THE RADIATION AND TRANSMISSION PROPERTIES OF A PAIR OF SEMI-INFINITE PARALLEL PLATES—I*

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1. Introduction. We are concerned here with the following problem. A plane monochromatic electromagnetic wave is incident upon a pair of semi-infinite parallel metallic plates of zero thickness and perfect conductivity (see Fig. 1 for a side view). The edges of the plates are infinite straight lines which are parallel to the y axis of an xyz rectangular coordinate system. (The y axis is perpendicular to the plane of the paper in Fig. 1). The plates extend indefinitely in the direction of the positive z axis and are spaced a units apart. It is assumed, as in CHI[1], that the electric field of the incident wave has only one component, namely the one which is parallel to the y axis. Since the incident electric field is independent of y, and the boundary conditions on the plates are fulfilled independently of y, no other components of the electric field will be excited. There will be two components of the magnetic field; these in turn may be derived from the single component of the electric field through the Maxwell equations. The angle θ, the direction of the propagation vector of the incident wave, is measured with respect to the positive z axis.

We have just described the manner in which Fig. 1 is excited from free space. It is now necessary to indicate the mode of excitation which the parallel plate region can sustain. We assume that for $z \gg 0$, $0 \leq x \leq a$, the y component of the electric field is asymptotic to $(\rho_1 e^{i\kappa z} + \rho_2 e^{-i\kappa z}) \sin (\pi x/a)$. That is, the parallel plate region can sustain a mode which is consistent with the polarization which we consider here. It is to be understood that there are no other means of excitation in the finite part of the xyz space. κ is the propagation constant in the parallel plate region and is equal to $(k^2 - (\pi/a)^2)^{1/2}$, where $k = 2\pi/\lambda$ and λ is the free space wave length. In order that a

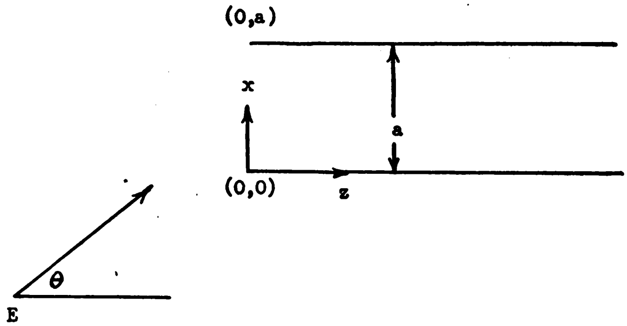

FIG. 1.

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[1]Carlson and Heins, *The reflection of an electromagnetic plane wave by an infinite set of plates I*, this Quarterly, 4, 313-329 (1947). Hereafter we shall refer to this as CHI. We employ here the same form of the Maxwell equations which were used in CHI. The time dependence is taken as exp $(-ikct)$ where c is the velocity of light.

single mode propagate in the parallel plate region, it is necessary to assume further that $\frac{1}{2} < a/\lambda < 1$. The constant ρ_1 is the amplitude of the wave going to the right in the parallel plate region, while the ρ_2 is the amplitude of the wave going to the left in the same region.

This structure may be viewed as a two dimensional antenna in the sense that it can receive and transmit energy. The problem can be formulated mathematically as a pair of simultaneous integral equations of the faltung type which are closely related to those of the Wiener-Hopf type.[2] The unknown functions in these integral equations are the surface current densities on the plates. The solution of these equations will give us the functional form of the current densities, their asymptotic form for $z \to 0^+$, $z \to \infty$, as well as the relation between the amplitudes of the various waves. We shall divide this problem into two parts. In the first part we shall assume excitation from free space. Then the parallel plate region sustains the wave travelling to the right, since there are no obstacles in the parallel plate region which would give rise to a wave travelling to the left. In this case, we find the magnitude as well as the phase of the parallel plate wave which is travelling to the right. In the second part, we shall assume that the parallel plate region has been excited. Here we shall find the reflection coefficient, that is, the ratio of ρ_1/ρ_2. This second problem breaks down into a single integral equation due to the presence of a symmetry in the field components. The first problem we treat considers the parallel plate region as a receiving antenna, while the second one considers it as a transmitting antenna. We shall see that their properties are not completely independent.

