A PRACTICAL METHOD FOR SOLVING HILL'S EQUATION*

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1. Introduction. The differential equation known as Mathieu-Hill's equation can be written

$$y'' + J(x)y = 0, \tag{1}$$

where J is a periodic function of x. The period is usually taken equal to π for historical reasons. The first equation of this type was discovered by Mathieu in connection with the problem of vibrations within an elliptic boundary, when

$$J(x) = \eta + \gamma \cos 2x$$

has a period π.

Floquet proved that the general solution of Eq. (1) can be written

$$y = D_1 e^{\mu x}\Phi(x) + D_2 e^{-\mu x}\Phi(-x), \tag{2}$$

where Φ is a periodic function, with the same period π as $J(x)$. This general solution contains two terms with the exponents $\pm\mu$. Floquet's theorem can be expressed in a slightly different way. Let us consider the term with $+\mu$:

$$f(x) = e^{\mu x}\Phi(x). \tag{3}$$

The condition on $f(x)$ is

$$f(x + n\pi) = e^{\mu n\pi} f(x) = \xi^n f(x)$$

with

$$\xi = e^{\mu\pi}.$$

The general solution is

$$y = D_1 f(x) + D_2 f(-x). \tag{4}$$

We shall look for a solution in an interval of length π and use condition (3) to extend the solution from $-\infty$ to $+\infty$. This means that we shall have to meet some boundary conditions in order to match the solutions in two consecutive intervals. These matching conditions will be essential in fixing the value of μ.

The method developed in this paper is based upon these general considerations and shall be explained more completely in Sec. 2. It differs completely from the classical method, as found in most textbooks.[1] The standard procedure is to expand J in Fourier series

$$J = \sum_n \theta_n e^{i2nx} \tag{5}$$

and to look for a similar expansion for the unknown periodic function Φ

$$\Phi = \sum_n b_n e^{i2nx}. \tag{6}$$

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[1]Whittaker and Watson, *Modern analysis*, Cambridge University Press, 4th edition, 1927, p. 414.

Equation (1) results in an infinite system of simultaneous linear homogeneous equations for the unknown b_n s. A non-trivial solution can only be obtained if the corresponding infinite determinant is zero. This last condition is used to determine the value of the exponent μ.

Whittaker was able to discuss this condition in the case when the series of the coefficients θ_n in expansion (5) is absolutely convergent, and obtained a formula

$$\sin^2\left(\frac{\pi}{2}\, i\mu\right) = -\sinh^2\left(\frac{\pi}{2}\,\mu\right) = \Delta_1(0)\sin^2\left(\frac{\pi}{2}\,\theta_0^{1/2}\right), \tag{7}$$

where $\Delta_1(0)$ is another infinite determinant with the following coefficients

$$\Delta_1(0) = |\, B_{mp}\,|, \qquad B_{mm} = 1,$$
$$B_{mp} = \frac{\theta_{m-p}}{\theta_0 - 4m^2}, \qquad (m \neq p) \tag{8}$$

This result is not very encouraging. First, the condition of absolute convergence for the series of the coefficients of θ_n is a very restrictive one. Second, we still have to compute an infinite determinant, and the computation proves very difficult unless the θ_n terms decrease very rapidly when n increases.

This infinite determinant takes on infinite value whenever

$$\theta_0 = 2n \qquad n \text{ integer} \tag{9}$$

These θ_0 values correspond to double poles of the determinant, since both rows $m = \pm n$ obtain infinite terms. These double poles are canceled out in formula (7) by the double zeros of $\sin^2(\pi\theta_0^{1/2}/2)$ and do not correspond to any singular values of μ.

Altogether, the method of Fourier expansion is not very practical, and leads to complicated computations.

The method presented in this paper does not involve the restrictions of the classical method, and leads to a practical solution of Hill's equation, even in such exceptional cases as periodic functions $J(x)$ containing discontinuities or δ functions.

