THE GENERAL PROBLEM OF ANTENNA RADIATION AND THE FUNDAMENTAL INTEGRAL EQUATION, WITH APPLICATION TO AN ANTENNA OF REVOLUTION—PART II*

BY
J. L. SYNGE
Carnegie Institute of Technology

5. The gap. Let us assume the gap to be cylindrical, of radius $R = a$; let it extend from $z = -\eta$ to $z = \eta$. Let us suppose that on the gap

$$E = -\tfrac{1}{2}V/\eta, \qquad (V = \text{const.}) \tag{5.1}$$

so that V is the potential difference across the gap. Then, by (3.18),

$$M(z_0) = \pi c k^2 V N(z_0), \qquad N(z_0) = -\frac{a^2}{k\eta}\int_{-\eta}^{\eta}\frac{1}{r}\frac{d\psi}{dr}\,dz. \tag{5.2}$$

Now

$$\begin{gathered} r^2 = a^2 + (z - z_0)^2, \qquad \psi = r^{-1}e^{ikr}, \\ \frac{1}{r}\frac{d\psi}{dr} = -\frac{1}{r^3} - \frac{k^2}{2r} + k^3(\chi_1 + i\chi_2), \\ \chi_1 = \frac{1}{k^3r^3}\left(1 - \tfrac{1}{2}k^2r^2 - \cos kr\right) + \frac{1}{k^2r^2}(kr - \sin kr), \\ \chi_2 = \frac{1}{k^3r^3}(kr - \sin kr) - \frac{1}{k^2r^2}(1 - \cos kr). \end{gathered} \tag{5.3}$$

Note that χ_1, χ_2 are power series in positive powers of kr. We obtain at once

$$\begin{aligned} N(z_0) = \frac{1}{k\eta}&\left\{\frac{\eta - z_0}{[a^2 + (\eta - z_0)^2]^{1/2}} + \frac{\eta + z_0}{[a^2 + (\eta + z_0)^2]^{1/2}}\right\} \\ &+ \frac{1}{2}\frac{k^2a^2}{k\eta}\left\{\ln k(\eta - z_0 + [a^2 + (\eta - z_0)^2]^{1/2})\right. \\ &\left. + \ln k(\eta + z_0 + [a^2 + (\eta + z_0)^2]^{1/2}) - \ln k^2a^2\right\} \\ &- \frac{k^2a^2}{\eta}\int_{-\eta}^{\eta}(\chi_1 + i\chi_2)\,dz. \end{aligned} \tag{5.4}$$

We shall now make two important simplifying assumptions. The first is

$$ka \ll 1. \tag{5.5}$$

*Received March 31, 1947. This paper is a continuation of the paper by G. E. Albert and J. L. Synge which appeared in this Quarterly 6, 117-131 (1948).

This means that the radius of the antenna at the gap is small compared with the wave length. Then we have approximately

$$N(z_0) = \frac{1}{k\eta}\left\{\frac{\eta - z_0}{[a^2 + (\eta - z_0)^2]^{1/2}} + \frac{\eta + z_0}{[a^2 + (\eta + z_0)^2]^{1/2}}\right\}, \tag{5.6}$$

provided a/η is not large. The second assumption is

$$a/\eta \ll 1. \tag{5.7}$$

This means that the gap is long compared with the radius of the antenna at the gap. Then (5.6) gives approximately

$$N(z_0) = \frac{1}{k\eta}\left\{\frac{\eta - z_0}{| \eta - z_0 |} + \frac{\eta + z_0}{| \eta + z_0 |}\right\}, \tag{5.8}$$

and so

$$\begin{aligned} N(z_0) &= \frac{2}{k\eta} \qquad \text{for } | z_0 | < \eta, \\ N(z_0) &= 0 \qquad \text{for } | z_0 | > \eta. \end{aligned} \tag{5.9}$$

Substitution in (5.2) gives

$$\begin{aligned} M(z_0) &= 2\pi ckV/\eta \qquad \text{for } | z_0 | < \eta \qquad \text{(in gap)} \\ M(z_0) &= 0 \qquad \text{for } | z_0 | > \eta \qquad \text{(outside gap)} \end{aligned} \tag{5.10}$$

6. Impedance and relative current. In this section we introduce the impedance Z and the relative current $\phi(z)$, and show how Z is found when $\phi(z)$ is known. The argument is exact; we understand by N the exact expression (5.2) rather than the approximation (5.9).

We define the impedance of the antenna to be

$$Z = V/I(0) \tag{6.1}$$

It may seem unnatural to use $I(0)$ in defining impedance. The point $z = 0$ is at the center of the gap, and there is no current there. In fact, $I(0)$ means $2\pi caH(0)$, according to (3.14). It might seem better to use the currents at the ends of the gap. But it appears simpler to use $I(0)$ as basic; we can easily pass to the other definitions, if required. In the case of a very short gap (Sec. 8), these subtle distinctions disappear, for we have then

$$I(-\eta) = I(0) = I(\eta)$$

approximately.

Let us write

$$\phi(z) = I(z)/I(0); \tag{6.2}$$

this will be called the *relative current*.

On dividing (3.17) by $I(0)$, we get

$$\int_{l_1}^{l_2} K(z, z_0)\phi(z)\, dz = i\pi ck^2 Z\, N(z_0), \qquad (l_1 < z_0 < l_2) \tag{6.3}$$

With this equation we associate the boundary conditions

$$\phi(0) = 1, \phi(l_1) = \phi(l_2) = 0. \tag{6.4}$$

The equation (6.3) contains the unknown function $\phi(z)$ and the unknown constant Z. If we knew $\phi(z)$, we could calculate Z at once, giving any value to z_0. If we have only a rough idea of $\phi(z)$, it would be better not to take a definite value of z_0, but to introduce a weighting factor $f(z_0)$, and calculate Z from

$$Z = -\frac{i}{\pi c k^2} \frac{\int_{l_1}^{l_2} f(z_0)\, dz_0 \int_{l_1}^{l_2} K(z, z_0)\phi(z)\, dz}{\int_{l_1}^{l_2} N(z_0) f(z_0)\, dz_0}. \tag{6.5}$$

For the present let us leave the weighting function $f(z)$ arbitrary except for the assumptions that it is continuous, has a continuous derivative, and satisfies the end conditions

$$f(l_1) = f(l_2) = 0. \tag{6.6}$$

Let us write J for the numerator in (6.5) and understand the limits of integration (l_1, l_2). Then

$$\begin{aligned} J &= \iint K(z, z_0)\phi(z) f(z_0)\, dz\, dz_0 \\ &= -\iint \frac{\partial^2 \psi}{\partial z\, \partial z_0} \phi(z) f(z_0)\, dz\, dz_0 + k^2 \iint \psi(z, z_0)\phi(z) f(z_0)\, dz\, dz_0, \end{aligned} \tag{6.7}$$

and on integration by parts

$$J = \iint \psi(z, z_0)\chi(z, z_0)\, dz\, dz_0, \tag{6.8}$$

where

$$\chi(z, z_0) = -\phi'(z) f'(z_0) + k^2 \phi(z) f(z_0). \tag{6.9}$$

Now we may write (6.8) as follows:

$$\begin{aligned} J &= J_1 + J_2, \\ J_1 &= \iint \frac{1}{r(z, z_0)} \chi(z, z)\, dz\, dz_0, \\ J_2 &= \iint \frac{1}{r(z, z_0)} \{\chi(z, z_0) \exp(ikr(z, z_0)) - \chi(z, z)\}\, dz\, dz_0, \\ [r(z, z_0)]^2 &= [R(z)]^2 + (z - z_0)^2. \end{aligned} \tag{6.10}$$

For a thin antenna, r is small for $z = z_0$. However, the integrand in J_2 remains finite as $R \to 0$. This is the reason for splitting J as above, and forms the basis of later approximations. For the present the argument remains exact.

The integral J_1 gives

$$J_1 = J_{11} + J_{12} + J_{13}, \tag{6.11}$$

where

$$J_{11} = L \int_{l_1}^{l_2} \chi(z, z)\, dz, \qquad L = -\ln(k^2 a^2),$$

$$J_{12} = \int_{l_1}^{l_2} \chi(z, z) \ln \frac{a^2}{(R(z))^2}\, dz, \tag{6.12}$$

$$J_{13} = \int_{l_1}^{l_2} \chi(z, z) \ln \{k^2[l_2 - z + (R^2 + (l_2 - z)^2)^{1/2}][z - l_1 + (R^2 + (z - l_1)^2)^{1/2}]\}\, dz.$$

Note, for later approximation, that J_{12} , J_{13} remain finite for an infinitely thin antenna.

We have, by (6.6),

$$\begin{aligned} J_{11} &= L \int_{l_1}^{l_2} \{-\phi'(z)f'(z) + k^2\phi(z)f(z)\}\, dz \\ &= L \int_{l_1}^{l_2} \{f''(z) + k^2 f(z)\}\phi(z)\, dz. \end{aligned} \tag{6.13}$$

Now (6.5) may be written

$$Z = -\frac{i}{\pi c k^2} \frac{J_{11}(\phi, f) + J_{12}(\phi, f) + J_{13}(\phi, f) + J_2(\phi, f)}{\int_{l_1}^{l_2} N(z)\, f(z)\, dz}. \tag{6.14}$$

This notation puts in evidence the dependence of the J's on the two functions—the relative current ϕ and the weighting function f.

The function f is at our disposal. We see from (6.13) that J_{11} would vanish for an f sinusoidal in kz. However, unless the whole length of the antenna is a multiple of $\frac{1}{2}\lambda$, there exists no such function with continuous derivative, satisfying the end conditions (6.6). So we approach a sinusoidal f by a limiting process, in which (on attaining the limit) the continuity of the derivative is lost, but (6.14) remains true.

