THE EXTRUSION OF PLASTIC SHEET THROUGH FRICTIONLESS ROLLERS*

By G. F. CARRIER (*Brown University*)

1. Introduction: The Saint Venant-Mises theory of slow plane plastic flow has repeatedly been applied to problems concerning the deformation process which occurs when sheets are formed by passing the material between two fixed cylindrical surfaces with parallel axes. These problems include, of course, the problems of the rolling, extrusion, and drawing of such sheets. The analyses given are of two kinds. In one [1],[1] [6], a simple one-dimensional theory is given; in the other [2], a more laborious scheme is used wherein the flow field is determined numerically by the method of characteristics (now very familiar to engineers because of its use in supersonic aerodynamic theory). It seems desirable to present information which either justifies the use of the simple theory, or modifies the simple theory so that its results become as accurate as those obtained in the numerical process mentioned above. When the thickness t of the formed sheet and the diameter R of the cylindrical forming surfaces are of the same order of magnitude, it appears that the numerical scheme mentioned above is the most efficient procedure. However, when the sheet is thin ($t/R \ll 1$) this procedure becomes very tedious. In this paper, we develop, from the fundamental equation of the Saint Venant-Mises theory, an approximation technique which leads directly to a justification of the one-dimensional theory for the cases where the cylindrical surfaces are frictionless and $t/R \ll 1$. The less idealized case will be treated in a subsequent paper.

2. Formulation of the problem: As is well known [3], the stress analysis in the case of problems of plane plastic strain is based on the yield condition

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2 \tag{1}$$

and the equations of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \tag{2}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \tag{3}$$

The following substitution of variables simplifies the procedure. We set

$$\sigma_x/k = 2\omega + \sin 2\theta \tag{4}$$

$$\sigma_y/k = 2\omega - \sin 2\theta \tag{5}$$

$$\tau_{xy}/k = -\cos 2\theta. \tag{6}$$

We note that this manner of expressing the stress components implies the satisfaction of Eq. (1).

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[1]Numbers in brackets refer to the bibliography at the end of this paper.

Omitting details which can be found in Reference [4], we observe that under fairly general conditions Eqs. (2) and (3) are equivalent to the system

$$y_\xi + x_\eta \cot \theta = 0, \tag{7}$$

$$y_\xi - x_\eta \tan \theta = 0, \tag{8}$$

where

$$\xi = \omega + \theta, \qquad \eta = \omega - \theta.$$

It is evident that this system of equations is linear in the functions x, y which are to be determined as functions of ξ and η. It must be noted that this transformation can be applied only when the region in the x, y plane does not correspond to a line or point (degenerate region) in the ξ, η plane. This will occur in a limited part of our flow region [4].

The boundary conditions are related to the stress conditions at the cylindrical surface and at the inlet and exit sections. In the absence of friction the shear stress τ must vanish at the roll surfaces. On each of these surfaces then