2. Principle of the method. Let us consider a differential equation

$$y'' + F(x)y = 0 \tag{10}$$

with a given function $F(x)$. We may find two independent solutions u and v and obtain the general solution

$$y = Au + Bv \tag{11}$$

containing two constants A and B. We obviously have

$$uv'' = vu'' = Fuv,$$

hence

$$uv' - vu' = C,$$

and a suitable normalization of u and v is used to make the constant C unity:

$$uv' - vu' = 1 \tag{12}$$

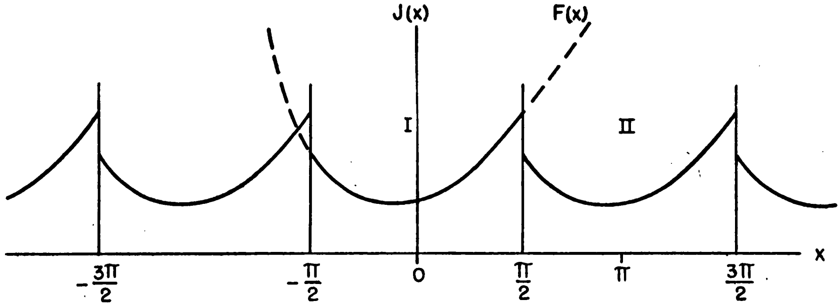

FIG. 1

We now want to discuss Hill's equation

$$y'' + J(x)y = 0 \tag{13}$$

with a periodic function J of period π defined in the following way (see Fig. 1):

$$J = \begin{cases} F(x) & -\pi/2 < x < \pi/2, \\ F(x - \pi) & \pi/2 < x < 3\pi/2, \\ \dots\dots\dots\dots & \\ F(x - n\pi) & n\pi - \pi/2 < x < n\pi + \pi/2. \end{cases} \tag{14}$$

Floquet's theorem assures us of the existence of two independent solutions y_1, y_2 characterized by the following properties (Eq. 5):

$$\begin{aligned} y_1(x + \pi) &= e^{\mu\pi}\, y_1(x) = \xi y_1(x), \\ y_2(x + \pi) &= e^{-\mu\pi} y_2(x) = \xi^{-1} y_2(x), \\ \xi &= e^{\mu\pi}. \end{aligned} \tag{15}$$

Let us consider y_1 and discuss a practical method for obtaining μ (or ξ). In the first interval $(-\pi/2 < x < \pi/2)$ the function y_1 may be represented by a formula (11), with a set of A and B coefficients. In the second interval $(\pi/2 < x < 3\pi/2)$ the coefficients will be ξA and ξB, according to (15). We must now write the continuity conditions for y_1 and y_1' across the border $\pi/2$:

$$\begin{aligned} Au_1 + Bv_1 &= \xi Au_2 + \xi Bv_2, \\ Au_1' + Bv_1' &= \xi Au_2' + \xi Bv_2', \end{aligned} \tag{16}$$

$$u_1 = u(\pi/2), \qquad u_2 = u(-\pi/2), \qquad v_1 = v(\pi/2), \qquad v_2 = v(-\pi/2).$$

We have obtained a set of simultaneous equations for the two unknowns A and B. This can be solved only when the determinant is zero:

$$D = \begin{vmatrix} u_1 - \xi u_2 & v_1 - \xi v_2 \\ u_1' - \xi u_2' & v_1' - \xi v_2' \end{vmatrix} \tag{17}$$

$$= \xi^2 + \xi(u_1'v_2 + u_2'v_1 - u_1v_2' - u_2v_1') + 1 = 0,$$

where we have used Eq. (12) at both points $\pm\pi/2$. Equation (17) in ξ fixes the Floquet coefficients ξ and ξ^{-1}. The product of the two roots is unity and their sum is given by

$$2 \cosh \mu\pi = \xi + \xi^{-1} = -u_1'v_2 - u_2'v_1 + u_1v_2' + u_2v_1'; \tag{18}$$

hence

$$4 \sinh^2 \left(\mu \frac{\pi}{2}\right) = 2 \cosh \mu\pi - 2 = -(u_1 - u_2)(v_1' - v_2') + (u_1' - u_2')(v_1 - v_2) \tag{19}$$

with the help of Eq. (12). Once the μ exponent is obtained, the coefficients A and B result from (16) and the solution of Hill's equation (13) is achieved.