Let ϵ be any small positive number. We define a function $f_1(z, \epsilon)$ as follows:

$$l_1 \leq z \leq -\epsilon : K(\epsilon)f_1(z, \epsilon) = -\sin kl_2 \sin k(l_1 - z), \tag{6.15a}$$

$$\begin{aligned} -\epsilon \leq z \leq \epsilon \ : K(\epsilon)f_1(z, \epsilon) &= K(\epsilon) \cos kz + \tfrac{1}{2} \sin kz \sin k(l_1 + l_2) \\ &\quad - \tfrac{1}{2} \operatorname{cosec} k\epsilon \sin k(l_2 - l_1)(1 - \cos kz), \end{aligned} \tag{6.15b}$$

$$\epsilon \leq z \leq l_2 \quad : K(\epsilon)f_1(z, \epsilon) = -\sin kl_1 \sin k(l_2 - z), \tag{6.15c}$$

$$K(\epsilon) = -\sin kl_1 \sin kl_2 - \tfrac{1}{2} \tan \tfrac{1}{2}k\epsilon \cdot \sin k(l_2 - l_1). \tag{6.16}$$

To avoid complicating the argument, we assume

$$\sin kl_1 \neq 0, \qquad \sin kl_2 \neq 0. \tag{6.17}$$

This means that neither arm of the antenna, measured from the center of the gap, is a multiple of $\frac{1}{2}\lambda$. Such critical cases must be approached by a special limiting process.

We note that $f_1(z, \epsilon)$ is continuous, with continuous first derivative, and satisfies

$$|z| > \epsilon : f_1''(z, \epsilon) + k^2 f_1(z, \epsilon) = 0, \tag{6.18a, c}$$

$$|z| < \epsilon : f_1''(z, \epsilon) + k^2 f_1(z, \epsilon) = -\frac{k^2 \sin k(l_2 - l_1)}{2K(\epsilon) \sin k\epsilon}. \tag{6.18b}$$

Then, by (6.13), in an obvious notation,

$$J_{11}(\phi, f_1(\epsilon)) = -\frac{k^2 \sin k(l_2 - l_1)}{2K(\epsilon) \sin k\epsilon} \int_{-\epsilon}^{\epsilon} \phi(z)\, dz. \tag{6.19}$$

Let us write

$$f_1(z) = \lim_{\epsilon \to 0} f_1(z, \epsilon), \tag{6.20}$$

so that

$$l_1 \le z \le 0: \qquad f_1(z) = \sin k(l_1 - z)/\sin kl_1\,, \tag{6.21a}$$

$$0 \le z \le l_2: \qquad f_1(z) = \sin k(l_2 - z)/\sin kl_2\,. \tag{6.21c}$$

Proceeding to the limit $\epsilon \to 0$ in (6.19), we get (since $\phi(0) = 1$)

$$J_{11}(\phi, f_1) = kL\Gamma, \qquad \Gamma = \frac{\sin k(l_2 - l_1)}{\sin kl_1 \sin kl_2}. \tag{6.22}$$

Let us now put $f_1(z, \epsilon)$ for $f(z)$ in (6.14) and proceed to the limit $\epsilon \to 0$. In this limiting process, $f_1(z, \epsilon)$ and its first derivative remain finite in the whole range (l_1, l_2), and so the contributions to J_{12}, J_{13}, J_2 from the range $(-\epsilon, \epsilon)$ vanish in the limit. Thus we get

$$Z = -\frac{i}{\pi c k^2} \frac{kL\Gamma + J_{12}(\phi, f_1) + J_{13}(\phi, f_1) + J_2(\phi, f_1)}{\int_{l_1}^{l_2} N(z) f_1(z)\, dz}, \tag{6.23}$$

where $f_1(z)$ is as in (6.21) and Γ as in (6.22); $L = -\ln(k^2 a^2)$.

This is accurate, and indeed holds for the general N of (5.4) as well as the more particular N of (5.9). The fact that $f_1(z)$ has a discontinuous first derivative at $z = 0$ creates no trouble. It is of course understood that in evaluating J_2 by (6.10) and J_{12}, J_{13} by (6.12), we are to put

$$\chi(z, z_0) = -\phi'(z) f_1'(z_0) + k^2 \phi(z) f_1(z_0). \tag{6.24}$$

On substituting for N from (5.9) in the denominator of (6.23), we obtain

$$Z = -\frac{i}{4\pi c k h} \{kL\Gamma + J_{12}(\phi, f_1) + J_{13}(\phi, f_1) + J_2(\phi, f_1)\},$$

$$h = \frac{\sin k\eta}{k\eta} + \tfrac{1}{2}\Gamma \frac{1 - \cos k\eta}{k\eta}. \tag{6.25}$$

This is the formula we shall use in the later work. It is accurate except for the approximation involved in (5.9).

Note: Do not confuse the range $(-\epsilon, \epsilon)$ with the gap $(-\eta, \eta)$. The former is merely a mathematical device, introduced to eliminate ϕ from the first term in the numerator of (6.23). As a matter of fact, we shall make no further use of this ϵ; it has done its work in providing the formulae (6.23), (6.25).

7. The thin antenna. We cannot expect to get results for an antenna of general form without a considerable amount of calculation. But if the antenna is thin, the largeness of L may be used as a basis of approximation. In the present section we obtain the principal parts of the current and impedance for a thin antenna. This current is,

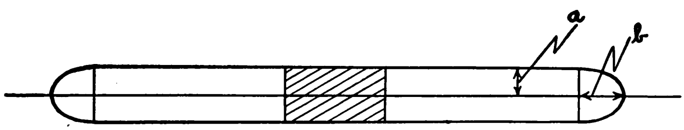

FIG. 13. Cylindrical antenna with spheroidal ends.

in fact, the familiar sinusoidal distribution. The present derivation of this distribution may be of interest because previous derivations have been by no means clear. Moreover, in the present method, it is not necessary to assume that the antenna is cylindrical; it is merely necessary that the radius R be small throughout. The ends of the antenna require no special treatment. Further, our formula for impedance contains a shape term. The method of the present section does not open up a process of successive approximations; that will be given in Sec. 10.

It will be well to mention here assumptions which will be introduced explicitly later:

The antenna is thin at the gap ($ka \ll 1$). (7.1)

The antenna is thin throughout ($kR(z) \ll 1$, $l_1 < z < l_2$). (7.2)

The gap is long compared with the radius ($a/\eta \ll 1$). (7.3)

The gap is short compared with the wave length ($k\eta \ll 1$). (7.4)

Obviously (7.2) contains (7.1).

We turn back to the exact formula (6.14), in which $f(z)$ is arbitrary. By (6.13) we have

$$J_{11} = L \int_{l_1}^{l_2} \{\phi''(z) + k^2\phi(z)\} f(z)\, dz. \tag{7.5}$$

We rearrange (6.14) in the form

$$\int_{l_1}^{l_2} \{\phi''(z) + k^2\phi(z) - i\pi c k^2 Z L^{-1} N(z)\} f(z)\, dz = -L^{-1}(J_{12} + J_{13} + J_2). \tag{7.6}$$

Now make the assumption (7.2). Then L is large and the right hand side of (7.6) is small of order L^{-1}, provided $\phi(z)$ and $\phi'(z)$ remain bounded as L tends to infinity. Since f is arbitrary except for the end conditions (6.6), it follows from (7.6) that

$$\phi''(z) + k^2\phi(z) - i\pi c k^2 Z L^{-1} N(z) = L^{-1}\mathfrak{F}(z), \tag{7.7}$$

where $\mathfrak{F}(z)$ is some finite function. Integration gives

$$\phi(z) = \alpha \cos kz + \beta \sin kz + i\pi c k Z L^{-1} \int_0^z N(t) \sin k(z - t)\, dt + L^{-1}\mathcal{G}(z), \tag{7.8}$$

where $\mathcal{G}(z)$ is some finite function. This solution is subject to the three conditions (6.4), and if we knew $\mathcal{G}(z)$ we could find α, β, Z. But we do not know $\mathcal{G}(z)$, and can merely make use of the fact that the last term is of order L^{-1}.

Whatever $\mathcal{G}(z)$ may be, the three equations for α, β, Z are consistent provided

$$\begin{vmatrix} \sin kl_1 & \int_0^{l_1} N(t) \sin k(l_1 - t)\, dt \\ \sin kl_2 & \int_0^{l_2} N(t) \sin k(l_2 - t)\, dt \end{vmatrix} \neq 0. \tag{7.8a}$$

Let us make the assumption (7.3), so that we may use (5.9) for N. Then the above condition for consistency reads

$$(1 - \cos k\eta) \sin k(l_2 - l_1) + 2 \sin k\eta \sin kl_1 \sin kl_2 \neq 0. \tag{7.8b}$$

Assuming that $k\eta$, kl_1, kl_2 are such that this inequality is satisfied, we obtain α, β, Z from (6.4), and substitution in (7.8) gives accurately

$$\phi(z) = \phi_1(z) + L^{-1}\Omega(z), \tag{7.9}$$

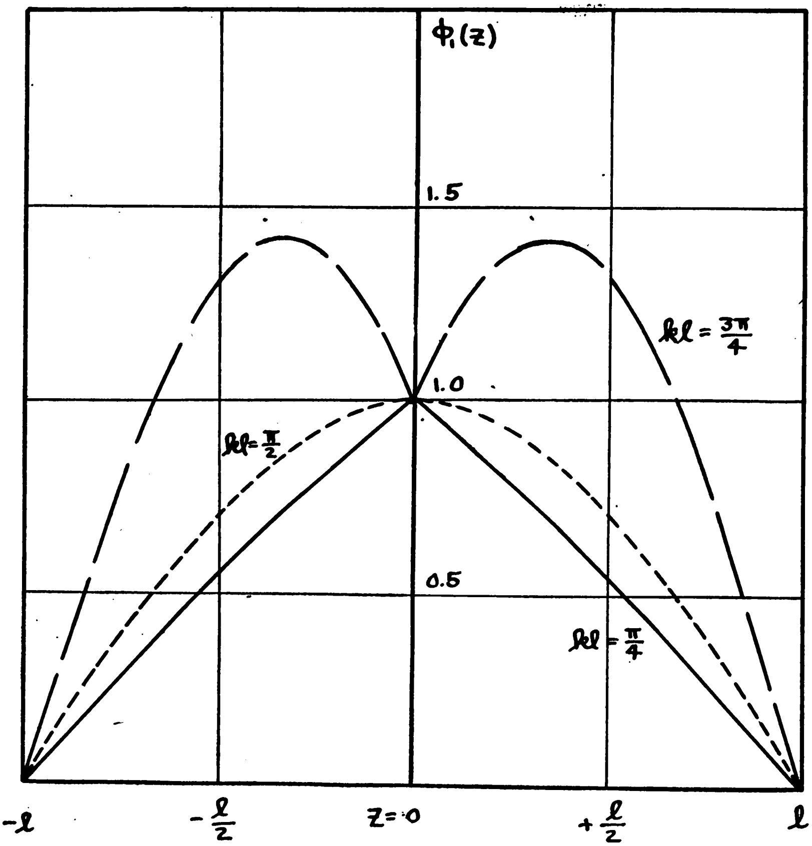

FIG. 14a. Infinitesimal gap at center.