The formulation of the pair of simultaneous integral equations which we have just mentioned can be carried out by the same method employed by Carlson and Heins (CHI). An application of Green's integral theorem in two dimensions with a free space Green's function as a kernel gives us the y component of the electric field in terms of the surface current density on each plate. Thus if $E_y(x, z)$ is the y component of the electric field, and $I_0(z)$ and $I_1(z)$ the surface current densities on the lower and upper plates respectively, we have the following relation

$$E_y(x, z) = E_y^{inc}(x, z) + \frac{i}{4}\int_0^\infty \{I_0(z')H_0^{(1)}[k(x^2 + (z - z')^2)^{1/2}] + I_1(z')H_0^{(1)}[k((x - a)^2 + (z - z')^2)^{1/2}]\}\, dz', \tag{1.1}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and $E_y^{inc}(x, z) = \exp[ik(x \sin\theta + z\cos\theta)]$. The boundary conditions on $E_y(x, z)$ give us the simultaneous integral equations. Indeed, since $E_y(x, z)$ is the component of the electric field which is tangent to the planes $x = 0$, $z \geq 0$ and $x = a$, $z \geq 0$, we have

$$0 = E_y^{inc}(0, z) + \frac{i}{4}\int_0^\infty \{I_0(z')H_0^{(1)}[k \mid z - z' \mid] + I_1(z')H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\}\, dz', \tag{1.2a}$$

[2]Paley and Wiener, *The Fourier transform in the complex domain,* Am. Math. Society Colloquium Publication, 1934, ch. IV. Actually, the integral equations we are required to solve are singular cases of the Wiener-Hopf theory but they are still susceptible to Fourier techniques.

and

$$0 = E_y^{inc}(a, z) + \frac{i}{4}\int_0^\infty \{I_0(z')H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}] + I_1(z')H_0^{(1)}[k \mid z - z' \mid]\} \, dz' \tag{1.2b}$$

for $z \geq 0$.

We can simplify these last equations by performing the arithmetical operations of addition and subtraction. Upon adding, we get

$$0 = E_y^{inc}(0, z) + E_y^{inc}(a, z) + \frac{i}{4}\int_0^\infty J_0(z')\{H_0^{(1)}[k \mid z - z' \mid] + H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\}, \tag{1.3a}$$

while subtraction gives us immediately

$$0 = E_y^{inc}(0, z) - E_y^{inc}(a, z) + \frac{i}{4}\int_0^\infty J_1(z')\{H_0^{(1)}[k \mid z - z' \mid] - H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\}. \tag{1.3b}$$

Here $J_0(z) = I_0(z) + I_1(z)$ and $J_1(z) = I_0(z) - I_1(z)$. In view of the z dependence of the kernels and the particular limits of the integrals in Eqs. (1.3a) and (1.3b), we have here two integral equations which may be solved rigorously with the Fourier transform in the complex domain. This implies, of course, that we seek solutions of appropriate growth, and the kernels possess the correct growth. We shall now show that such is indeed the case.

2. The Fourier transform solution of equations (1.3a) and (1.3b). Let us now write Eqs. (1.3a) and (1.3b) in a form which makes them amenable to Fourier transform methods. We define $E_y^{inc}(0, z)$ and $E_y^{inc}(a, z)$, $J_0(z)$ and $J_1(z)$ to be identically zero for $z < 0$. We further extend the Eqs. (1.3a) and (1.3b) for $z < 0$ to read

$$\phi_0(z) = \frac{i}{4}\int_0^\infty J_0(z')\{H_0^{(1)}[k \mid z - z' \mid] + H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\} \, dz', \tag{2.1a}$$

$$\phi_1(z) = \frac{i}{4}\int_0^\infty J_1(z')\{H_0^{(1)}[k \mid z - z' \mid] - H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\} \, dz', \tag{2.1b}$$

where $\phi_0(z)$ and $\phi_1(z)$ are defined to be identically zero for $z > 0$. Upon noting our assumptions on $J_0(z)$, $J_1(z)$, $E_y^{inc}(0, z)$, $E_y^{inc}(a, z)$, $\phi_0(z)$ and $\phi_1(z)$ we have for all z

$$\phi_0(z) = E_y^{inc}(0, z) + E_y^{inc}(a, z) + \frac{i}{4}\int_{-\infty}^\infty J_0(z')\{H_0^{(1)}[k \mid z - z' \mid] + H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\} \, dz', \tag{2.2a}$$

$$\phi_1(z) = E_y^{inc}(0, z) - E_y^{inc}(a, z) + \frac{i}{4}\int_{-\infty}^\infty J_1(z')\{H_0^{(1)}[k \mid z - z' \mid] - H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\} \, dz'. \tag{2.2b}$$

We assume as in CHI that k has a small positive imaginary part.