A very important case is obtained when

$$F(x), \qquad J(x) = \text{even}. \tag{20}$$

One may choose correspondingly for u, v an even and an odd function:

$$\begin{aligned} u(x) &= u(-x), & u_2 &= u_1 , & u_2' &= -u_1' , \\ v(x) &= -v(-x), & v_2 &= -v_1 , & v_2' &= v_1' . \end{aligned} \tag{21}$$

Eq. (18) then reads

$$\cosh \mu\pi = u_1v_1' + u_1'v_1 \tag{22}$$

from which we obtain

$$\sinh^2 (\mu\pi/2) = u_1'v_1 , \tag{23}$$

a formula that will be found useful for a comparison with Whittaker's theory of Hill's equation.

The whole method is a generalization of the discussion that was previously given for the special example of a rectangular $J(x)$ function.[2] One advantage of the method is that it works with periodic $J(x)$ functions exhibiting a finite number of discontinuities.

3. Some special examples. The case of a rectangular $J(x)$ function can be easily investigated along these lines, and the results agree completely with those of a previous discussion[2] where a slightly different method was followed.

Let us consider the case of a parabolic function

$$F = a - b^2x^2. \tag{24}$$

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[2]L. Brillouin, *Wave propagation in periodic structures*, McGraw-Hill, New York, 1946, Ch. VIII, pp. 180-186, and Ch. IX, pp. 218-226.

We may find the solutions as power series expansions

$$
\begin{aligned}
u &= 1 + u_2x^2 + \cdots\, u_{2n}x^{2n} \cdots, \\
v &= x + v_3x^3 + \cdots\, v_{2n+1}x^{2n+1} \cdots.
\end{aligned} \tag{25}
$$

Substituting in Eq. (10) we obtain the recurrence formulas

$$
\begin{aligned}
2u_2 + a &= 0, \qquad (2n+2)(2n+1)u_{2n+2} + au_{2n} - b^2u_{2n-2} = 0, \\
6v_3 + a &= 0, \qquad (2n+3)(2n+2)v_{2n+3} + av_{2n+1} - b^2v_{2n-1} = 0,
\end{aligned} \tag{26}
$$

and the u, v functions satisfy the normalizing condition (12). The following special cases may be of interest

$$
\begin{aligned}
a = -b, \qquad u &= e^{bx^2/2}, \\
v &= e^{bx^2/2}I(x) \qquad \text{with } I(x) = \int_0^x e^{-bx^2}\,dx,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
a = -3b, \qquad u &= e^{-bx^2/2} + 2bx\,e^{bx^2/2}I(x), \\
v &= xe^{bx^2/2},
\end{aligned} \tag{28}
$$

as may be checked by direct computation.

In all these cases, there is no discontinuity of the function F on the limits $\pm\pi/2$ of the interval, and the curve on Fig. 1 is a continuous curve with a discontinuous derivative at $\pm\pi/2$.

The corresponding Hill's problem is immediately solved with the help of Equation (22) or (23). For instance, the cases indicated above under (27) and (28) yield:

$$
a = -b, \qquad \sinh^2\left(\mu\frac{\pi}{2}\right) = u_1'v_1 = b\,\frac{\pi}{2}\,e^{b\pi^2/4}I\left(\frac{\pi}{2}\right) \tag{29}
$$

$$
a = -3b, \qquad \sinh^2\left(\mu\frac{\pi}{2}\right) = b\,\frac{\pi^2}{4} + b\pi e^{b\pi^2/4}I\left(\frac{\pi}{2}\right) + b^2\,\frac{\pi^3}{4}\,e^{b\pi^2/4}I\left(\frac{\pi}{2}\right) \tag{30}
$$

These results could not have been obtained by any other method of solution.