where $\Omega(z)$ is an unknown finite function and $\phi_1(z)$ is given by

$$l_1 \leq z \leq -\eta : K(\eta)\phi_1(z) = -\sin kl_2 \sin k(l_1 - z), \tag{7.10a}$$

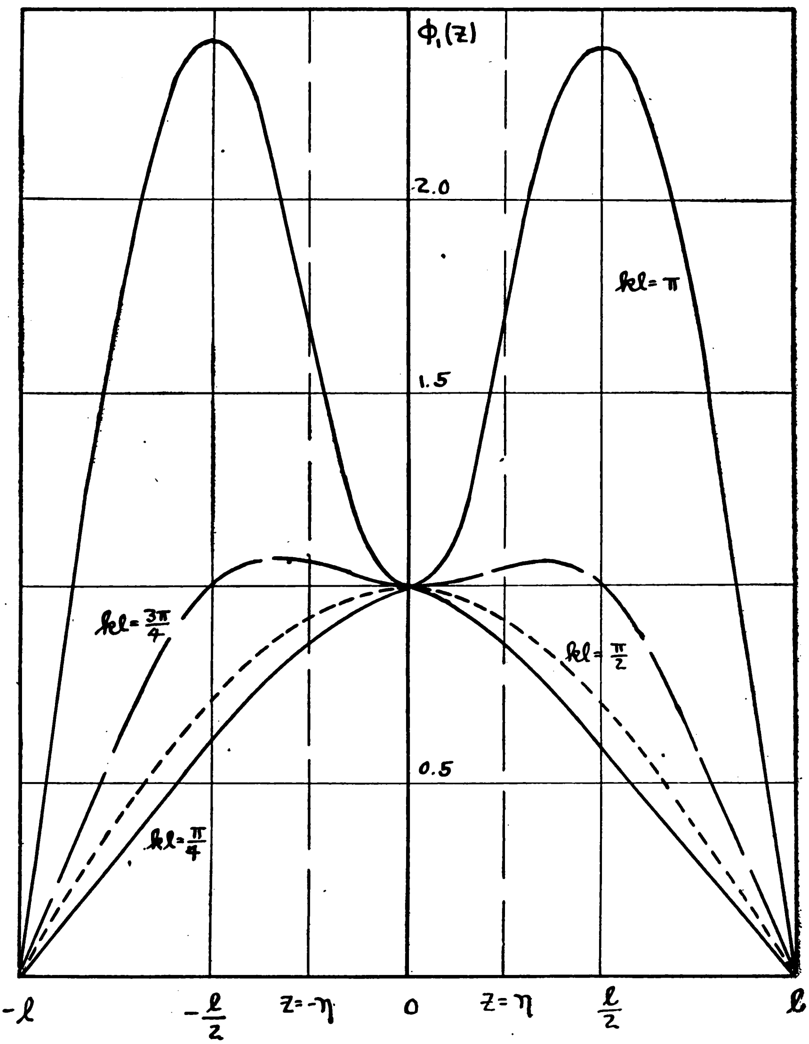

FIG. 14b. Finite gap ($\eta = l/4$) at center.

$$-\eta \leq z \leq \eta \ : K(\eta)\phi_1(z) = K(\eta) \cos kz + \tfrac{1}{2} \sin k(l_1 + l_2) \sin kz - \tfrac{1}{2} \operatorname{cosec} k\eta \sin k(l_2 - l_1)(1 - \cos kz), \tag{7.10b}$$

$$\eta \leq z \leq l_2 \quad : K(\eta)\phi_1(z) = -\sin kl_1 \sin k(l_2 - z), \tag{7.10c}$$

$$K(\eta) = -\sin kl_1 \sin kl_2 - \tfrac{1}{2} \tan \tfrac{1}{2}k\eta \sin k(l_2 - l_1). \tag{7.11}$$

(These formulae should be compared with (6.15), (6.16). Note that $l_2 - l_1$ is the length of the antenna.)

Physically, $\phi_1(z)$ represents the principal part of the relative current for a thin antenna with a gap which is long compared with the radius of the antenna, and becomes a better approximation the thinner the antenna becomes. Graphs of $\phi_1(z)$ are given in Figs. 14a–g (for discussion, see Appendix, p. 155). Outside the gap, $\phi_1(z)$ has the *sinusoidal distribution,* so basic in antenna theory.

To get the impedance, we substitute for ϕ from (7.9) in (6.25). This gives

$$Z = -\frac{i}{4\pi ckh}\{kL\Gamma + J_{12}(\phi_1, f_1) + J_{13}(\phi_1, f_1) + J_2(\phi_1, f_1) + L^{-1}[J_{12}(\Omega, f_1) + J_{13}(\Omega, f_1) + J_2(\Omega, f_1)]\}. \tag{7.12}$$

Here everything is known, except Ω.

Let us sum up:

For a thin antenna with a gap much longer than its radius, the relative current is given by (7.10), *with an error of order* L^{-1}, *and the impedance is*

$$Z_1 = -\frac{i}{4\pi ckh}\{kL\Gamma + J_{12}(\phi_1, f_1) + J_{13}(\phi_1, f_1) + J_2(\phi_1, f_1)\}, \tag{7.13}$$

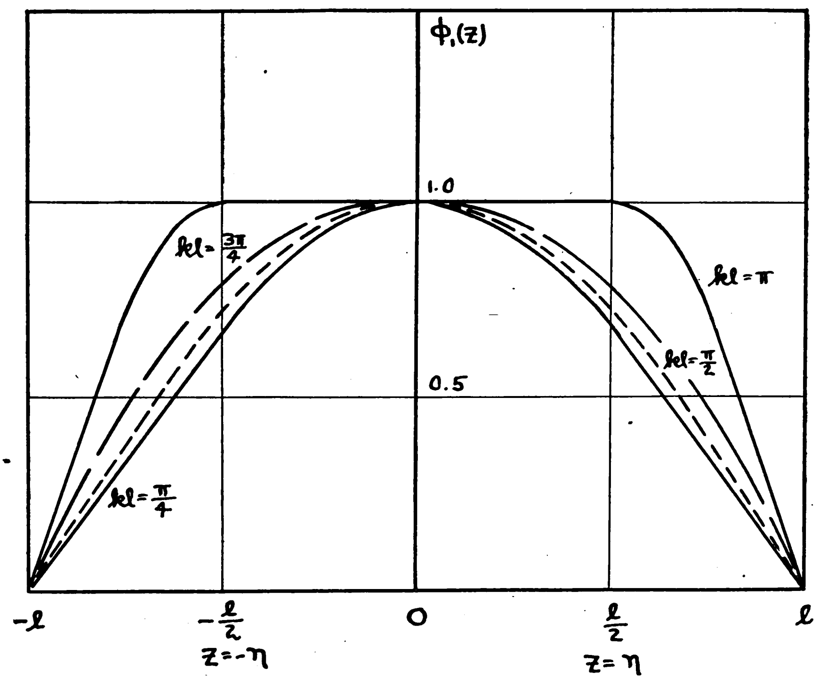

FIG. 14c. Finite gap ($\eta = l/2$) at center.

with an error of order L^{-1}. Here

$$L = -\ln(k^2a^2), \qquad a = \text{radius at gap}, \qquad k = 2\pi/\lambda,$$

$$h = \frac{\sin k\eta}{k\eta} + \tfrac{1}{2}\Gamma\frac{1-\cos k\eta}{k\eta}, \qquad (-\eta < z < \eta) \text{ is gap}, \tag{7.14}$$

$$\Gamma = \frac{\sin k(l_2 - l_1)}{\sin kl_1 \sin kl_2} = \cot kl_1 - \cot kl_2 .$$

We must remember that f_1 is given by (6.21) and does not depend on ϵ, whereas ϕ_1 is given by (7.10) and does depend on η.

On account of the assumed thinness of the antenna, we can simplify the expressions (6.10) for J_2 and (6.12) for J_{13}. In fact, we shall put $R = 0$ in J_2 and J_{13}, since the consequent error is of order ka, and so is negligible in comparison with L^{-1}. Thus we write

$$\begin{aligned} J_{12} &= \int_{l_1}^{l_2} \chi(z, z) \ln \frac{a^2}{(R(z))^2}\, dz, \\ J_{13} &= \int_{l_1}^{l_2} \chi(z, z) \ln [4k^2(l_2 - z)(z - l_1)]\, dz, \\ J_2 &= \int_{l_1}^{l_2} \int_{l_1}^{l_2} | z - z_0 |^{-1}\{\chi(z, z_0) \exp ik\, | z - z_0 | - \chi(z, z)\}\, dz\, dz_0\,, \\ \chi(z, z_0) &= -\phi_1'(z)f_1'(z_0) + k^2\phi_1(z)f_1(z_0). \end{aligned} \tag{7.15}$$

We note that the shape of the antenna is now involved only in J_{12}, and that there is no contribution to J_{12} from cylindrical parts of the antenna.

Putting $Z_1 = R_1 - iX_1$, the approximate resistance and reactance are (with an error of order L^{-1})

$$\begin{aligned} R_1 &= \frac{1}{4\pi ckh} J_{22}\,, \\ X_1 &= \frac{1}{4\pi ckh} (kL\Gamma + J_{12} + J_{13} + J_{21}), \end{aligned} \tag{7.16}$$

where

$$J_{21} + iJ_{22} = J_2\,. \tag{7.17}$$

Thus the resistance is independent of shape.