$$\tau(x, y_0) = -\cos [2(\theta - \gamma)] = 0,$$

$$\tan \gamma = dy_0/dx,$$

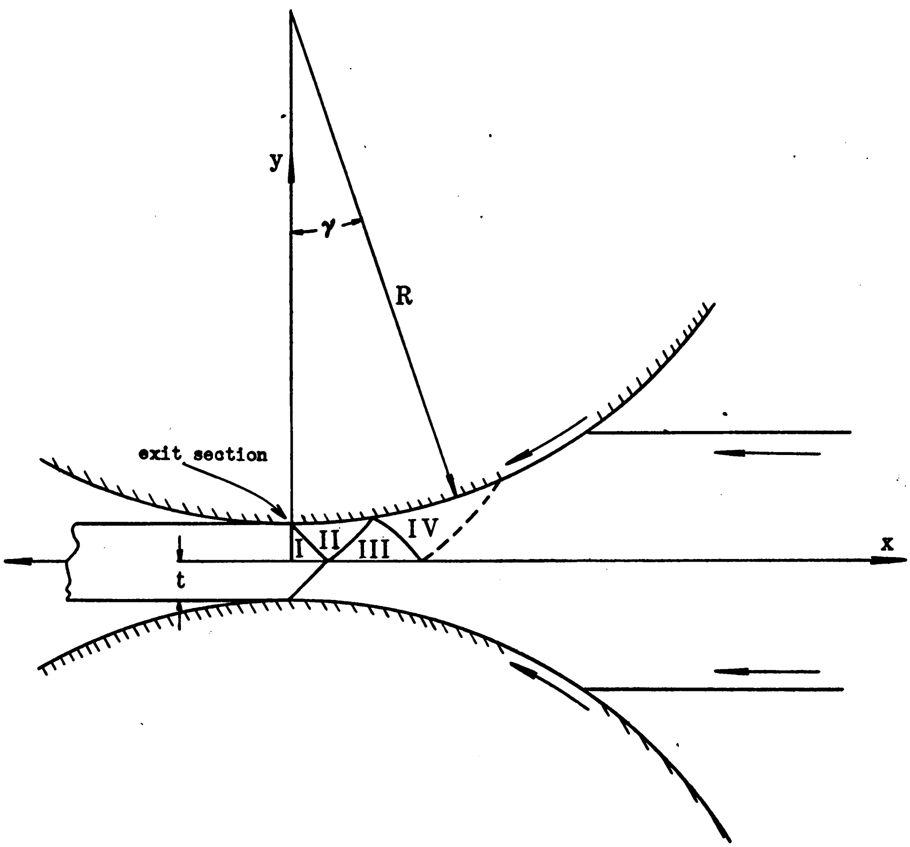

Fig. 1.

where $y = y_0(x)$ is the equation of the roll surface. If we consider that the material enters and leaves as a rigid body (i.e. if we assume the elastic deformations before entrance and after exit to be negligible), we must demand that the material leave the roll (see Fig. 1) with uniform horizontal velocity. Since the velocities must be continuous, a uniform state must then exist in the plastic region adjacent to exit. In fact, it is known (see, for instance, Reference [2]) that region I (Fig. 1) is associated with single values of ξ and η or of ω and θ, namely: $\omega = \omega_{exit} = \omega_e$, $\theta = \pi/4$. The former number is associated with the pull exerted externally on the exit section and the latter value follows from the condition that τ_{xy} must vanish on $y = 0$. It is also known (see for instance Reference [5]) that since region II contains a straight characteristic (its boundary with region I) it is a region in which $\xi = \text{const.} = \pi/4 + \omega_e$, but where η varies.[2] The mapping of the x, y plane onto the ξ, η plane is shown qualitatively in Fig. 2.

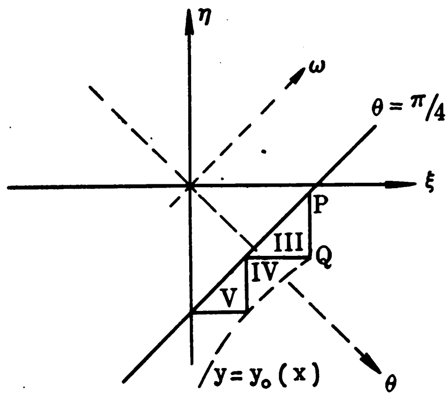

Fig. 2.

In view of the degeneracy of the mapping of regions I and II and in view of the previously stated remark on such mappings, it is evident that any approximate solution of Eqs. (7) and (8) could not apply to regions I and II. We should also note at this point that a boundary value problem of this type cannot have a single analytic solution for the entire region. In fact, the derivatives ξ_x, ξ_y, η_x, η_y will, in general, be discontinuous at the boundaries separating the individual regions. Thus, *it is not possible to write with rigor* either

$$\xi = \sum_0^\infty a_i f_i(x, y), \qquad \eta = \sum_0^\infty b_i f_i(x, y).$$

$$x = \sum_0 \alpha_i \varphi_i(\xi, \eta), \qquad y = \sum_0 \beta_i \omega_i(\xi, \eta),$$

or any other such development.

However, we *can* write

$$x = \sum_0^N \alpha_i \varphi_i(\xi, \eta) = \sum_0^N [g_i(\omega)](\theta - \pi/4)^i \tag{9}$$

$$y = \sum_0^N \beta_i \varphi_i(\xi, \eta) = \sum_0^N [f_i(\omega)](\theta - \pi/4)^i \tag{10}$$

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[2]This remark would be slightly modified for the case where non-vanishing wall friction exists, but it applies rigorously to our present problem.

where *by proper choice of N, the f_i, and the g_i we can make these functions approach as closely as we wish*[3] *the functions $x(\omega, \theta)$ and $y(\omega, \theta)$*. We observe that the actually existing discontinuous derivatives will not appear in these functions, but we also note that no differentiation of these functions is required in order to give the complete state of stress.