Another example can be solved with the help of Bessel functions, which satisfy the equation

$$
z\frac{d}{dz}\left(z\frac{d}{dz}y\right) + (z^2 - n^2)y = 0, \tag{31}
$$

$$
y = AJ_n(z) + BJ_{-n}(z), \qquad n \text{ non integer} \tag{32}
$$

Taking a new variable

$$
x = \log z, \qquad z = e^x,
$$

we obtain an equation of type (10):

$$
\frac{d^2}{dx^2}y + (e^{2x} - n^2)y = 0. \tag{33}
$$

Thus,

$$y = AJ_n(e^x) + BJ_{-n}(e^x) \tag{34}$$

and we have a solution corresponding to an unsymmetrical

$$F = e^{2x} - n^2 \tag{35}$$

for which our general formulas (18), (19) should be used in connection with the corresponding Hill's equation.

4. Solution with the B. W. K. method. The point of departure of our method is a solution of an equation of type (10). An approximate solution can be found with the B. W. K. procedure,[3] if $F(x)$ is exhibiting only small variations about a large average value. A more precise statement of the conditions involved will result from the following discussion. We rewrite Eq. (10) as follows:

$$y'' + G^2(x)y = 0, \qquad F(x) = G^2(x). \tag{36}$$

We now consider a function

$$y = G^{-1/2}\, e^{iS}, \qquad S = \int_0^x G\, dx, \tag{37}$$

which yields

$$\frac{y''}{y} = \frac{3}{4}\left(\frac{G'}{G}\right)^2 - \frac{1}{2}\frac{G''}{G} - G^2. \tag{38}$$

The function y represents an approximate solution of Eq. (36) if the first two terms in (38) are negligible in comparison to the last one. This is the case if

$$\frac{G'}{G^2} \sim \epsilon, \qquad \frac{G''}{G^3} \sim \epsilon^2, \qquad \epsilon^2 \ll 1, \tag{39}$$

and terms in ϵ^2 are neglected, while ϵ terms are retained. The second condition (39) is very restrictive, however, since it allows only for variations of G' of the order of ϵ^2. The function (37) becomes a *rigorous solution* of Eq. (36) when

$$\frac{3}{4}\left(\frac{G'}{G}\right)^2 - \frac{1}{2}\frac{G''}{G} = 0, \tag{40}$$

$$G = \frac{A}{(a + x)^2}. \tag{41}$$

Solutions u, v normalized in accordance with (12) are easily found:

$$u = (2iG)^{-1/2}e^{-iS}, \qquad v = (2iG)^{-1/2}e^{+iS} \tag{42}$$

An interesting example is shown in Fig. 2; it corresponds to the *even* function

$$G = \frac{A}{(a + |\, x \,|)^2} \tag{43}$$

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[3]The initials B. W. K. refer to the three authors L. Brillouin, Journ. de Phys. **7**, 353 (1926); G. Wentzel, Zts. f. Phys. **38**, 518 (1926); H. A. Kramers, Zts. f. Phys. **39**, 828 (1926).