8. Thin antenna with short gap. The calculation of the impedance from (7.13) is direct. The result will, of course, depend on the gap-length 2η, and will be very involved on account of the complexity of (7.10). Let us therefore, for simplicity, introduce the assumption (7.4); we consider the gap short compared with the wave-length, but still long compared with the radius of the antenna.

If $k\eta$ is small, $h = 1$ approximately. Further, by (7.10), we have approximately

$$l_1 \leq z \leq 0: \qquad \phi_1(z) = \sin k(l_1 - z)/\sin kl_1\,, \tag{8.1a}$$

$$0 \leq z \leq l_2: \qquad \phi_1(z) = \sin k(l_2 - z)/\sin kl_2\,. \tag{8.1c}$$

(Note: the consistency condition (7.8b) becomes $\sin kl_1 \sin kl_2 \neq 0$ for small $k\eta$.) On comparison with (6.21), we see that $\phi_1 = f_1$. This greatly simplifies the work by introducing symmetry. We have

$$\chi(z, z_0) = -\phi_1'(z)\phi_1'(z_0) + k^2\phi_1(z)\phi_1(z_0), \tag{8.2}$$

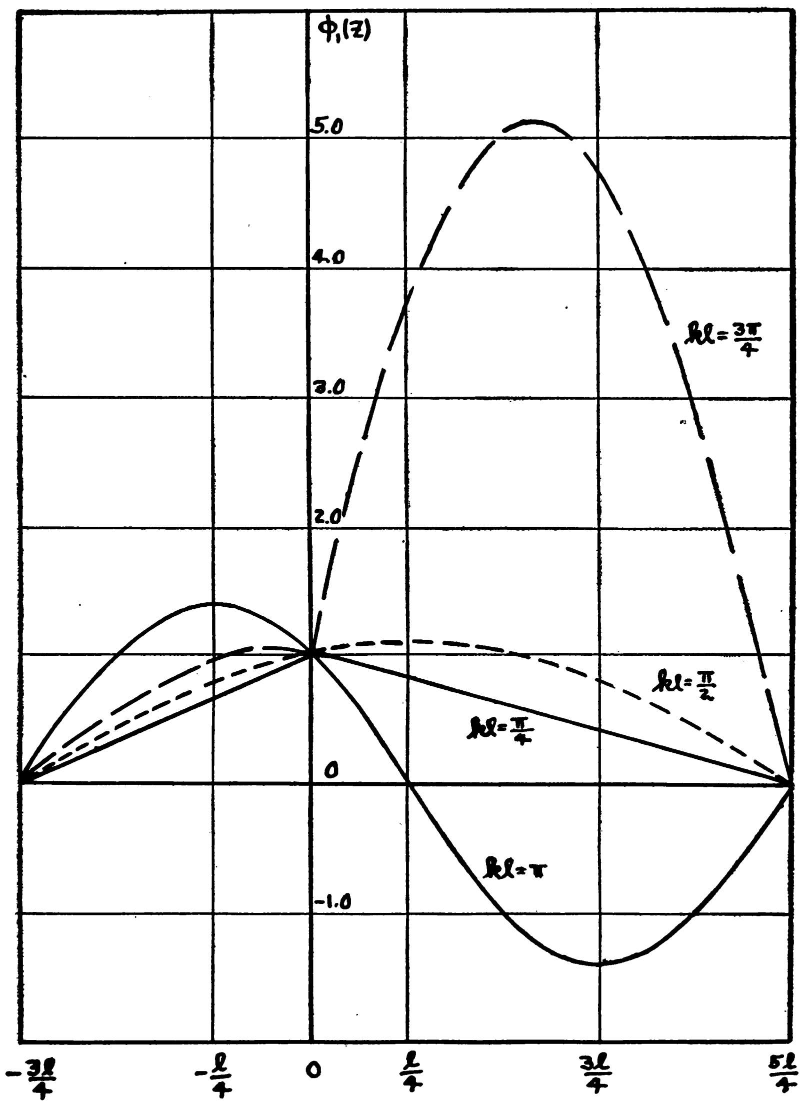

FIG. 14d. Infinitesimal gap at point dividing antenna in ratio 3:5.

and so

$$l_1 \leq z < 0 : \chi(z, z) = -k^2 \operatorname{cosec}^2 kl_1 \cos 2k(l_1 - z), \tag{8.3a}$$

$$0 < z \leq l_2 : \chi(z, z) = -k^2 \operatorname{cosec}^2 kl_2 \cos 2k(l_2 - z), \tag{8.3c}$$

$$z < 0, z_0 < 0 : \chi(z, z_0) = -k^2 \operatorname{cosec}^2 kl_1 \cos k(2l_1 - z - z_0), \tag{8.4aa}$$

$z > 0$, $z_0 < 0$, or $z < 0$, $z_0 > 0$:

$$\chi(z, z_0) = -k^2 \operatorname{cosec} kl_1 \operatorname{cosec} kl_2 \cos k(l_1 + l_2 - z - z_0), \tag{8.4ac}$$

$$z > 0, z_0 > 0 : \chi(z, z_0) = -k^2 \operatorname{cosec}^2 kl_2 \cos k(2l_2 - z - z_0). \tag{8.4cc}$$

It is best to combine J_{13} and J_2 in (7.15). We have

$$\iint_{(\epsilon)} | z - z_0 |^{-1} \chi(z, z)\, dz\, dz_0$$
$$= \int_{l_1}^{l_2} \chi(z, z) \ln[4k^2(l_2 - z)(z - l_1)]\, dz - 2\ln(2k\epsilon) \int_{l_1}^{l_2} \chi(z, z)\, dz. \tag{8.5}$$

Here $\iint_{(\epsilon)}$ means integration over the square (l_1, l_2) with omission of the strip $| z - z_0 | < \epsilon$. Now

$$J_2 = \lim_{\epsilon \to 0} \iint_{(\epsilon)} | z - z_0 |^{-1} \{\chi(z, z_0) \exp ik | z - z_0 | - \chi(z, z)\}\, dz\, dz_0 , \tag{8.6}$$

and so

$$J_{13} + J_2 = \lim_{\epsilon \to 0} \left[\iint_{(\epsilon)} | z - z_0 |^{-1} \chi(z, z_0) \exp ik | z - z_0 |\, dz\, dz_0 + 2\ln(2k\epsilon) \int_{l_1}^{l_2} \chi(z, z)\, dz \right]. \tag{8.7}$$

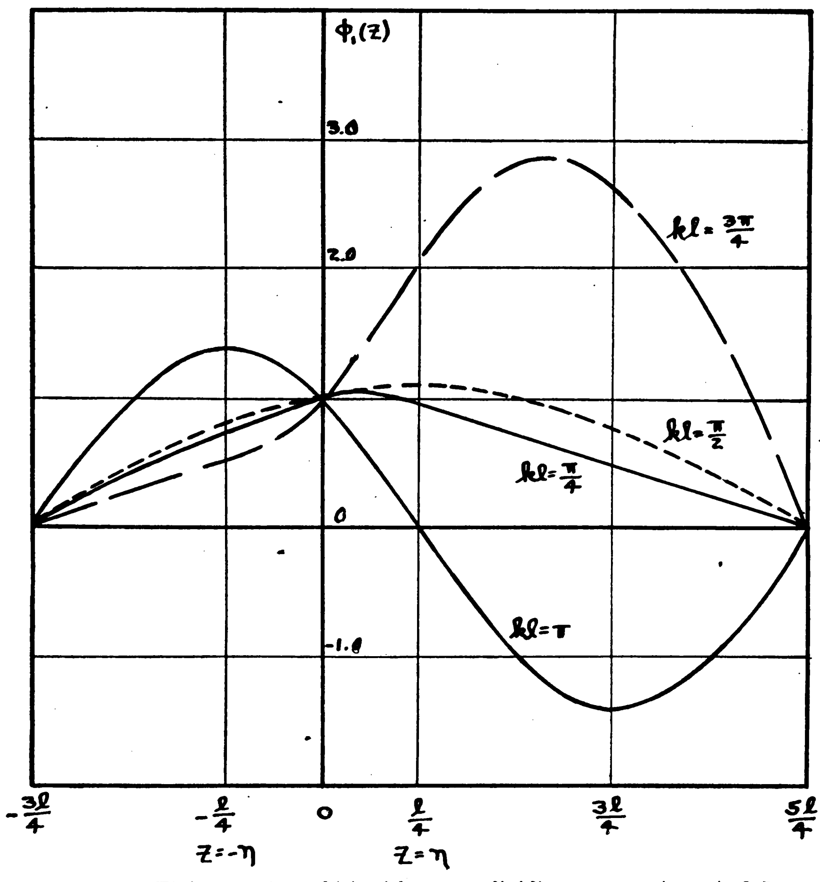

FIG. 14e. Finite gap ($\eta = l/4$) with center dividing antenna in ratio 3:5.