The equation of the upper roll boundary can be written as

$$y_0(x)/R = \epsilon + 1 - (1 - x^2/R^2)^{1/2} = \epsilon + 1 - \cos\gamma.$$

On the roll surface (i.e. at $y = y_0$)

$$\tau = -\cos 2(\theta - \gamma) \equiv 0,$$

i.e.

$$\theta = \pi/4 + \gamma, \tag{11}$$

and on $y = 0$, τ_{xy} vanishes; that is, on $y = 0$, $\theta = \pi/4$. We now let the quantities x, y in Eqs. (9) and (10) be the dimensionless coordinates[4] x/R, y/R, so that on the roll boundary we have [from Eq. (11)]

$$\theta = \pi/4 + \gamma,$$

$$y = \epsilon + 1 - \cos\gamma,$$

$$x = \sin\gamma,$$

that is,

$$\sin\gamma = \sum_0^N f_i(\omega)(\theta - \pi/4)^i = \sum_0^N g_i(\omega)\gamma^i, \tag{12}$$

$$\epsilon + 1 - \cos\gamma = \sum_{i=1}^N f_i(\omega)\gamma^i. \tag{13}$$

We now note that Eqs. (7) and (8) can be written

$$(1 + \tan\gamma)y_\xi + (1 - \tan\gamma)x_\xi = 0,$$

$$(1 - \tan\gamma)y_\eta - (1 + \tan\gamma)x_\eta = 0,$$

or

$$(1 + \gamma + \gamma^3/3 + \cdots)y_\xi + (1 - \gamma + \cdots)x_\xi = 0, \tag{14}$$

$$(1 - \gamma - \cdots)y_\eta - (1 + \gamma + \cdots)x_\eta = 0, \tag{15}$$

and when we use Eqs. (9) and (10), we obtain [noting that $y(\omega)$ is odd in $\theta - \pi/4$]

$$f_1 = -g_0', \tag{16}$$

$$g_2 = g_0' + g_0''/2, \tag{17}$$

$$f_3 = \frac{1}{3}(2g_0' + g_0'' - g_0'''/2), \tag{18}$$

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[3] In the integral mean square sense.

[4] R is the radius of the cylinder.

$$g_4 = \frac{1}{4}\left(\frac{2g_0'}{3} - \frac{4g_0''}{3} + \frac{g_0'''}{3} + \frac{g_0^{\text{iv}}}{3}\right), \tag{19}$$

. . . .

When these are substituted in Eqs. (14), (15) we obtain two differential equations in the functions $\gamma(\omega)$, $g_0(\omega)$. These equations can be written in the form

$$\begin{aligned}\frac{\gamma\gamma'}{\gamma^2/2 + \epsilon} + 1 = \frac{1}{(\gamma^2/2 + \epsilon)}&\left\{\left(\frac{4}{3}g_0'' + \frac{2}{3}g_0' - \frac{1}{3}g_0'''\right)\gamma^3\right.\\ &+ \left(\frac{2}{9}g_0' - \frac{1}{45}g_0'' - 23g_0''' + \frac{1}{6}g_0^{\text{iv}} + \frac{1}{60}g_0^{\text{v}}\right)\gamma^5\\ &\left.+ 2\left(g_0 + \frac{1}{2}g_0''\right)\gamma^2\gamma' + \left(\frac{2}{3}g_0' - \frac{4}{3}g_0'' + \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\text{iv}}\right)\gamma^4\gamma' + \cdots\right\},\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{g_0g_0'}{g_0^2/\epsilon + 1} + 1 = \frac{1}{2(g_0^2/2 + \epsilon)}&\left\{-(2g_0' + g_0)\left(g_0' + \frac{1}{2}g_0''\right)\gamma^2\right.\\ &- \left(\frac{1}{3}g_0' + \frac{1}{6}g_0'' + \frac{1}{3}g_0'''\right)\gamma^3 - \frac{1}{4}(2g_0' + g_0)\left(\frac{2}{3}g_0' - \frac{4}{3}g_0''\right.\\ &\left.\left.+ \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\text{iv}}\right)\gamma^4 + \left(\frac{5}{18}g_0' - \frac{1}{45}g_0'' - \frac{13}{180}g_0''' + \frac{1}{12}g_0^{\text{iv}} - \frac{2}{15}g_0^{\text{v}}\right)\gamma^5 + \cdots\right\}.\end{aligned} \tag{21}$$

