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ENERGY DECAY AND SELF-PRESERVING CORRELATION FUNCTIONS IN ISOTROPIC TURBULENCE*

BY

G. K. BATCHELOR

Trinity College, Cambridge

1. Introduction. We consider those properties of turbulence in fluid of zero mean motion which can be deduced from the assumptions of spatial homogeneity and isotropy. The i-components of the velocity fluctuations at two points $P(\mathbf{x})$ and $P'(\mathbf{x}')$ will be written as u_i and u_i'. The spatial separation of P and P' is denoted by the vector

$$\xi = \mathbf{x}' - \mathbf{x}$$

of magnitude r, where

$$\xi_i \xi_i = r^2.$$

and repeated indices imply summation over the values 1, 2 and 3.

It may be shown without difficulty[1,2] that the condition of isotropy requires the correlation between u_i and u_j', where i and j have arbitrary values, to depend only on the geometrical configuration defined by ξ and unit vectors in the i- and j-directions, and a single scalar function of r^2. It can be represented as the typical component of a tensor of the second rank in the following manner:

$$\overline{u_i u_j'} = R_i^j = u'^2\left[-\frac{1}{2r}\frac{\partial f}{\partial r}\xi_i\xi_j + \left(f + \frac{1}{2}r\frac{\partial f}{\partial r}\right)\delta_{ij}\right] \tag{1.1}$$

where the over-bar denotes a spatial mean value, $f(r)$ is the scalar function, and

$$u'^2 = \overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2};$$

δ_{ij} is the unit tensor whose value is 1 if $i = j$, and 0 otherwise. The triple correlation between velocity components at P and P' has a similar dependence on a single scalar function;

$$\overline{u_i u_j u_k'} = T_{ij}^k$$

$$= u'^3\left[\left(\frac{k - r\,\partial k/\partial r}{2r^3}\right)\xi_i\xi_j\xi_k + \left(\frac{k + \frac{1}{2}r\,\partial k/\partial r}{2r}\right)(\delta_{ik}\xi_j + \delta_{kj}\xi_i) - \frac{k}{2r}\delta_{ij}\xi_k\right]. \tag{1.2}$$

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[1]T. v. Kármán and L. Howarth, *On the statistical theory of isotropic turbulence*, Proc. Roy. Soc. (A) **164**, 192-215 (1938).

[2]H. P. Robertson, *The invariant theory of isotropic turbulence*, Proc. Camb. Phil. Soc. **36**, 209-223 (1940).

The scalar function $k(r)$ is in this case an odd function of r, of order r^3 when r is small. It is clear from (1.1) and (1.2) that $f(r)$ and $k(r)$ are the correlation coefficients for particular values of the indices and the ξ_i, viz.

$$u'^2 \cdot f(r) = (\overline{u_i u'_i})_{\substack{i=j \\ \xi_i = r}},$$

$$u'^3 \cdot k(r) = (\overline{u_i u_j u'_k})_{\substack{i=j=k \\ \xi_i = r}}.$$

All higher order correlations which involve at least one velocity component at each of the points P and P' depend on more than one scalar function of r.

The equations of motion of a viscous incompressible fluid contain both linear and quadratic terms in the velocity components, and it is consequently possible to relate the double and triple velocity correlations. Expressed in terms of the scalar functions $f(r)$ and $k(r)$, this relation becomes the equation for the propagation of the double velocity correlation and has the form

$$\frac{\partial(u'^2 f)}{\partial t} = u'^3\left(k' + \frac{4}{r}k\right) + 2\nu u'^2\left(f'' + \frac{4}{r}f'\right), \tag{1.3}$$

where dashes to f and k denote differentiation with respect to r, and t is the time. Equation (1.3) has a simple and useful form for the particular value $r = 0$, when it describes the rate of decay of energy of the turbulence;

$$\frac{du'^2}{dt} = 10\nu u'^2 (f'')_{r=0} = -\frac{10\nu u'^2}{\lambda^2}. \tag{1.4}$$

where λ is the length parameter previously introduced by Taylor.[3]