Some remarks on the growth of $J_0(z)$, $J_1(z)$, $\phi_0(z)$ and $\phi_1(z)$ as z becomes either positively or negatively infinite are now in order. With this information we can find the half planes of regularity of the Fourier transforms of the functions with which we have to work. It is to be noted, in light of the definitions we have imposed upon $\phi_0(z)$ and $\phi_1(z)$, that they are asymptotic to $e^{-ikz}/z^{1/2}$ for z large and negative. This asymptotic form may be seen directly from the Hankel function. Thus the Fourier transforms of $\phi_0(z)$ and $\phi_1(z)$ are

$$\Phi_0(w) = \int_{-\infty}^{0} e^{-iwz}\phi_0(z)\,dz \qquad \text{and} \qquad \Phi_1(w) = \int_{-\infty}^{0} e^{-iwz}\phi_1(z)\,dz$$

and $\Phi_0(w)$ and $\Phi_1(w)$ are regular in an upper half plane $\mathfrak{Im}\,w > -\mathfrak{Im}\,k$.

The transforms of the Hankel functions $i/4\ H_0^{(1)}[k\mid z\mid]$ and $i/4\ H_0^{(1)}[k(a^2+z^2])^{1/2}$ are well known. For example[3]

$$\frac{i}{4}\int_{-\infty}^{\infty} H_0^{(1)}[k(a^2+z^2)^{1/2}]e^{-iwz}\,dz = \frac{i}{2}(k^2-w^2)^{-1/2}\exp\left[i\mid a\mid(k^2-w^2)^{1/2}\right]$$

and is regular in the strip $-\mathfrak{Im}\,k < \mathfrak{Im}\,w < \mathfrak{Im}\,k$. Furthermore the transforms of $E_y^{inc}(0, z)$ and $E_y^{inc}(a, z)$ are readily calculated since they have been annihilated for $z < 0$. Hence we have

$$\int_0^{\infty} e^{-iwz}E_y^{inc}(0, z)\,dz = \frac{1}{i(w - k\cos\theta)}$$

and

$$\int_0^{\infty} e^{-iwz}E_y^{inc}(a, z)\,dz = \frac{e^{iak\sin\theta}}{i(w - k\cos\theta)}$$

and these last two transforms are regular in the lower half plane $\mathfrak{Im}\,w < \mathfrak{Im}\,k\cos\theta$. We observe, that thus far the transforms of $\phi_0(z)$, $\phi_1(z)$, the Hankel functions, $E_y^{inc}(0, z)$ and $E_y^{inc}(a, z)$ are regular in a common strip $-\mathfrak{Im}\,k < \mathfrak{Im}\,w < \mathfrak{Im}\,k\cos\theta$.

We still have to discuss the growth properties of the surface current densities $I_0(z)$ and $I_1(z)$ for $z \gg 0$. Their dominant parts for $z \gg 0$ are terms of the type $e^{i\kappa z}$. All other terms in the asymptotic forms of $I_0(z)$ and $I_1(z)$ approach zero more rapidly than these imaginary exponentials. Now a term of the type $e^{i\kappa z}$ has the Fourier transform

$$\int_0^{\infty} e^{i\kappa z - iwz}\,dz$$

which is regular in some lower half plane bounded by a small but positive ordinate. It now follows that the Fourier transforms of $\phi_0(z)$, $\phi_1(z)J_0(z)$, $J_1(z)E_y^{inc}(a, z)$, $E_y^{inc}(0, z)$ and the Hankel functions are regular in the strip $-\mathfrak{Im}\,k < \mathfrak{Im}\,w < \mathfrak{Im}\,\kappa$ (or $\mathfrak{Im}\,k\cos\theta$). We are thus permitted to apply the Fourier transform to Eqs. (2.2a) and (2.2b) to get

$$\Phi_0(w) = \frac{(1 + e^{iak\sin\theta})}{i(w - k\cos\theta)} + \frac{i}{2}\frac{(1 + e^{ia(k^2-w^2)^{1/2}})}{(k^2 - w^2)^{1/2}}H_0(w), \tag{2.4a}$$

$$\Phi_1(w) = \frac{(1 - e^{iak\sin\theta})}{i(w - k\cos\theta)} + \frac{i}{2}\frac{(1 - e^{ia(k^2-w^2)^{1/2}})}{(k^2 - w^2)^{1/2}}H_1(w), \tag{2.4b}$$