This double integral is best evaluated by rotating the z, z_0 axes through half a right angle in the plane of the integral. It is convenient to introduce the function

$$\Phi(x) = \int_0^x \frac{e^{it} - 1}{t}\,dt = \mathrm{Ci}\,x + i\,\mathrm{Si}\,x - \ln \gamma x, \qquad \ln \gamma = 0.5772. \tag{8.8}$$

We find

$$\begin{aligned}(J_{13} + J_2)/k = {} & (c_2 - c_1) \ln \frac{(l_2 - l_1)^2}{4k^2 l_1^2 l_2^2} \\ & + \Phi(4kl)\{c_2 - c_1 - i(c_1 c_2 + 1)\} \\ & - \Phi(2kl_2)\{c_2 - c_1 + ic_2(c_2 - c_1)\} \\ & - \Phi(-2kl_1)\{c_2 - c_1 - ic_1(c_2 - c_1)\},\end{aligned} \tag{8.9}$$

$$c_2 = \cot kl_2 , \qquad c_1 = \cot kl_1 , \qquad 2l = l_2 - l_1 = \text{length of antenna.}$$

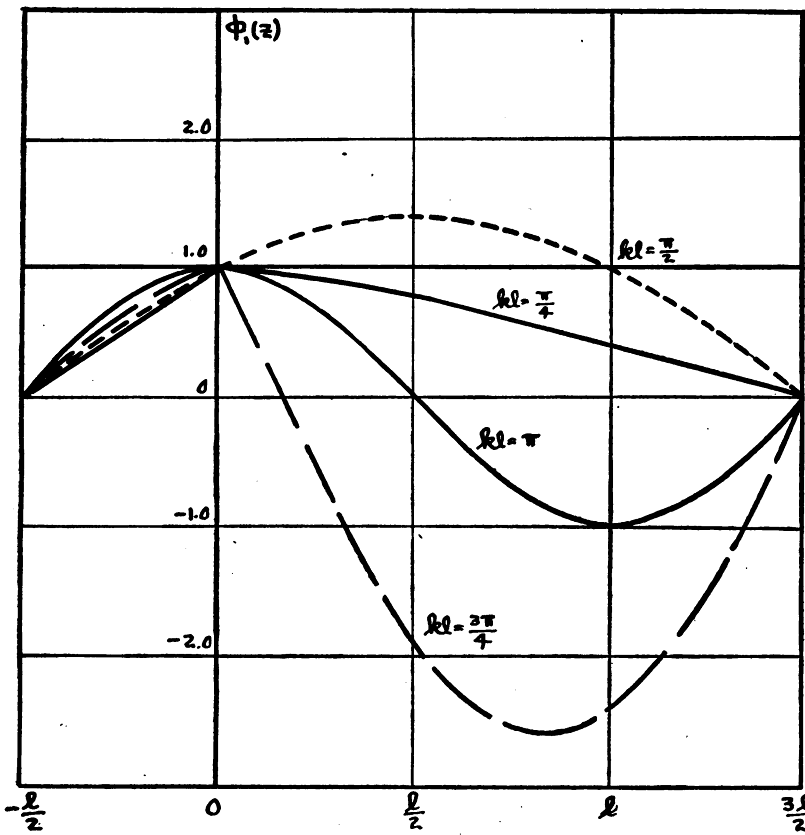

FIG. 14f. Infinitesimal gap at point dividing antenna in the ratio 1:3.

When we substitute from (8.9) in (7.13), we get the following expression for *the principal part of the impedance of a thin antenna*, expressed in ohms (1 Heaviside unit = 120 πc ohms):

$$
\begin{aligned}
Z_1 &= R_1 - iX_1 \\
&= 30 \frac{\sin 2kl}{\sin k \mid l_1 \mid \sin kl_2} \Big\{ 2i \ln \left| \frac{l_1 l_2}{al} \right| - \Phi(4kl)(i - \cot 2kl) \\
&\qquad + \Phi(2k \mid l_1 \mid)(i - \cot k \mid l_1 \mid) \\
&\qquad + \Phi(2kl_2)(i - \cot kl_2) \Big\} - iX_s ,
\end{aligned} \tag{8.10}
$$

where

$$
\begin{aligned}
X_s = 30J_{12}/k = -30k \operatorname{cosec}^2 kl_1 \int_{l_1}^{0} \cos 2k(l_1 - z) \ln \frac{a^2}{(R(z))^2}\, dz \\
-30k \operatorname{cosec}^2 kl_2 \int_0^{l_2} \cos 2k(l_2 - z) \ln \frac{a^2}{(R(z))^2}\, dz.
\end{aligned} \tag{8.11}
$$

Here $z = l_1$, $z = l_2$ are the ends of the antenna ($l_1 < 0 < l_2$); the gap is short and at $z = 0$; $2l$ is the length of the antenna; a is the radius at the gap, and $R(z)$ the radius at the general point; $k = 2\pi/\lambda$, where λ is the free wave length.

If we take the gap at the middle, then $-l_1 = l_2 = l$, and (8.10) becomes

$$
\begin{aligned}
Z_1 = 60 \cot kl \{2i \ln(l/a) - \Phi(4kl)(i - \cot 2kl) \\
+ 2\Phi(2kl)(i - \cot kl)\} - iX_s .
\end{aligned} \tag{8.12}
$$

Except for the shape term X_s, this agrees with the formula given by Brown and King[1] using the method of Labus.[2] There is also agreement with the principal part of Schelkunoff's formula[3] and with the formula of Hallén.[4] Schelkunoff's treatment of the influence of shape is difficult to follow. Hallén includes a shape term in his equation (26), but later specializes to a cylindrical antenna, so that a formula such as our (8.11) does not occur explicitly in his equation (39). Owing to the inadequate treatment of the gap in the work of Hallén and Schelkunoff, the validity of their higher approximations is open to question. It must be remembered that an error of the order L^{-1} is admitted in our formulae (8.10), (8.12).

In (8.12) it is not assumed that the antenna has $z = 0$ for equatorial plane of symmetry. Deviation from this symmetry influences X_s, but not the other terms.

Let us consider an antenna with the gap at the center and total length nearly $\frac{1}{2}\lambda$, so that

$$
kl = \tfrac{1}{2}\pi + \epsilon, \tag{8.13}
$$

where ϵ is small. Then, approximately, with errors of orders ϵ^3 and ϵ^2 respectively,

$$
\cot kl = -\epsilon, \qquad \cot kl \cot 2kl = -\tfrac{1}{2}. \tag{8.14}
$$

[1] G. H. Brown and R. King, Proc. I. R. E. 22, 457-480 (1934).
[2] J. Labus, Hochfrequenztechnik und Elektroakustik 41, 17-23 (1933).
[3] S. A. Schelkunoff, Proc. I. R. E. 29, 493-521 (1941).
[4] E. Hallén, Nova Acta Reg. Soc. Sci. Upsaliensis 11, No. 4 (1939).

Then we may neglect parts of (8.12) and write

$$\begin{aligned} Z_1 &= R_1 - iX_1 = -120i\epsilon \ln \frac{l}{a} - 30\Phi(2\pi) - iX_s, \\ R_1 &= 30(\log 2\pi\gamma - \text{Ci } 2\pi) = 73.13, \\ X_1 &= 120\epsilon \ln \frac{l}{a} + X_s + 30 \text{ Si } 2\pi, \\ &= 120\epsilon \ln \frac{l}{a} + X_s + 42.54. \end{aligned} \tag{8.15}$$

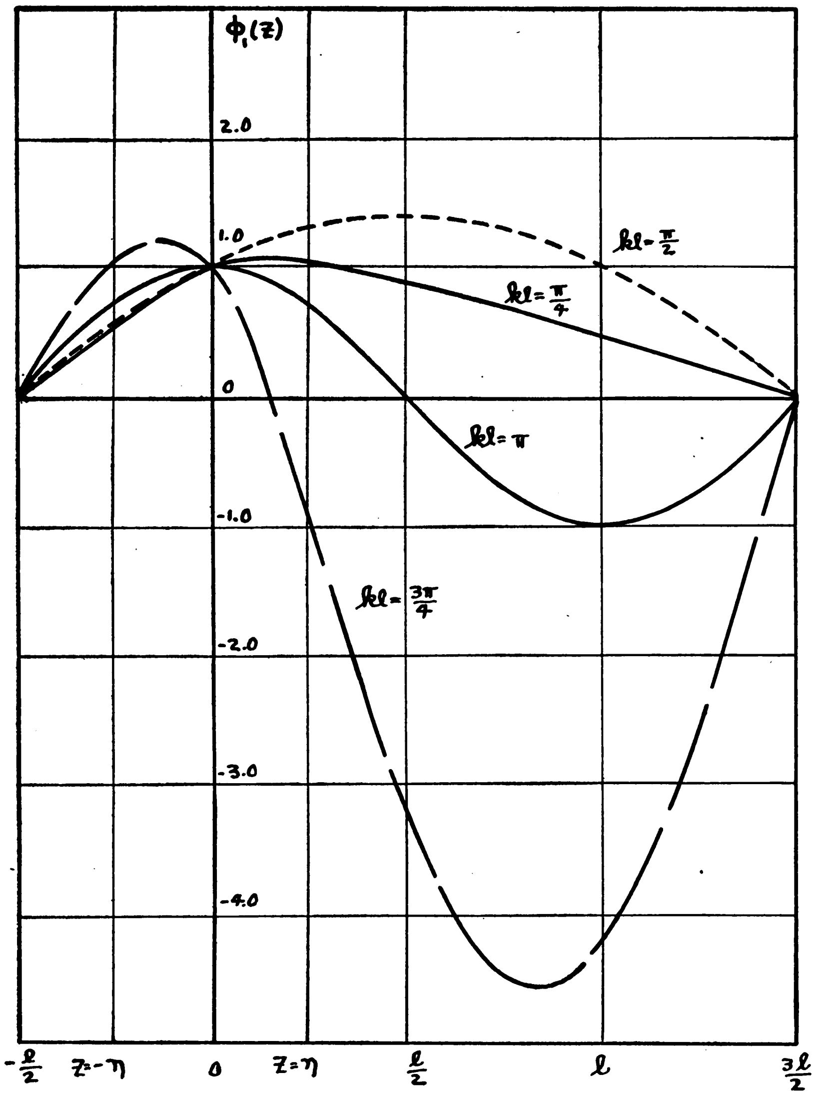

FIG. 14g. Finite gap ($\eta = l/4$) with center dividing antenna in the ratio 1:3.

The case where the antenna is approximately of total length $\frac{1}{2}\lambda$, but with the gap not at the center, is also of interest. Again we have (8.14), and substitution in (8.10) gives approximately

$$Z_1 = -30 \operatorname{cosec}^2 kl_2\left\{\Phi(2\pi) + 4i\epsilon \ln \left| \frac{l_1 l_2}{al} \right|\right\} - iX_s ,$$

$$R_1 = 30 \operatorname{cosec}^2 kl_2(\ln 2\pi\gamma - \operatorname{Ci} 2\pi) = 73.13 \operatorname{cosec}^2 kl_2 , \tag{8.16}$$

$$X_1 = 120\epsilon \operatorname{cosec}^2 kl_2 \ln \left| \frac{l_1 l_2}{al} \right| + X_s + 42.54 \operatorname{cosec}^2 kl_2 .$$

Thus, by moving the gap away from the center of the antenna, we increase the resistance, but the tuning of the antenna to make $X_1 = 0$ is more difficult, because the derivative of X_1 with respect to ϵ is greater.