Since the difficulties of measurement become very great as the order of the correlation increases, it is inevitable that Eq. (1.3) should occupy an important place in any practical theories of turbulence. The purpose of this paper is to extract as much information from it as is possible with a minimum of further assumptions, and in particular to deduce the rates of energy decay which are consistent with certain types of behaviour of the function $f(r)$. It is important to appreciate that Eq. (1.3) represents all the information about the function $f(r)$ which we possess, and is clearly insufficient to permit a complete solution to be obtained. This lack of information for the determination of correlation functions is inherent in averaged equations, and is the penalty paid for this use of the statistical method. Our plan is therefore (a) to discuss suitable limiting cases for which a solution of (1.3) is possible, and (b) to discuss the consequences of simple hypotheses so that their validity can be put to the test of experiment. Types of solution for which the function f is "self-preserving", i.e. for which f is a function only of r/L, where L is a length which depends on the time t, were introduced by v. Kármán and Howarth.[1] The consequent gain in simplicity of the mathematics is considerable and it is with such solutions that we shall largely be concerned here. There is available sufficient experimental evidence to indicate that the theoretical solutions correspond in some respects with reality and are worth pursuing.

2. Loitsiansky's invariant. We shall have need later of a simple deduction from the

[3] G. I. Taylor, *Statistical theory of turbulence*, Proc. Roy. Soc. (A) **151**, 421-478 (1935).

basic Eq. (1.3), which was pointed out first by Loitsiansky.[4] Multiplying both sides of the equation by r^4 and integrating over the range 0 to ∞, we find

$$\frac{\partial}{\partial t}\left(u'^2 \cdot \int_0^\infty r^4 f\, dr\right) = 0$$

provided that $r^4 f' \to 0$ and $r^4 k \to 0$ as $r \to \infty$. Physically it seems reasonable to suppose that the convergence conditions are satisfied; then

$$u'^2 \cdot \int_0^\infty r^4 f\, dr = \Lambda, \tag{2.1}$$

where Λ is a constant during the decay process.

Loitsiansky remarks that the relation (2.1) and Eq. (1.3) are consistent with an analogy between the propagation of fu'^2, and the propagation of heat in a spherically-symmetrical five-dimensional field. In such a space the Laplacian operator has the form

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r},$$

so that the last term of Eq. (1.3) represents the effect of molecular conduction provided that fu'^2 is the analogue of temperature. The first term on the right side of (1.3) must be interpreted as the effect of convection, and (2.1) shows that this convective effect is such as to leave the total quantity of heat constant. On the basis of this analogy Loitsiansky describes the constant Λ as a measure of the total quantity of disturbance to the fluid, which is uniquely determined by the initial conditions of the turbulence.

It is not without interest to notice the behaviour of the general moment of the function $f(r)$. Thus

$$\frac{\partial}{\partial t}\left(u'^2 \cdot \int_0^\infty r^m f\, dr\right) = (4 - m)u'^3 \int_0^\infty k r^{m-1}\, dr + 2(m - 1)(m - 4)\nu u'^2 \int_0^\infty r^{m-2} f\, dr \tag{2.2}$$

provided that $m > 1$ and $(r^{m-1}f)_\infty = (r^m k)_\infty = 0$. When $m = 1$ or 0, special formulae are necessary, viz.

$$\frac{\partial}{\partial t}\left(u'^2 \cdot \int_0^\infty rf\, dr\right) = 3u'^3 \cdot \int_0^\infty k\, dr - 6\nu u'^2, \tag{2.3}$$

$$\frac{\partial}{\partial t}\left(u'^2 \cdot \int_0^\infty f\, dr\right) = 4u'^3 \cdot \int_0^\infty \frac{k}{r}\, dr + 8\nu u'^2 \int_0^\infty \frac{f'}{r}\, dr. \tag{2.4}$$