[3]The branch of $(k^2 - w^2)^{1/2}$ is equal to k for $w = 0$.

where

$$H_0(w) = \int_0^\infty e^{-iwz} J_0(z)\, dz, \qquad H_1(w) = \int_0^\infty e^{-iwz} J_1(z)\, dz.$$

Equations (2.4a) and (2.4b) are now to be decomposed into two sets of equations, one of which will be analytic in the upper half plane $\mathfrak{Im}\, w > -\mathfrak{Im}\, k$ while the other of them will be regular in the lower half plane $\mathfrak{Im}\, w < \mathfrak{Im}\, k$ or ($\mathfrak{Im}\, k \cos\theta$). Let us first turn to Eq. (2.4a). The factor

$$(k^2 - w^2)^{-1/2}\{1 + \exp[ia(k^2 - w^2)^{1/2}]\}$$

may be written as

$$2(k^2 - w^2)^{-1/2} \exp\left[i\frac{a}{2}(k^2 - w^2)^{1/2}\right] \cos\left[\frac{a}{2}(k^2 - w^2)^{1/2}\right] = \frac{K_-(w)}{K_+(w)} = K(w).$$

Without indicating the precise form of $K_-(w)$ and $K_+(w)$ which we assume to be regular in the appropriate lower and upper half planes, we may proceed to the required decomposition of Eq. (2.4a). We have

$$\Phi_0(w)K_+(w) - \frac{(1 + e^{iak\sin\theta})[K_+(w) - K_+(k\cos\theta)]}{i(w - k\cos\theta)}$$
$$= \frac{(1 + e^{iak\sin\theta})K_+(k\cos\theta)}{i(w - k\cos\theta)} + \frac{i}{2}K_-(w)H_0(w). \tag{2.5a}$$

The left side of Eq. (2.5a) is regular in the upper half plane $\mathfrak{Im}\, w > -\mathfrak{Im}\, k$, while the right side is regular in the lower half plane $\mathfrak{Im}\, w < \mathfrak{Im}\, \kappa$ (or $\mathfrak{Im}\, k \cos\theta$) and both sides are regular in a common strip. It follows then that each side is equal to an integral function $\epsilon_0(w)$, i.e.

$$\Phi_0(w)K_+(w) - \frac{(1 + e^{iak\sin\theta})[K_+(w) - K_+(k\cos\theta)]}{i(w - k\cos\theta)} = \epsilon_0(w), \tag{2.6a}$$

$$\frac{(1 + e^{iak\sin\theta})K_+(k\cos\theta)}{i(w - k\cos\theta)} + \frac{i}{2}K_-(w)H_0(w) = \epsilon_0(w). \tag{2.6b}$$

In a similar fashion, we may decompose Eq. (2.4b). Let

$$(k^2 - w^2)^{-1/2}\{1 - \exp[ia(k^2 - w^2)^{1/2}]\} = \frac{L_-(w)}{L_+(w)} = L(w),$$

where $L_-(w)$ and $L_+(w)$ are regular in the appropriate lower and upper half planes. We have upon repeating the argument for separation

$$\Phi_1(w)L_+(w) - \frac{(1 - e^{iak\sin\theta})[L_+(w) - L_+(k\cos\theta)]}{i(w - k\cos\theta)} = \epsilon_1(w), \tag{2.7a}$$

$$\frac{i}{2}L_-(w)H_1(w) + \frac{(1 - e^{iak\sin\theta})L_+(k\cos\theta)}{i(w - k\cos\theta)} = \epsilon_1(w), \tag{2.7b}$$

where $\epsilon_1(w)$ is an integral function.