For a complete discussion and more references, see E. C. Kemble, *The fundamental principles of quantum mechanics*, McGraw-Hill, 1937, Ch. III, p. 91-112.

with a discontinuity of the derivative at $x = 0$. Here,

$$-S(-x) = S(x) = \int_0^{x>0} G\,dx = -A\left(\frac{1}{a+x} - \frac{1}{a}\right) = \frac{Ax}{a(a+x)}.$$

For the *odd* solution v we simply take

$$v = G^{-1/2}\sin S = \frac{a + |x|}{A^{1/2}}\sin S,$$
$$v' = -\frac{1}{2}G^{-3/2}G'\sin S + G^{1/2}\cos S. \tag{44}$$

Both v and v' are continuous at $x = 0$ since $G'(+0) = -G'(-0)$. The even solution u can be written as follows:

$$u = G^{-1/2}[\cos S \pm K\sin S],$$
$$u' = -\frac{1}{2}G^{-3/2}G'_{\pm}[\cos S \pm K\sin S] + G^{1/2}[-\sin S \pm K\cos S], \tag{45}$$

where the $+$ sign must be taken for $0 < x < \pi/2$ and the minus sign for $-\pi/2 < x < 0$. The function u is continuous at the origin, and u' becomes continuous when it is made to vanish at the origin. Hence,

$$K = \frac{G'_{+}(0)}{2G^2(0)} = -\frac{a}{A}.$$

Finally, after some reductions, one finds

$$u = \frac{a + |x|}{A^{1/2}}\left[\cos S \mp \frac{a}{A}\sin S\right],$$
$$u' = \frac{1}{A^{1/2}}\left[\frac{|x|}{a+|x|}\cos S - \frac{A^2 + a(a+|x|)}{A(a+|x|)}\sin S\right]. \tag{46}$$

It is easily verified that the coefficients have been chosen correctly so as to satisfy the normalizing conditions (12).

With these solutions u, v we may now compute the Floquet coefficient μ, with the help of Eq. (23). We find

$$\sinh^2\left(\mu\frac{\pi}{2}\right) = u_1'v_1 = \frac{|x|}{A}\sin S_1\cos S_1 - \sin^2 S_1\frac{A^2 + a(a+|x|)}{A^2},$$

where we must take

$$x = \pi/2, \qquad S_1 = \frac{A\pi/2}{a(a+\pi/2)}.$$

$$\sinh^2\left(\mu\frac{\pi}{2}\right) = \frac{\pi}{2A}\sin S_1\cos S_1 - \frac{A^2 + a^2 + a\pi/2}{A^2}\sin^2 S_1. \tag{47}$$

A simple check can be made on this formula. Keeping the same A value, we may replace a by

$$a' = -a - \frac{\pi}{2},$$

thus interchanging the position of the angular maxima and minima of the curve (see Fig. 2). Our formula is not affected by this change.

5. Successive approximations. Starting from the solutions obtained in the preceding sections, we may develop a method of successive approximations. Let us consider Eq. (10) with a function F and assume that it can be approximately represented by a G^2 function of the type (41).

$$G = \frac{A}{(a + x)^2}, \qquad F = G^2 - \epsilon H(x), \qquad \epsilon \text{ small.} \tag{48}$$

If a single function G does not yield a good enough approximation over the whole interval $-\pi/2, +\pi/2$, it may be convenient to divide this interval into two or more partial intervals using different G functions, and to join solutions at the boundaries, as shown on Figs. 2 or 3.