We see that the problem of matching the antenna to a coaxial line, as far as reactance is concerned, depends on the shape term X_s in an important way. This term is discussed in the next section.

9. The shape term in the reactance. Let us consider the term X_s , given in (8.11). For simplicity, let us assume that the gap is at the center of the antenna and that the antenna has an equatorial plane of symmetry ($z = 0$). Then (8.11) reads

$$X_s = -60k \operatorname{cosec}^2 kl \int_0^l \cos 2k(l - z) \ln \frac{a^2}{(R(z))^2}\, dz, \tag{9.1}$$

$$2l = \text{length of antenna.}$$

We note that X_s receives no contribution from cylindrical portions of the antenna.

We can now settle the vexed question of contribution from the ends of a cylindrical antenna, by supposing the antenna to be a cylinder ($R = a$), terminated by spheroids of semi-axis b (Fig. 13, p. 138). Then on the ends

$$\frac{(R(z))^2}{a^2} + \frac{(z - l + b)^2}{b^2} = 1. \tag{9.2}$$

No assumption is made at first about the magnitude of b. Equation (9.1) gives

$$X_s = 60k \operatorname{cosec}^2 kl \int_{l-b}^{l} \cos 2k(l - z) \ln[1 - (z - l + b)^2/b^2]\, dz. \tag{9.3}$$

Since the logarithm breaks into the sum of two logarithms, this integral is easily evaluated in terms of Ci and Si functions; we find

$$X_s = 30 \operatorname{cosec}^2 kl\{\sin 4kb(\operatorname{Ci} 4kb - \operatorname{Ci} 2kb) - \cos 4kb \operatorname{Si} 4kb - (1 - \cos 4kb) \operatorname{Si} 2kb\}. \tag{9.4}$$

If kb is small, this approximates to

$$X_s = 120\, kb.\ \operatorname{cosec}^2 kl.\ (\ln 2 - 1) \text{ ohms.} \tag{9.5}$$

Since this tends to zero with kb, we see that *there is no contribution to reactance from the ends of a cylindrical antenna cut off square at the ends.* (Of course, "contribution from the ends" implies some mathematical division of reactance into "contributions" of various sorts; our statement refers to the division we have made in using the integral (9.1).) In fact, $X_s = 0$ to our order of approximation if the ends are rounded for a length b comparable to the radius a of the cylindrical part even for non-central gap.

Spheroidal antenna. The impedance of a spheroidal antenna can be calculated accurately by means of spheroidal functions.[5] However for a thin spheroid, the present method may be used. Let us take the gap at the center. Then

$$(R(z))^2/a^2 = (l^2 - z^2)/l^2, \tag{9.6}$$

and so by (9.1)

$$\begin{aligned} X_s &= 60k \operatorname{cosec}^2 kl \int_0^l \cos 2k(l - z) \ln[(l^2 - z^2)/l^2]\, dz \\ &= 30 \operatorname{cosec}^2 kl\{\sin 4kl(\mathrm{Ci}\, 4kl - \mathrm{Ci}\, 2kl) \\ &\qquad - \cos 4kl(\mathrm{Si}\, 4kl - \mathrm{Si}\, 2kl) - \mathrm{Si}\, 2kl\}. \end{aligned} \tag{9.7}$$

For $l = \lambda/4$, $kl = \frac{1}{2}\pi$, we have

$$X_s = -30\, \mathrm{Si}\, 2\pi. \tag{9.8}$$

Referring to (8.15) with $\epsilon = 0$, we see that $X_1 = 0$; *the reactance of a thin spheroidal half-wave antenna (with the gap in middle) is zero.* This fact is mentioned by Schelkunoff (loc. cit.), but the reason for this statement is not clear.

Conical antenna. For a symmetrical thin conical antenna fed at the vertex, we put

$$R(z) = \beta z \tag{9.9}$$

where β is the semi-angle of the cone. Equation (9.1) gives at once for the shape term

$$X_s = 120 \cot kl \ln \frac{\beta}{ka} + 60. \tag{9.10}$$

For the approximation to be valid, we should take β of the same order as ka.

Graphs of impedance are given in Fig. 15 (see Appendix).

10. Successive approximations. The method of Sections 7 and 8 gives the principal part of the impedance for a thin antenna, but it does not open up a method of successive approximations. To get such a method, let us return to the exact integral equation (6.3) and write it in a slightly different notation as follows:

$$-\int_{l_1}^{l_2} \frac{\partial^2 \psi(t, z)}{\partial z\, \partial t} \phi(t)\, dt + k^2 \int_{l_1}^{l_2} \psi(t, z)\phi(t)\, dt = i\pi ck^2 ZN(z), \qquad (l_1 < z < l_2). \tag{10.1}$$

Let us introduce the integration operator S such that

$$Sf(z) = \int_0^z f(t)\, dt. \tag{10.2}$$

[5] J. A. Stratton and L. J. Chu, J. Applied Physics **12**, 241-248 (1941); L. Infeld, Q. Appl. Math. **5**, 113-132 (1947).

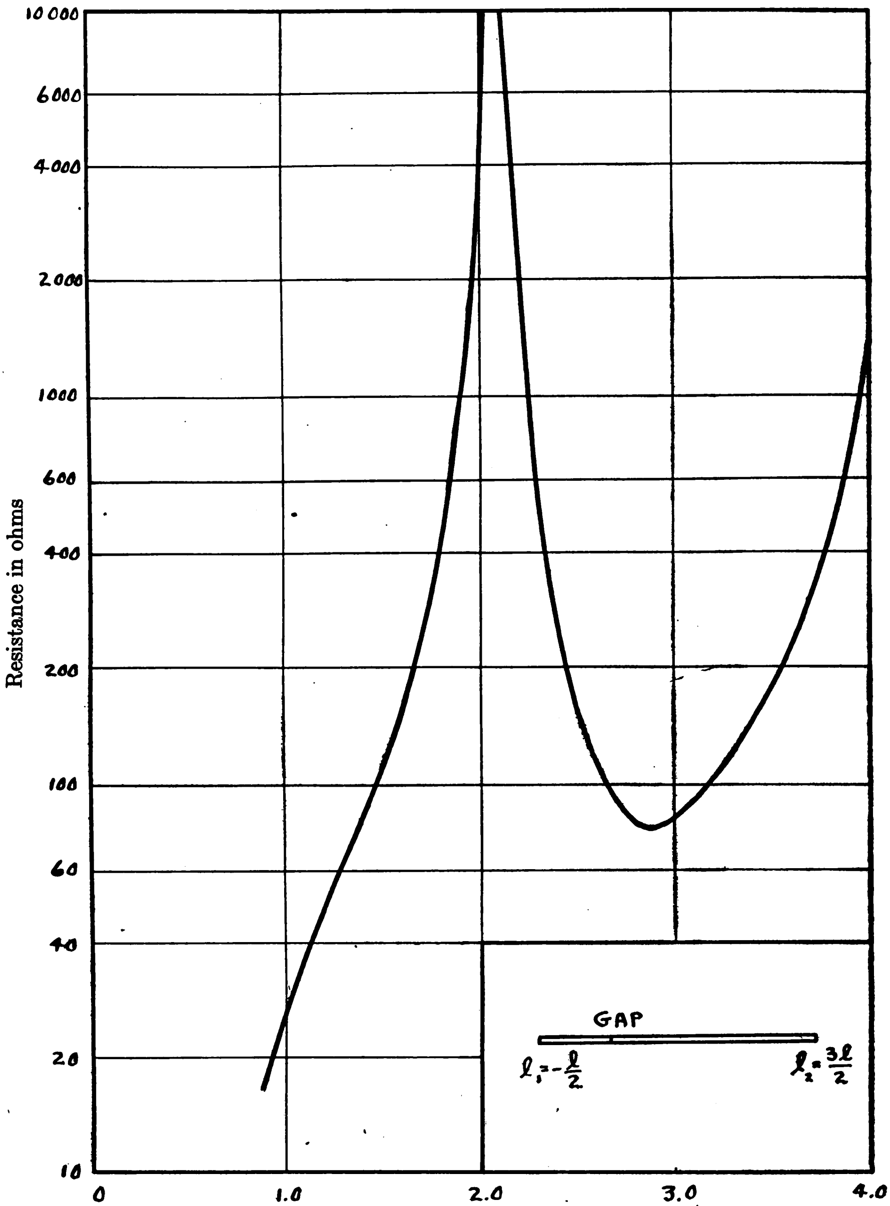

FIG. 15a. Resistance of thin antenna with infinitesimal gap at point of quadrisection.

Then

$$-S\int_{l_1}^{l_2}\frac{\partial^2\psi(t,z)}{\partial z\,\partial t}\phi(t)\,dt = -S\frac{d}{dz}\int_{l_1}^{l_2}\frac{\partial\psi(t,z)}{\partial t}\phi(t)\,dt$$

$$= -\int_{l_1}^{l_2}\frac{\partial\psi(t,z)}{\partial t}\phi(t)\,dt + \int_{l_1}^{l_2}\frac{\partial\psi(t,0)}{\partial t}\phi(t)\,dt \qquad (10.3)$$

$$= -\int_{l_1}^{l_2}\psi(t,z)\phi'(t)\,dt - A,$$

on account of (6.4). Here A is a constant,

$$A = \int_{l_1}^{l_2} \psi(t, 0)\phi'(t)\, dt. \tag{10.4}$$

Now we operate on (10.1) with S^2; this gives

$$S \int_{l_1}^{l_2} \psi(t, z)\phi'(t)\, dt - Az + k^2 S^2 \int_{l_1}^{l_2} \psi(t, z)\phi(t)\, dt = i\pi c k^2 Z S^2 N(z). \tag{10.5}$$