We are now compelled to indicate the precise forms of $K_-(w)$, $K_+(w)$, $L_-(w)$ and $L_+(w)$ if we are to evaluate the integral functions $\epsilon_0(w)$ and $\epsilon_1(w)$ and thereby find $H_0(w)$ and $H_1(w)$. In much the same manner which was indicated in CHI one finds that

$$K_-(w) = \frac{a^2}{\pi^2} \frac{(w - \kappa)}{(k - w)^{1/2}} \exp\left[\frac{ia}{\pi} (k^2 - w^2)^{1/2} \arctan \left(\frac{k + w}{k - w}\right)^{1/2} + \chi_0(w)\right] \prod_{n=1}^{\infty} \left[\left(1 - \frac{(ak)^2}{\pi^2(2n + 1)^2}\right)^{1/2} + \frac{iaw}{\pi(2n + 1)}\right] e^{-iaw/\pi(2n+1)}$$

is regular in the appropriate lower half plane.[4] $\chi_0(w)$ is an integral function which has been introduced into the product decomposition of $K(w)$ and is to be chosen such that $K_-(w)$ is of algebraic growth for $|w| \to \infty$ and $\Im m w$ in the correct lower half plane. Similarly

$$\frac{1}{K_+(w)} = \frac{2(w + \kappa)}{(k + w)^{1/2}} \exp\left[\frac{ia}{\pi} (k^2 - w^2)^{1/2} \arctan \left(\frac{k - w}{k + w}\right)^{1/2} - \chi_0(w)\right] \prod_{n=1}^{\infty} \left[\left(1 - \frac{(ak)^2}{\pi^2(2n + 1)^2}\right)^{1/2} - \frac{iaw}{\pi(2n + 1)}\right] e^{iaw/\pi(2n+1)}.$$

In order to determine $\chi_0(w)$, we simply calculate the asymptotic form of $K_-(w)$ and $K_+(w)$ as $|w| \to \infty$, $\Im m w$ in the appropriate half plane and choose $\chi_0(w)$ so that $K_-(w)$ and $K_+(w)$ will be of algebraic growth.[5] Let us first study $K_-(w)$. If we observe that the parameter ak may be neglected in the infinite product as $|w| \to \infty$, $\Im m w < \Im m k$, we have that $K_-(w)$ is asymptotic to

$$w^{1/2} \exp\left[\frac{iaw}{2\pi} \log\left(-\frac{2w}{k}\right) + \chi_0(w)\right] \prod_{n=1}^{\infty} \left[1 + \frac{iaw}{\pi(2n + 1)}\right] e^{-iaw/\pi(2n+1)}$$
$$= cw^{-1/2} \frac{\Gamma(iaw/2\pi)}{\Gamma(iaw/\pi)} \exp\left[\frac{iaw}{\pi}\left(1 - \frac{\gamma}{2} + \frac{1}{2} \log - \frac{2w}{k}\right) + \chi_0(w)\right] \tag{2.8}$$

where c is a constant whose precise form does not interest us and γ is the Euler-Mascheroni constant. We may now apply the Stirling formula to (2.8) to obtain that $K_-(w)$ and $K_+(w)$ are of algebraic growth for $|w| \to \infty$, $\Im m w$ in the appropriate half planes, if

$$\chi_0(w) = \frac{iaw}{2\pi}\left[-3 + \gamma - \log \frac{\pi}{ak} - \frac{i\pi}{2}\right].$$

With $\chi_0(w)$ so chosen, $K_-(w)$ is asymptotic to $w^{-1/2}$ for $|w| \to \infty$ and $\Im m w$ in the appropriate lower half plane, while $K_+(w)$ is asymptotic to $w^{1/2}$ for $|w| \to \infty$ and $\Im m w$ in the upper half plane $\Im m w > -\Im m k$.

We are now in a position to determine the integral function $\epsilon_0(w)$. Let us note in Eq. (2.6b) that $H_0(w)$ is the unilateral Fourier transform of $I_0(z) + I_1(z) = J_0(z)$. As such, since $J_0(z)$ has appropriate growth for z large and positive, and since it is integrable over any finite interval of z including the origin, $H_0(w)$ possesses the property that

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[4]Henceforth principal determinations of inverse trigonometric functions and logarithms are understood.