Next we use an expansion

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots .$$

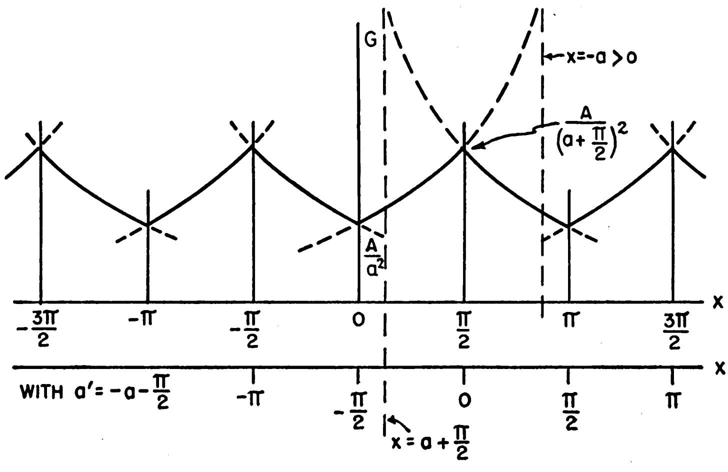

Fig. 2

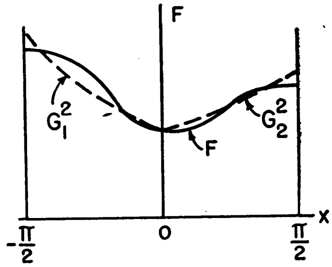

Fig. 3

Grouping terms with the same power of ϵ, we obtain

$$y_0'' + G^2 y_0 = 0, \tag{49a}$$

$$y_1'' + G^2 y_1 = H y_0 , \tag{49b}$$

$$y_2'' + G^2 y_2 = H y_1 . \tag{49c}$$

The solutions of (49a) are the u_0 and v_0 obtained in (42). Thus, the solution of (49b) reads

$$y_1 = u_0 \int_0^x Y_0 v_0' \, dx - v_0 \int_0^x Y_0 u_0' \, dx \qquad \text{with } Y_0 = \int_0^x H y_0 \, dx \tag{50}$$

and the further approximations can be obtained in a similar way.

As an example, let us assume an even function F which can be approximated with the even G function (43). The zero order approximation is represented by u_0 and v_0 of equations (44), (45). We thus have,

$$U_0 = \int_0^x H u_0 \, dx, \qquad V_0 = \int_0^x H v_0 \, dx. \tag{51}$$

H being even, U_0 is odd and V_0 is even. Next, we obtain

$$\begin{aligned} u_1 &= u_0 \int_0^x U_0 v_0' \, dx - v_0 \int_0^x U_0 u_0' \, dx \qquad \text{(even)}, \\ v_1 &= u_0 \int_0^x V_0 v_0' \, dx - v_0 \int_0^x V_0 u_0' \, dx \qquad \text{(odd)}, \end{aligned} \tag{52}$$

and the first order solutions are

$$\begin{aligned} u &= u_0 + \epsilon u_1 + \cdots \qquad \text{(even)}, \\ v &= v_0 + \epsilon v_1 + \cdots \qquad \text{(odd)}. \end{aligned} \tag{53}$$

These functions being automatically normalized according to (12), as a direct check easily shows.

A solution of Eq. (10) can thus be obtained step by step to any desired degree of approximation, and may be used to the solution of Hill's equation as shown in Sec. 2.

The same method could be applied to any other known solution of Eq. (10). We used the G^2 functions in the zero order approximation, but any other known solution would do just as well.

6. Hill's equation containing delta function. Let us consider our fundamental Eq. (10) and assume the function F to be a delta function:

$$F = B\delta(x), \qquad \delta = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}, \qquad \int_{-\infty}^{+\infty} \delta(x) \, dx = 1 \tag{54}$$

Equation (10) can be readily integrated:

$$y'' = \begin{cases} 0, & (x \neq 0), \\ B\delta(x) y(x) & (x = 0), \end{cases}$$

Thus,

$$y' = \begin{cases} a, & (x < 0) \\ a + By(0), & (x > 0) \end{cases} \tag{55}$$

and we obtain our even u or odd v solutions:

$$u = 1 + \frac{1}{2} B \mid x \mid \quad \text{(even)}, \qquad v = x \quad \text{(odd)}. \tag{56}$$

These u, v functions are normalized according to (12). The solution of the corresponding Hill's equation is obtained from Eq. (23):

$$\sinh^2 \left(\mu \frac{\pi}{2}\right) = (u'v)_{x=\pi/2} = \left(\frac{1}{2} Bx\right)_{x=\pi/2} = \frac{\pi}{4} B \tag{57}$$