We note that these equations are of an order which depends on the value of N [see Eq. (12)]. Thus, many boundary conditions are indicated. Recalling, however, the given information concerning regions I, II, we see that at $\theta = \pi/4$ [i.e. $y = 0$], $g_0(\omega_e) = \epsilon$. Also at $\gamma = 2\epsilon$ (the upper corner of region II), we have $\omega + \theta = \omega_e + \pi/4$, $\theta = \pi/4 + 2\epsilon$ hence $\omega = \omega_e - 2\epsilon$. If we now define $s = \omega_e - \omega$, the differential equations (20), (21) become

$$\begin{aligned}\frac{\gamma\gamma'}{\epsilon + \gamma^2/2} = 1 - \frac{1}{(\gamma^2/2 + \epsilon)}&\left\{\left(\frac{4}{3}g_0'' + \frac{2}{3}g_0' - \frac{1}{3}g_0'''\right)\gamma^3\right.\\ &+ \left(\frac{2}{9}g_0' - \frac{1}{45}g_0'' - 23g_0''' + \frac{1}{6}g_0^{\text{iv}} + \frac{1}{60}g_0^{\text{v}}\right)\gamma^5\\ &\left.+ 2\left(g_0 + \frac{1}{2}g_0''\right)\gamma^2\gamma' + \left(\frac{2}{3}g_0' - \frac{4}{3}g_0'' + \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\text{iv}}\right)\gamma^4\gamma' + \cdots\right\},\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{g_0g_0'}{g_0^2/\epsilon + 1} = 1 - \frac{1}{2(g_0^2/2 + \epsilon)}&\left\{-(2g_0' + g_0)\left(g_0' + \frac{g_0''}{2}\right)\gamma^2\right.\\ &- \left(\frac{1}{3}g_0' + \frac{1}{6}g_0'' + \frac{1}{3}g_0'''\right)\gamma^3 - \frac{1}{4}(2g_0' + g_0)\left(\frac{2}{3}g_0' - \frac{4}{3}g_0''\right.\\ &\left.\left.+ \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\text{iv}}\right)\gamma^4 + \left(\frac{5}{18}g_0' - \frac{1}{45}g_0'' - \frac{13}{180}g_0''' + \frac{1}{12}g_0^{\text{iv}} - \frac{2}{15}g_0^{\text{v}}\right)\gamma^5 + \cdots\right\},\end{aligned} \tag{23}$$

where the primes now indicate differentiation with respect to s.

Rigorously, we have more boundary conditions along the line ξ = const. which represents region II of the physical plane. If these were applied, the problem would be over-determined and, in fact, we could at best obtain a solution valid for region III only. If we relax these conditions and define $\varphi = \ln(1 + \gamma^2/2\epsilon)$, $\psi = \ln(1 + g_0^2/2\epsilon)$, Eqs. (22) and (23) can be put into the form

$$\begin{aligned}\varphi = s - \int_0^s \frac{1}{(\gamma^2/2 + \epsilon)} \Big\{ &\left(\frac{4}{3}g_0'' + \frac{2}{3}g_0' - \frac{1}{3}g_0'''\right)\gamma^3 \\ &+ \left(\frac{2}{9}g_0' - \frac{1}{45}g_0'' - 23g_0''' + \frac{1}{6}g_0^{\text{iv}} + \frac{1}{60}g_0^{\text{v}}\right)\gamma^5 \\ &+ 2\left(g_0 + \frac{1}{2}g_0''\right)\gamma^2\gamma' + \left(\frac{2}{3}g_0' - \frac{4}{3}g_0'' + \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\text{iv}}\right)\gamma^4\gamma' + \cdots \Big\}\, ds,\end{aligned} \tag{24}$$

$$\begin{aligned}\psi = s + \int_0^s \frac{1}{2(g_0^2/2 + \epsilon)} \Big\{ &(2g_0' + g_0)\left(g_0' + \frac{g_0''}{2}\right)\gamma^2 \\ &+ \left(\frac{1}{3}g_0' + \frac{1}{6}g_0'' + \frac{1}{3}g_0'''\right)\gamma^3 + \frac{1}{4}(2g_0' + g_0)\left(\frac{2}{3}g_0' - \frac{4}{3}g_0''\right. \\ &\left. + \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\text{iv}}\right)\gamma^4 - \left(\frac{5}{18}g_0' - \frac{1}{45}g_0'' - \frac{13}{180}g_0''' + \frac{1}{12}g_0' - \frac{2}{15}g_0\right)\gamma^5 + \cdots \Big\}\, ds\end{aligned} \tag{25}$$