[5]J. S. Schwinger, *Theory of guided waves*, Radiation Laboratory publication, forthcoming.

it approaches zero for $|w| \to \infty$, $\mathfrak{Im}\, w < \mathfrak{Im}\, \kappa$ or $\mathfrak{Im}\, k \cos\theta$. If we let $|w| \to \infty$, w in the correct lower half plane we see from Eq. (2.6b) that $\epsilon_0(w) = O(w^{-\alpha-1/2})$, $\alpha > 0$. On the other hand $\Phi_0(w)$ approaches zero for $|w| \to \infty$ $\mathfrak{Im}\, w > -\mathfrak{Im}\, k$ because $\Phi_0(w)$ is the unilateral Fourier transform of a function defined for negative z, is integrable over any finite negative range of z including the origin and possesses appropriate growth for z large and negative. Hence for $|w| \to \infty$, $\mathfrak{Im}\, w > -\mathfrak{Im}\, k$, we see from Eq. (2.6a) that $\epsilon_0(w) = O(w^{1/2-\beta})$, $\beta > 0$. It follows, therefore, by a theorem of Liouville, that $\epsilon_0(w)$ is a polynomial of degree less than minus one half, and hence identically zero. We have finally

$$H_0(w) = \frac{2(1 + e^{iak\sin\theta})K_+(k\cos\theta)}{(w - k\cos\theta)K_-(w)}, \tag{2.9}$$

the Fourier transform of $J_0(z)$.

We can obtain some information regarding $J_0(z)$ for $z \to 0^+$ from $H_0(w)$ as $|w| \to \infty$, $\mathfrak{Im}\, w < \mathfrak{Im}\, \kappa$ or $\mathfrak{Im}\, k \cos\theta$. For now we have that

$$H_0(w) = O(w^{-1/2})$$

and this tells us immediately that

$$J_0(w) = O(z^{-1/2})$$

for $z \to 0^+$. This verifies the integrability of $J_0(z)$ for finite and positive z.

We now turn to the determination of $H_1(w)$ and for this we consider Eq. (2.4b). We note that

$$\frac{L_-(w)}{L_+(w)} = -2i(k^2 - w^2)^{-1/2} \exp\left[\frac{ia}{2}(k^2 - w^2)^{1/2}\right] \sin\left[\frac{a}{2}(k^2 - w^2)^{1/2}\right]$$

where now

$$L_-(w) = -ia \exp\left[\frac{ia}{\pi}(k^2 - w^2)^{1/2} \operatorname{arc\,tan}\left(\frac{k + w}{k - w}\right)^{1/2} + \chi_1(w)\right] \prod_{n=1}^{\infty}\left[\left(1 - \left(\frac{ak}{2\pi n}\right)^2\right)^{1/2} + \frac{iaw}{2\pi n}\right]e^{-iaw/2\pi n}$$

and

$$\frac{1}{L_+(w)} = \exp\left[\frac{ia}{\pi}(k^2 - w^2)^{1/2} \operatorname{arc\,tan}\left(\frac{k - w}{k + w}\right)^{1/2} - \chi_1(w)\right] \prod_{n=1}^{\infty}\left[\left(1 - \left(\frac{ak}{2\pi n}\right)^2\right)^{1/2} - \frac{iaw}{2\pi n}\right]e^{iaw/2\pi n}$$

Here we have the $L_-(w)$ which is regular in the lower half plane $\mathfrak{Im}\, w < \mathfrak{Im}\, k$ and the $L_+(w)$ which is regular in the upper half plane $\mathfrak{Im}\, w > -\mathfrak{Im}\, k$. Again we choose $\chi_1(w)$, an integral function, such that $L_-(w)$ and $L_+(w)$ will behave algebraically as $|w| \to \infty$ and $\mathfrak{Im}\, w$ in either of the appropriate half planes just described. We proceed as we did above to determine $\chi_1(w)$. For $|w| \to \infty$, $\mathfrak{Im}\, w < \mathfrak{Im}\, k$, $L_-(w)$ is asymptotic to

$$\exp\left[\chi_1(w) + \frac{iaw}{2\pi}\log - \frac{2w}{k}\right]\prod_{n=1}^{\infty}\left[1 + \frac{iaw}{2\pi n}\right]e^{-iaw/2\pi n}$$

$$= \frac{2\pi \exp [\chi_1(w) - iaw\gamma/2\pi + \{iaw/2\pi\} \log (-2w/k)]}{iaw\Gamma(iaw/2\pi)}.$$