This result can be checked by different methods. The case of a rectangular J function was discussed in the author's book,[2] p. 181, assuming

$$J = \begin{cases} -\chi_1^2 & (-l_1 < x < 0) \\ -\chi_2^2 & (0 < x < l_2) \end{cases} \qquad (l_1 + l_2 = \pi) \tag{58}$$

The solution was given by equation (44.12), *loc. cit.* p. 181:

$$\cosh \mu\pi = \cosh \chi_1 l_1 \cosh \chi_2 l_2 + \frac{1}{2}\left(\frac{\chi_1}{\chi_2} + \frac{\chi_2}{\chi_1}\right) \sinh \chi_1 l_1 \sinh \chi_2 l_2 \,. \tag{59}$$

We now take

$$\chi_1 = 0, \qquad \chi_2 \to \infty, \qquad l_2 \to 0, \qquad l_2\chi_2^2 = B,$$

and obtain

$$\cosh \mu\pi = 1 + \frac{1}{2} \chi_2 l_1 \chi_2 l_2 = 1 + \frac{B}{2} l_1 \tag{60}$$

where $l_1 = \pi$ when $l_2 = 0$. This checks with our previous result (57). Other types of J functions can be used and lead to similar results. Such a problem is completely outside the reach of the Fourier series expansion method.

7. Comparison between the present method and the classical one. We discussed in Sec. 1 the classical method of solution, and underlined its limitations. The Fourier expansion of the periodic function J must be such that the series of the Fourier coefficients be absolutely convergent. This rules out functions with discontinuities, whose Fourier coefficients decrease as slowly as $1/n$, but the classical method should apply to a continuous function J with discontinuous derivative. This is the case for the problem discussed in Sec. 4, Eq. (43):

$$F = G^2 = \frac{A^2}{(a + \mid x \mid)^4} \tag{61}$$

Within a period π this function oscillates between A^2/a^4 and $A^2/((a + \pi/2)^4)$. The corresponding periodic function J can be analysed in Fourier series and Whittaker's solution obtained. The question now is to compare solutions computed one way or the other.

The comparison is easier when the variation of the function is small; hence we assume

$$A = Ba^2, \qquad a \gg \frac{\pi}{2} \tag{62}$$

and compute expansions with respect to the small quantity

$$\epsilon = \frac{\pi}{2a} \ll 1.$$

Let us start with the solution obtained in Sec. 4. It contains the quantity S_1 (Eq. 47).

$$S_1 = \frac{A\pi/2}{a(a + \pi/2)} = \frac{B\pi/2}{1 + \pi/2a} = B\frac{\pi}{2}\left(1 - \frac{\pi}{2a} + \left(\frac{\pi}{2a}\right)^2 \cdots\right) \tag{63}$$

we shall keep terms up to $\pi/2a$ only, since the computation of second order terms proves rather cumbersome. The solution is given in Eq. (47), and contains

$$\sin S_1 = \sin(\pi B/2) - B(\pi^2/4a)\cos(\pi B/2) + \cdots,$$

$$\cos S_1 = \cos(\pi B/2) + B(\pi^2/4a)\sin(\pi B/2) + \cdots.$$

Using these expansions in Eq. (47), we note a first term in $(\pi/2A)\sin S_1 \cos S_1$ that can be dropped, since A is of the order of a^2. We thus are left with

$$\sinh^2\left(\mu\frac{\pi}{2}\right) = -\sin^2 S_1\left(1 + \frac{a(a + \pi/2)}{B^2a^4}\right)$$

and the term in parenthesis is again of second order. Thus,

$$\begin{aligned} \sinh^2\left(\mu\frac{\pi}{2}\right) &= -\sin^2 S_1 = -\sin^2(\pi B/2) + 2B\frac{\pi^2}{4a}\sin(\pi B/2)\cos(\pi B/2) \\ &= -\sin^2 B\frac{\pi}{2} + B\frac{\pi^2}{4a}\sin B\pi \end{aligned} \tag{64}$$

Terms in $1/a^2$ could be computed here without much trouble, but they lead to serious complications with the Whittaker's formula, that we want to discuss now.