The equations are of the conventional non-linear Voltera type and may be treated by successive approximations. Without going into details, it may be stated that the first approximation and the subsequent ones are in excellent agreement for $s > 2\epsilon$, whereas in the neighborhood of $s = 0$, the behavior is fairly erratic. However, $\gamma(s)$ is of interest only for $s \geq 2\epsilon$ and $g_0(s)$ only for $s > \epsilon$. Hence, the solution

$$\varphi(s) \simeq s, \qquad \psi(s) \simeq s \tag{26}$$

provides an excellent estimate of the state of stress existing in the flow field. We can write in fact, [this is equivalent to Eq. (26)]

$$\omega_e - \omega = \ln(1 + \gamma^2/2\epsilon) \tag{27}$$

along the boundary, and

$$\omega_e - \omega = \ln(1 + g_0^2/2\epsilon) = \ln(1 + x^2/2\epsilon) \tag{28}$$

along $y = 0$. These formulae are good for $s > 2\epsilon$ and are essentially Sachs' formulae [1].

The corresponding results for the case where the wall friction does not vanish require certain numerical work and will be presented in a later paper. Detailed interpretative remarks concerning the foregoing result do not seem to be in order here since they would necessarily coincide with the findings in [1] regarding this specific problem.

References

1. G. Sachs and L. J. Klinger, *The flow of metals through tools of circular contour*, J. Appl. Mech. **14**, No. 2 (1947).

2.* K. H. Shevchenko, *The plastic state of stress and the flow of metals in cold rolling and drawing*, Izvestia Ak. Nauk SSSR, OTN **1946**, No. 3, pp. 329-354.

3. See, for example, A. Nadai, *Plasticity*, McGraw Hill Book Co., Inc., New York, 1931.

4.* S. Khristianovich, *The plane problem of the mathematical theory of plasticity in the case where the external forces are given along a closed contour*, Mat. Sbornik **1**, 511-534 (1936).

5. R. Courant, *Supersonic flow and shock waves*, Applied Mathematics Panel, Report 38.2R, p. 39.

6.* A. Iu. Ishlinskii, *Rolling and drawing at high speeds*, Prikl. Mat. Mekh. **7**, 226-230 (1943).

ADDITIONAL CORRECTIONS TO OUR PAPER

THE CYLINDRICAL ANTENNA: CURRENT AND IMPEDANCE

QUARTERLY OF APPLIED MATHEMATICS **3**, 302-335 (1946) and **4**, 199-200 (1946)

BY RONOLD KING AND DAVID MIDDLETON (*Harvard University*)

page 305, Change the number of Eq. (13a) to (13); delete "so that" following Eq. (13a); delete Eq. (13b).

page 306, Change the number of Eq. (14a) to (14); delete "where" following Eq. (14a); delete Eq. (14b).
Eq. (16)—add superscript -1 on R_{1h} in the integrand.

page 317, Fig. 10—The value $|\psi_2(h - \lambda/4)|$ should be at 16.6 instead of 17.4 with appropriate changes in the several curves.

page 321, Figs. 12 and 13—All the curves are somewhat in error for $\beta h < \pi/2$. The correct values are obtained from (74), using the corrected values for ψ obtained from Fig. 11a on page 200 of volume 4.

page 327, Table II—First line: Insert $-$ between $\pi/2$ and βh_{res}
Second line: Replace 800 by 820.
Fourth line: Replace 67 by 73.

page 328, Eqs. (14a), (14b)—Insert $-$ after $=$.

page 330, Eq. (23)—Change sign of lower limits on all three integrals by inserting $-$ sign. This change is in addition to corrections on page 200 of volume 4.
Eq. (27)—Change first $-$ sign to $+$; change last $+$ sign to $-$.

page 335, Eq. (45)—Change all upper limits in four integrals to u_2.
Change all lower limits in four integrals to $-u_1$.
Eq. (46)—Last integral only: Change upper limit to u_2, lower limit to $-u_1$.
Eq. (47)—Delete superscript bars in second integral of the first member of the equation. Change $-$ to $+$ before this second integral.

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