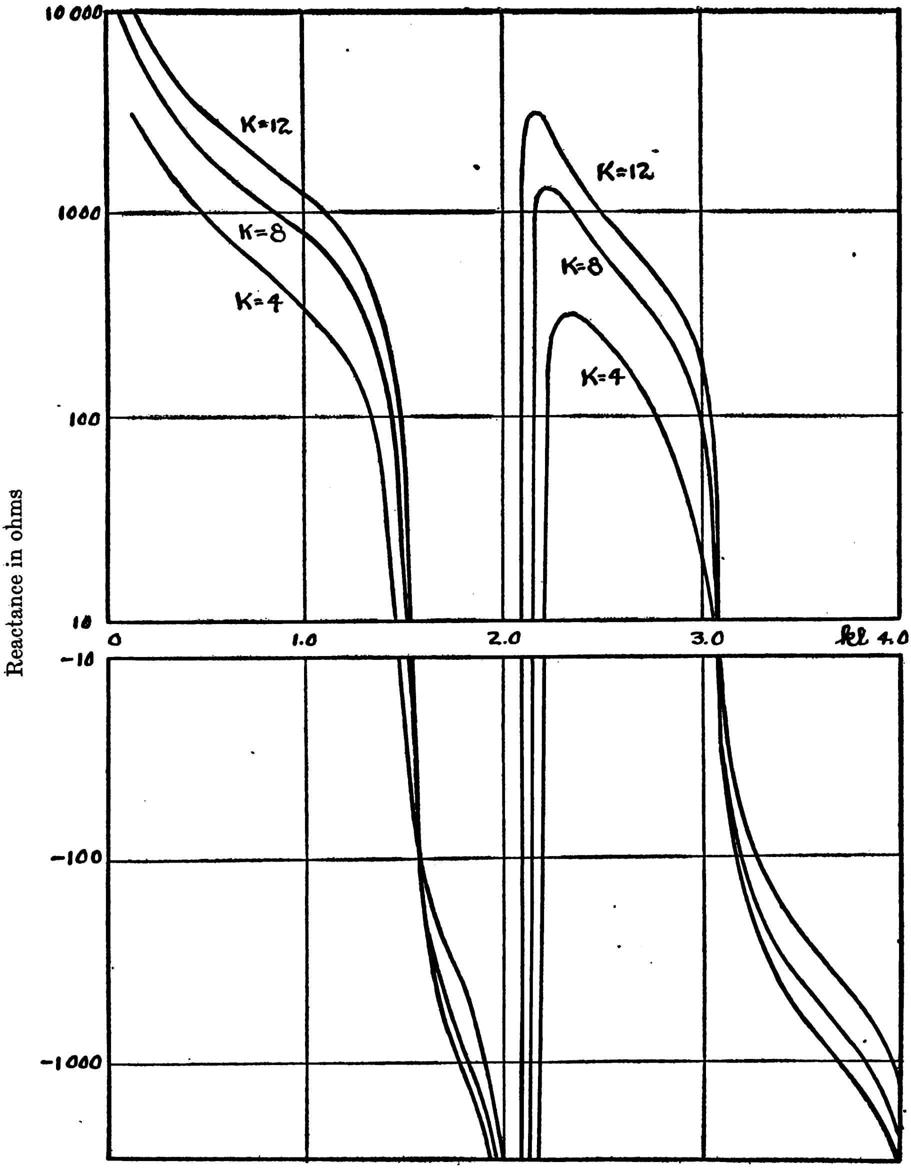

Fig. 15b. Reactance of a thin cylindrical antenna with infinitesimal gap at point of quadrisection.

To treat these integrals, consider the operator B, defined by

$$\begin{aligned} Bf(z) &= S \int_{l_1}^{l_2} \psi(t, z) f(t)\, dt \\ &= \int_{l_1}^{l_2} f(t)\, dt \int_0^z \psi(t, y)\, dy \\ &= B_1 f(z) + B_2 f(z), \end{aligned} \tag{10.6}$$

where

$$\begin{aligned} B_1 f(z) &= \int_{l_1}^{l_2} f(t)\, dt \int_0^z \frac{dy}{r(t, y)}, \qquad [r(t, y)]^2 = [R(t)]^2 + (y - t)^2, \\ B_2 f(z) &= \int_{l_1}^{l_2} f(t)\, dt \int_0^z \frac{\exp ikr(t, y) - 1}{r(t, y)}\, dy. \end{aligned} \tag{10.7}$$

Then

$$\begin{aligned} B_1 f(z) &= \int_{l_1}^{l_2} \ln \frac{z - t + \{[R(t)]^2 + (z - t)^2\}^{1/2}}{-t + \{[R(t)]^2 + t^2\}^{1/2}} f(t)\, dt \\ &= \left(\int_{l_1}^{z} + \int_{z}^{l_2}\right) \ln k[z - t + \{[R(t)]^2 + (z - t)^2\}^{1/2}] f(t)\, dt \\ &\qquad - \left(\int_{l_1}^{0} + \int_{0}^{l_2}\right) \ln k[-t + \{[R(t)]^2 + t^2\}^{1/2}] f(t)\, dt \\ &= \left(\int_{l_1}^{z} - \int_{z}^{l_2}\right) \ln k[\, | z - t | + \{[R(t)]^2 + (z - t)^2\}^{1/2}] f(t)\, dt \\ &\qquad - \left(\int_{l_1}^{0} - \int_{0}^{l_2}\right) \ln k[\, | t | + \{[R(t)]^2 + t^2\}^{1/2}] f(t)\, dt \\ &\qquad + \int_{z}^{l_2} \ln[kR(t)]^2 f(t)\, dt - \int_{0}^{l_2} \ln[kR(t)]^2 f(t)\, dt \\ &= B_3 f(z) + B_4 f(z) - \ln(k^2 a^2) S f(z), \end{aligned} \tag{10.8}$$

where

$$\begin{aligned} B_3 f(z) &= \left(\int_{l_1}^{z} - \int_{z}^{l_2}\right) \ln k[\, | z - t | + \{[R(t)]^2 + (z - t)^2\}^{1/2}] f(t)\, dt \\ &\qquad - \left(\int_{l_1}^{0} - \int_{0}^{l_2}\right) \ln k[\, | t | + \{[R(t)]^2 + t^2\}^{1/2}] f(t)\, dt, \\ B_4 f(z) &= \int_0^z \ln[a/R(t)]^2 f(t)\, dt. \end{aligned} \tag{10.9}$$

In the notation of (10.6), (10.5) reads

$$B\phi'(z) - Az + k^2 SB\phi(z) = i\pi c k^2 Z S^2 N(z), \tag{10.10}$$

or, since $S\phi'(z) = \phi(z) - 1$,

$$L[\phi(z) - 1 + k^2S^2\phi(z)] + F\phi(z) - Az = i\pi ck^2ZS^2N(z), \tag{10.11}$$

where $L = -\ln k^2a^2$, and F is the operator

$$F = (B_2 + B_3 + B_4)D + k^2S(B_2 + B_3 + B_4). \tag{10.12}$$

Here D is the derivative symbol. Equation (10.11) gives

$$(1 + k^2S^2)\phi(z) = 1 + L^{-1}Az - L^{-1}F\phi(z) + L^{-1}i\pi ck^2ZS^2N(z). \tag{10.13}$$

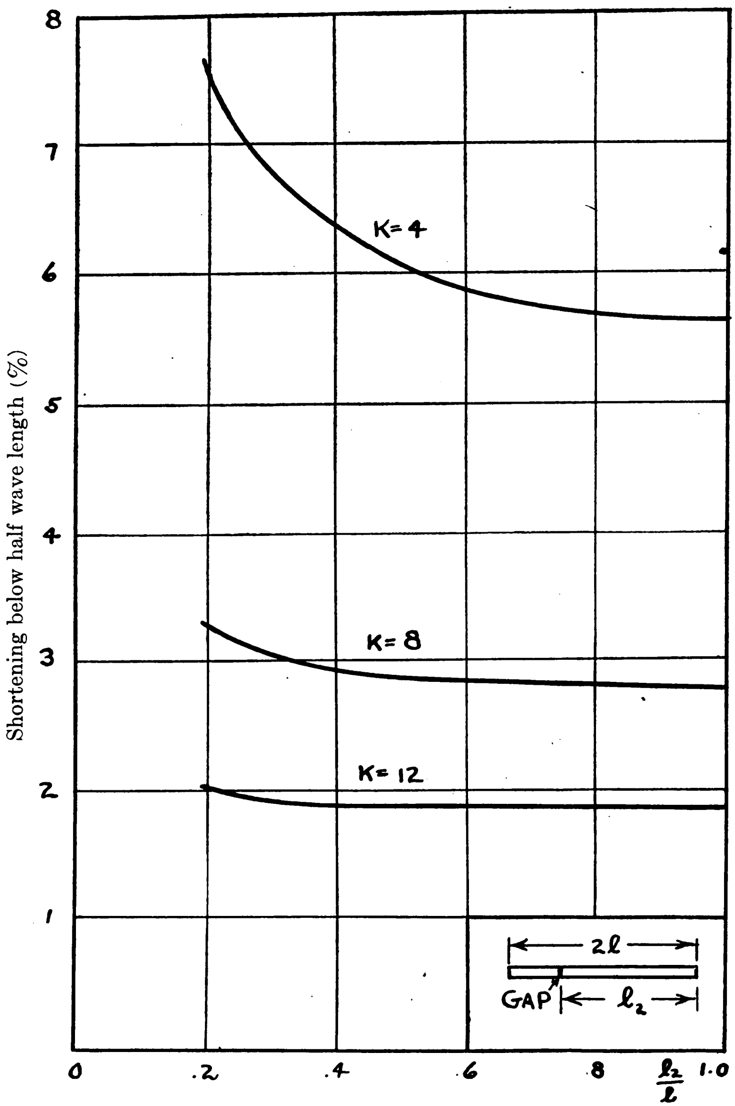

FIG. 15c. Resonant length of a thin cylindrical antenna with flat ends and infinitesimal gap at different points.

Now the operator P inverse to $1 + k^2S^2$ is easily found. It is

$$(1 + k^2S^2)^{-1}f(z) = Pf(z) = f(z) - k\int_0^z \sin k(z - t)f(t)\,dt. \tag{10.14}$$

This means that for an arbitrary function $f(z)$ we have $P(1 + k^2S^2)f(z) = (1 + k^2S^2)Pf(z) = f(z)$; this is easily verified.