Upon applying Stirling's expansion theorem we obtain immediately

$$\frac{2\pi}{iaw} \exp [\chi_1(w) + \{iaw/2\pi\}\{1 - \gamma + \log (-2w/k)\}]\left(\frac{iaw}{2\pi}\right)^{1/2 - iaw/2\pi}.$$

Hence, if we choose

$$\chi_1(w) = -\frac{iaw}{2\pi}\left[1 - \gamma + \log\frac{4\pi}{ak} + \frac{i\pi}{2}\right]$$

$L_-(w)$ will have algebraic growth in the lower half plane $\Im m w < \Im m k$ for $|w| \to \infty$ and will be asymptotic to $w^{-1/2}$. If we repeat the argument in the upper half plane, $\Im m w > -\Im m k$ for the term $L_+(w)$ we find that the same $\chi_1(w)$ will render $L_+(w)$ algebraic in growth for $|w| \to \infty$ and now $L_+(w)$ will be asymptotic to $w^{1/2}$ for $w \to \infty$. Finally, reasoning as we did for Eqs. (2.6a) and (2.6b) we find that $\epsilon_1(w)$ is $O(w^{\alpha_1})$, $\alpha_1 < -\frac{1}{2}$ for $|w| \to \infty$, $\Im m w < \Im m k$ and is $O(w^{\beta_1})$, $\beta_1 < \frac{1}{2}$ for $|w| \to \infty$, $\Im m w > -\Im m k$. Applying Liouville's theorem once again, we find that $\epsilon_1(w)$ is identically zero. Thus we now have

$$H_1(w) = \frac{2(1 - e^{iak\sin\theta})L_+(k \cos \theta)}{(w - k \cos \theta)L_-(w)}$$

the Fourier transform of $I_0(z) - I_1(z) = J_1(z)$.

From $H_0(w)$ and $H_1(w)$ we can obtain the Fourier transforms of $I_0(z)$ and $I_1(z)$ by a simple addition and subtraction. We can also see how $J_1(z)$ behaves for $z \to 0^+$. Since $H_1(w)$ is now $O(w^{-1/2})$ for $|w| \to \infty$, $\Im m w < \Im m k$, $J_1(z) = O(z^{-1/2})$ for $z \to 0^+$ and this verifies the integrability of $J_1(z)$ for finite and positive z. In closing, we note that the precise forms of $I_0(z)$ and $I_1(z)$ are of no interest to us since we are only interested in the far fields.

3. The calculation of the far fields. In order to calculate the far fields we first express Eq. (1.3) by a Fourier integral representation. We have

$$E_y(x, z) = \exp [ik(x \sin \theta + z \cos \theta)]$$

$$+ \frac{i}{8\pi}\int_C e^{iwx}(k^2 - w^2)^{-1/2}[\{H_0(w) + H_1(w)\}e^{i|z|(k^2-w^2)^{1/2}}$$

$$+ \{H_0(w) - H_1(w)\}e^{i|z-a|(k^2-w^2)^{1/2}}]\, dw$$

where C is a path of integration drawn within the strip of regularity of all the Fourier transforms which appear in the above integral. The path is closed either above or below depending upon whether $z > 0$ or $z < 0$. Care must be taken in closing the path so that it does not intersect the branch cuts which are introduced due to the presence of the branch points k and $-k$. The dominant terms arise from the residues due to the two poles $k \cos \theta$ and κ. Furthermore, since contributions from other poles or branch points give rise to terms which are small compared to the terms arising from the poles $k \cos \theta$ and κ for $|z| \to \infty$, we need only calculate these dominant effects, at least insofar as we are concerned with the far field.[6]

There are four separate regions of interest (a) $z < 0$, $-\infty < x < \infty$ (b) $z > 0$, $x \geq a$ (c) $z \geq 0$, $x \leq 0$ and (d) $z \geq 0$, $0 \leq x \leq a$. Let us consider region (a). For $z \leq 0$, $E_y(x, y)$ is asymptotic to $\exp [ik(z \cos \theta + x \sin \theta)]$, as it should. For region (b), $z \gg 0$, $x > a$, $E_y(x, z)$ has no term comparable in magnitude with the plane wave term. Thus region (b) is the region of the geometrical shadow. For region (c) $z \gg 0$, $x < 0$, $E_y(x, z)$ is asymptotic to

$$2i \exp [ikz \cos \theta] \sin [kx \sin \theta].$$

Thus for $x < 0$, $z \gg 0$, the lower plate acts as a perfect reflector.

