choices can be made *relative to the coefficient function, f, of D_f*, rather than in a way which, as in $\lambda = t$, is independent of f.

Proof of the sufficiency. This part of the italicized criterion can be deduced from the following fact, which is a corollary of a general theorem[4]: If $p = p(t) \neq 0$ and $q = q(t)$ are continuous functions for large positive t, then the condition

$$\int^{\infty} \mid q(t) \mid \left(\int_t^{\infty} \mid p(u) \mid^{-1} du \right) dt < \infty \tag{4}$$

is sufficient in order that *some* solution $y = y(t)$ of the differential equation

$$(py')' + qy = 0 \qquad (' = d/dt) \tag{5}$$

[1] A. Kneser, *Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen,* Mathematische Annalen **42**, 409-435 (1893), p. 411.

[2] T. v. Kármán and M. A. Biot, *Mathematical methods in engineering,* New York and London, 1940, Chapter VII and the references on p. 322.

[3] M. Bôcher, *On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic,* Transactions of the American Mathematical Society **1**, 40-52 (1900), pp. 48-52.

[4] A. Wintner, *Asymptotic integrations of the adiabatic oscillator in its hyperbolic range,* to appear in the Duke Mathematical Journal **15**, (1948).

should tend to a finite limit, as $t \to \infty$, and that this limit, $y(\infty)$, be distinct from 0.

It follows that, if $\lambda = \lambda(t)$ is any positive function possessing a continuous second derivative, then the case

$$p = \lambda^2, \qquad q = \lambda\lambda'' + f\lambda^2 \tag{6}$$

of (5) must have some solution $y = y(t)$ which does not vanish from a certain t onward, if condition (4) is satisfied by the functions (6). But it is clear from the definitions (1), (2) that the case (6) of (4) is identical with (3). Since $\lambda(t)$ is positive, it follows that (3) implies the non-oscillatory character of that differential equation for $x = x(t)$ which results when $y = x/\lambda$ is substituted into the case (6) of (5).

The result of this substitution is seen to be the differential equation

$$(x'\lambda - x\lambda')' + (\lambda'' + f\lambda)x = 0.$$

Since the latter can be contracted into $(x'' + fx)\lambda = 0$, where $\lambda > 0$, it is equivalent to $x'' + fx = 0$ and so, in view of (2), to $D_f(x) = 0$. This completes the proof of the sufficiency of (3).

Proof of the necessity. This part of the criterion is of theoretical interest only, and its verification is straightforward. As a matter of fact, $\lambda(t)$ can now be chosen to be a solution $x(t)$ of $D_f(x) = 0$.

In order to see this, suppose that the differential equation is non-oscillatory. Then there exist a constant t_0 and a solution $x(t)$ of $D_f(x) = 0$ such that $x(t) > 0$ when $t_0 \leqq t < \infty$. Let t^0 be any value exceeding t_0, restrict t to the half-line $t^0 \leqq t < \infty$, and put

$$\lambda(t) = x(t) \int_{t_0}^{t} (du)/x^2(u). \tag{7}$$

Then $\lambda(t)$ is positive, since $x(t)$ is. Furthermore, it is easily verified from (2) and (7) that $D_f(\lambda) = 0$, since $D_f(x) = 0$. Hence, in order to prove that condition (3) is satisfied by the function (7), all that remains to be ascertained is that the function (1) exists in the case (7), i.e., that

$$\int^{\infty} (du)/\lambda^2(u) < \infty \tag{8}$$

holds by virtue of (7). But this can be ascertained by an elementary argument used by Hartman[5].

In fact, it is readily verified from (7) that the Wronskian, $x\lambda' - \lambda x'$, of $x(t)$ and $\lambda(t)$ is the constant 1. Hence, the derivative of x/λ is identical with $-1/\lambda^2$, and so

$$x(t)/\lambda(t) = \text{const.} - \int_{t_0}^{t} (du)/\lambda^2(u).$$

Since $x(t) > 0$ and $\lambda(t) > 0$, it follows that

$$0 < \text{const.} - \int_{t_0}^{t} (du)/\lambda^2(u).$$

This proves (8).

[5] P. Hartman, *On differential equations with non-oscillatory eigenfunctions*, to appear soon.