In particular, we have

$$P.1 = \cos kz, \qquad Pz = k^{-1}\sin kz. \tag{10.15}$$

Thus (10.13) gives

$$\phi(z) = \cos kz + L^{-1}Ak^{-1}\sin kz - L^{-1}PF\phi(z) + L^{-1}i\pi ck^2ZPS^2N(z). \tag{10.16}$$

This is a *transform of the basic integral equation* (6.3). It is exact because in deriving it, we have not actually used the approximation (5.9). With (10.16) we are to associate the boundary conditions (6.4). The first of these is satisfied automatically. The others give

$$\begin{aligned}\cos kl_1 + L^{-1}Ak^{-1}\sin kl_1 - L^{-1}T_{l_1}PF\phi(z) + L^{-1}i\pi ck^2ZT_{l_1}PS^2N(z) &= 0,\\ \cos kl_2 + L^{-1}Ak^{-1}\sin kl_2 - L^{-1}T_{l_2}PF\phi(z) + L^{-1}i\pi ck^2ZT_{l_2}PS^2N(z) &= 0.\end{aligned} \tag{10.17}$$

Here T_{l_1}, T_{l_2} are substitution operators, meaning "put $z = l_1$, or $z = l_2$, finally." We now eliminate A and Z from (10.16), (10.17); this gives for $\phi(z)$ the equation

$$\begin{vmatrix} \phi(z) - \cos kz + L^{-1}PF\phi(z) & \sin kz & PS^2N(z) \\ -\cos kl_1 + L^{-1}T_{l_1}PF\phi(z) & \sin kl_1 & T_{l_1}PS^2N(z) \\ -\cos kl_2 + L^{-1}T_{l_2}PF\phi(z) & \sin kl_2 & T_{l_2}PS^2N(z) \end{vmatrix} = 0. \tag{10.18}$$

Remembering that $N(z)$ is known, the plan of solving by successive approximations is now obvious. We are to substitute $\phi = \phi_1$ in the PF column, ϕ_1 being some initial approximation, and solve, obtaining ϕ_2. Then ϕ_2 is to be substituted in the PF column, and the equation solved, giving ϕ_3; and so on. At any stage, we might get Z from (10.17) but it seems probable that a better value will be given by (6.23) or (6.25).

Under the assumptions (7.1), (7.3) we have (5.9), and hence we can calculate $PS^2N(z)$. In general

$$\begin{aligned} Pf(z) &= f(z) - k\int_0^z f(t)\sin k(z - t)\,dt \\ &= f(0)\cos kz + \int_0^z Df(t)\cdot\cos k(z - t)\,dt \\ &= f(0)\cos kz + k^{-1}f'(0)\sin kz + k^{-1}\int_0^z D^2f(t)\cdot\sin k(z - t)\,dt. \end{aligned} \tag{10.19}$$

Hence

$$PS^2N(z) = k^{-1}\int_0^z N(t)\sin k(z - t)\,dt, \tag{10.20}$$

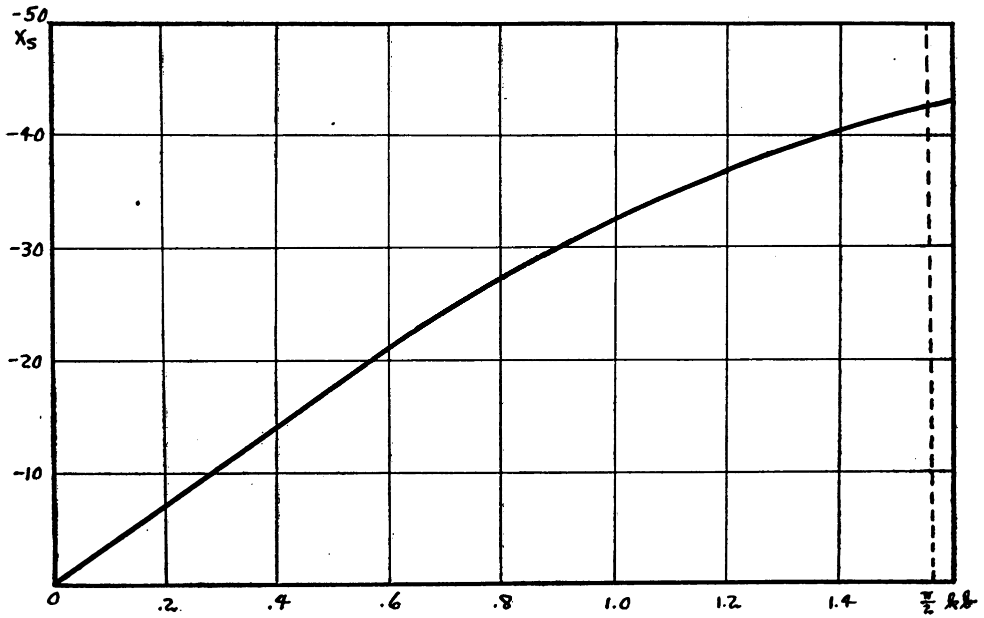

FIG. 15d. Shape term in the reactance of a thin cylindrical half wave antenna terminated by spheroids of semi-axis b (center gap).

and substituting from (5.9) we have

$$l_1 \leq z \leq -\eta : PS^2N(z) = -2k^{-3}\eta^{-1}[\cos k(z+\eta) - \cos kz], \qquad (10.21a)$$

$$-\eta \leq z \leq \eta : PS^2N(z) = -2k^{-3}\eta^{-1}(1 - \cos kz), \qquad (10.21b)$$

$$\eta \leq z \leq l_2 : PS^2N(z) = -2k^{-3}\eta^{-1}[\cos k(z-\eta) - \cos kz]. \qquad (10.21c)$$

It is natural to take as first approximation $\phi_1(z)$, the function obtained by deleting the PF terms from (10.18). This $\phi_1(z)$ is precisely that given by (7.10). Then the second approximation $\phi_2(z)$ will be given by

$$\begin{vmatrix} \phi_2(z) - \cos kz + L^{-1}PF\phi_1(z) & \sin kz & PS^2N(z) \\ -\cos kl_1 + L^{-1}T_{l_1}PF\phi_1(z) & \sin kl_1 & T_{l_1}PS^2N(z) \\ -\cos kl_2 + L^{-1}T_{l_2}PF\phi_1(z) & \sin kl_2 & T_{l_2}PS^2N(z) \end{vmatrix} = 0, \qquad (10.22)$$

and the higher approximations by similar formulae.

APPENDIX: NOTES ON FIGURES 14 AND 15.

Fig. 14. Relative current in a thin antenna. The following graphs are drawn from equations (7.10), which were obtained on the following assumptions:

(*i*) the tangential electric field is constant over the gap;
(*ii*) the antenna is very thin, but not necessarily cylindrical with flat ends;
(*iii*) when the gap is infinitesimal (as in Figs. 14 a, d, f) its length is still much greater than the infinitesimal radius of the antenna.

The ordinate $\phi_1(z)$ in each graph is the principal part of the relative current, $I(z)/I(0)$. The abscissa z represents position on the antenna. The ends of the antenna are $z = l_1$, $z = l_2$, and the ends of the gap are $z = -\eta$, $z = \eta$, so that $z = 0$ is the center of the gap. Also

$$k = 2\pi/\lambda, \qquad \lambda = \text{wave length},$$

$$2l = l_2 - l_1 = \text{total length of antenna}.$$

In each figure graphs are drawn for various values of kl up to $kl = \pi$.

The following points are of interest:

1) When the gap is finite, the derivative of $\phi_1(z)$ is continuous; when the gap is infinitesimal, the derivative is discontinuous at the gap, unless $kl = \frac{1}{2}\pi$ or π.
2) The curve for $kl = \frac{1}{2}\pi$ (that is, $2l = \frac{1}{2}\lambda$) is always a single sine curve, whether the gap is finite or infinitesimal.
3) Certain curves are not shown, because they go to infinity. This means that $I(0) = 0$ and the impedance is infinite (to this order of approximation). This occurs for $kl = \pi$ (that is, $2l = \lambda$) in Fig. 14a. This infinity disappears when the gap is widened in Fig. 14b. This does not mean that the widening of the gap eliminates infinite impedance; in Fig. 14b there is infinite impedance for $kl = 8\pi/7$, since this makes $K(\eta)$ vanish in (7.11). However, when the gap is finite, it is questionable whether the impedance is correctly defined (for matching purposes) by $Z = V/I(0)$.

   For infinitesimal eccentric gaps, we get infinite impedance for $kl = 4\pi/5$ in Fig. 14d and $kl = 2\pi/3$ in Fig. 14f.
4) Fig. 14c has a remarkable feature: the current is constant in the gap for $kl = \pi$.

Fig. 15. Impedance of a thin antenna. These graphs are based on equations (8.10) and (9.1).

Fig. 15a shows the resistance of a thin antenna plotted against kl. The gap is infinitesimal and situated at the point of quadrisection. It is interesting to compare this curve with those given by King[6] and Schelkunoff (loc. cit.) for an antenna with central gap. The effect of moving the gap from the center to the point of quadrisection is to change the point of great or infinite resistance from $kl = \pi$ to $kl = 2\pi/3$. This comes from the factor $\sin kl_2$ in the denominator in (8.10). Further, the resistance of a half-wave antenna ($kl = \frac{1}{2}\pi$) is changed by this shift of gap from about 70 to about 140 ohms.

Fig. 15b shows the reactance of a thin cylindrical antenna plotted against kl. As in the case of Fig. 15a, the gap is infinitesimal and at the point of quadrisection. Graphs are plotted for several values of the thickness parameter $K = \ln (l/a)$, where $2l$ is the length of the antenna and a its radius. The reactance vanishes not only in the neighbourhood of $kl = \frac{1}{2}\pi$ and $kl = \pi$, but also in the neighbourhood of $kl = 2\pi/3 = 2.09$.

The resonant length of a thin cylindrical antenna (i.e. the length making the reactance vanish) is a little less than half a wave length. The shortening below the half wave length is a function of the position of the gap. This dependence is shown graphically in Fig. 15c.

Fig. 15d shows the effect of rounding the ends of a cylindrical antenna, as in Fig. 13. The antenna is half a wave length long and the gap is infinitesimal and at the center. Flat ends correspond to $kb = 0$ and a completely spheroidal antenna to $kb = \frac{1}{2}\pi$.

---

[6] L. V. King, Phil. Trans. Roy. Soc. (A) **236**, 381-422 (1937).