Region (d) is the interesting one. We now have a means of finding the amplitude of the transmitted wave guide mode. We know that in this region $E_y(x, z)$ is asymptotic to $e^{i\kappa z} \sin \pi x/a$ for $z \gg 0$, $0 \leq x \leq a$. On the other hand when we evaluate the integral and take out its dominant terms, we find that we are left with

$$-\frac{ai}{\pi}(1 + e^{iak\sin\theta})\frac{K_+(k \cos \theta)}{\kappa - k \cos \theta} \lim_{w\to\kappa} \frac{w - \kappa}{K_-(w)} e^{i\kappa z} \sin \pi x/a.$$

The amplitude of the transmitted wave is the coefficient of the factor $e^{i\kappa z} \sin \pi x/a$. It may be simplified if we now take k to be real. In the first place

$$\lim_{w\to\kappa} \frac{w - \kappa}{K_-(w)} = 2[\pi^3(k - \kappa)a^{-2}]^{1/2} \exp\left[i\Theta_1 - \chi_0(\kappa) - i \arctan \left(\frac{k + \kappa}{k - \kappa}\right)^{1/2}\right]$$

where

$$\Theta_1 = -\sum_{n=1}^{\infty}\left[\arcsin \frac{\kappa a}{\pi[(2n + 1)^2 - 1]^{1/2}} - \frac{\kappa a}{\pi(2n + 1)}\right]$$

while

$K_+(k \cos \theta)$

$$= \frac{a[k(1 + \cos \theta)(k^2 \cos^2 \theta - \kappa^2)]^{1/2} \exp [i\Theta_2 + \chi_0(k \cos \theta) - (iak\theta \sin \theta)/2\pi]}{2\pi(k \cos \theta + \kappa)[\cos \{(ak \sin \theta)/2\}]^{1/2}}$$

and

$$\Theta_2 = \sum_{n=1}^{\infty}\left[\arcsin \frac{ak \cos \theta}{[\pi^2(2n + 1)^2 - (ak \sin \theta)^2]^{1/2}} - \frac{ak \cos \theta}{\pi(2n + 1)}\right].$$

Thus, the transmission coefficient is

$$2\left[\frac{k\pi(1 + \cos \theta) \cos (ak/2 \sin \theta)}{a^2(k^2 \cos^2 \theta - \kappa^2)(k + \kappa)}\right]^{1/2} e^{i\Psi + a(k\cos\theta - \kappa)/4}, \qquad 0 \leq \theta \leq \pi$$

[6] E. T. Copson, Oxford Quart. Math. 17, 19-34 (1946). There is a detailed discussion in this paper on the choice of a path similar to C.

where

$$\Psi = \Theta_1 + \Theta_2 + \text{arc tan}\left(\frac{k - \kappa}{k + \kappa}\right)^{1/2} + \frac{ak}{2}\left(1 - \frac{\theta}{\pi}\right)\sin\theta$$

$$+ \frac{a(k\cos\theta - \kappa)}{2\pi}[\gamma - 3 - \log(\pi/ak)].$$

The square of the absolute magnitude of the amplitude of the transmission coefficient is proportional to the power gain, as a direct consequence of the Lorentz reciprocity theorem. Thus insofar as the angular variation is concerned, the radiation pattern is

$$\frac{(1 + \cos\theta)\cos(ak/2\sin\theta)e^{(ak\cos\theta)/2}}{(k^2\cos^2\theta - \kappa^2)}.$$

We have thus found how the parallel plates act as a receiving antenna. It is to be noted that the radiation pattern arises from the excitation of the parallel plates for $z \gg 0$, $0 \leq x \leq a$. The reciprocity theorem has enabled us to give a partial solution of the second part of the problem. The reflection coefficient which we have described earlier has yet to be calculated.