We first compute the Fourier coefficients of the periodic function J derived from F according to Eq. (14):

$$J = \sum_{n=-\infty}^{+\infty} \theta_n e^{i2nx} \tag{65}$$

We obtain

$$\theta_n = \theta_{-n} = \frac{B^2}{\pi}\int_{-\pi/2}^{\pi/2}\frac{e^{-i2nx}}{(1 + |x|/a)^4}\,dx = \Re e\,\frac{2B^2}{\pi}\int_0^{\pi/2}\frac{e^{-i2nx}}{(1 + x/a)^4}\,dx$$

The result is

$$\theta_0 = \frac{2B^2}{3\pi}\left[1 - \left(1 + \frac{\pi}{2a}\right)^{-3}\right] = B^2\left(1 - \frac{\pi}{a} + \frac{10}{3}\left(\frac{\pi}{2a}\right)^2 \cdots\right),$$

$$\theta_n = \theta_{-n} = \frac{2B^2}{\pi a n^2}\left[1 - (-1)^n + (-1)^n \frac{5\pi}{2a}\right]. \tag{66}$$

Comparing (66) and (63), we note that to the first order

$$S_1 \approx \frac{\pi}{2}\,\theta_0^{1/2}. \tag{67}$$

We also see that θ_n decreases as $1/n^2$ as needed, and that it is of the first order in $1/a$

Whittaker's formula (7) reads

$$\sinh^2\left(\mu\frac{\pi}{2}\right) = -\Delta_1(0)\sin^2\left(\frac{\pi}{2}\,\theta_0^{1/2}\right). \tag{68}$$

We shall prove easily that $\Delta_1(0)$ is practically unity and equation (67) shows that Whittaker's formula (68) checks completely with our solution (64).

Let us now discuss the infinite determinant (8):

$$\Delta_1(0) = |\,B_{mp}\,|, \qquad B_{mm} = 1,$$

$$B_{mp} = \frac{\theta_{m-p}}{\theta_0 - 4m^2}, \qquad (m \neq p). \tag{69}$$

Diagonal elements are all equal to unity while non-diagonal terms are proportional to θ_{m-p}; hence, according to (66), these non-diagonal terms are all very small, of the order $1/a$.

Such a determinant can be computed in the following way: we first take the product of the diagonal elements, that is 1. Next we take all the diagonal elements but two, namely n, n and m, m, which we replace by the non-diagonal terms B_{nm} and B_{mn}, then we take all the diagonals but three $(n, n; m, m; p, p)$ which we replace by B_{nm}, B_{mp}, B_{pm}, etc. Thus, we obtain

$$\Delta = |\,B_{mp}\,| = 1 - \Sigma_{nm}B_{nm}B_{mn} + \Sigma_{n,m,p}B_{nm}B_{mp}B_{pn} - \cdots \qquad n \neq m \neq p \neq n \cdots\cdots \tag{70}$$

The rule is obvious, and the expansion is ordered with respect to powers of $1/a$, with no term in $1/a$ and terms in $1/a^2$, $1/a^3$ $\cdots$. We decided not to use terms in $1/a^2$ our expansions, hence our determinant is practically unity. Some difficulty may occur when $\theta_0 - 4m^2$ becomes very small (of the order of $1/a$), when the determinant becomes very large. We already noticed in Sec. 1 the inadequacy of Whittaker's formula (68) near the poles of the determinant. There is a compensation, when $\theta_0 = 4m^2$, between the determinant having a double pole and the $\sin^2(\pi/2\,(\theta_0)^{1/2})$ a double zero. Our formula (64) does not exhibit any such trouble.

Otherwise, both methods check completely. It is hoped that the general method developed in this paper will be found useful for practical discussion of many problems reducing to Hill's equation.