Now all the experimental evidence suggests that the function $f(r)$ is everywhere positive and monotonic decreasing, and the signs of the integrals containing f and f' can be predicted with safety. There is not very much data about $k(r)$ but measurements at the Cavendish Laboratory show it to be everywhere negative (or zero). It must certainly be negative for small values of r in order to give a positive contribution to the rate of change of mean square vorticity due to random extension of the vortex lines.[5] If the

[4] L. G. Loitsiansky, *Some basic laws of isotropic turbulent flow*, Central Aero- and Hydro-dynamic Institute, Moscow, Report No. 440, 1939; also N. A. C. A. Tech. Memo. 1079.

[5] G. K. Batchelor and A. A. Townsend, *Decay of vorticity in isotropic turbulence*, Proc. Roy. Soc. (A) **190**, 534-550 (1947).

signs of the integrals involving k are predicted on this basis, Eqs. (2.2), (2.3) and (2.4) show that

$$\left.\begin{aligned} \frac{\partial}{\partial t}\left(u'^2\cdot\int_0^\infty r^m f\,dr\right) &> 0 \qquad \text{if} \qquad m > 4,\\ &= 0 \qquad \text{if} \qquad m = 4,\\ &< 0 \qquad \text{if} \qquad m = 3, 2, 1, 0. \end{aligned}\right\} \tag{2.5}$$

These results have a bearing on the behaviour, during decay, of the spectrum function describing the spatial structure of the energy of the turbulence field. For if $u'^2\cdot F(\mu)\,d\mu$ is the amount of energy lying within a small range about the wave number μ the functions $f(r)$ and $F(\mu)/2(2\pi)^{1/2}$ are Fourier transforms,[6] and

$$F(\mu) = 4\int_0^\infty \cos 2\pi\mu r\cdot f(r)\,dr. \tag{2.6}$$

The relations (2.5) thus become, when m is even,

$$\frac{\partial}{\partial t}\left[u'^2\cdot\left(\frac{\partial^m F}{\partial\mu^m}\right)_{\mu=0}\right] \gtreqless 0 \qquad \text{if} \qquad m \gtreqless 4. \tag{2.7}$$

If the energy spectrum function is expanded in powers of μ^2, viz.

$$u'^2\cdot F(\mu) = u'^2\cdot F(0) + \frac{\mu^2}{2!}\left(\frac{\partial^2 u'^2 F}{\partial\mu^2}\right)_0 + \frac{\mu^4}{4!}\left(\frac{\partial^4 u'^2 F}{\partial\mu^4}\right)_0 + \cdots,$$

then the effect of decay is to decrease the coefficients of μ'^0 and μ^2, to leave the coefficient of μ^4 constant and to increase all other coefficients.

3. Self-preserving solutions at Reynolds number which are not large. By Reynolds number which are not large is meant a state of affairs in which λ is not small compared with other lengths associated with the function $f(r)$. Since the length λ is already present in the basic Eq. (1.3) (in view of the expression for du'^2/dt), it will be convenient mathematically to use λ as the scale factor L in any solution for $f(r)$ which preserves its shape over a range of r which includes small values of r. Several possibilities can be considered, the simplest of which is that the correlation functions preserve their shape for *all* values of r.

The hypothesis whose consequences are to be examined is that

$$f(r) \equiv f(\psi), \qquad k(r) \equiv k(\psi) \tag{3.1}$$

for all values of r, where $\psi = r/\lambda$. Equation (1.3) then becomes

$$-5f - \frac{\lambda}{2\nu}\frac{d\lambda}{dt}\psi\frac{df}{d\psi} = \frac{1}{2}\frac{u'\lambda}{\nu}\left(\frac{dk}{d\psi} + \frac{4k}{\psi}\right) + \left(\frac{d^2f}{d\psi^2} + \frac{4}{\psi}\frac{df}{d\psi}\right)$$

or, in terms of the number $R_\lambda = u'\lambda/\nu$,

$$\left(\frac{d^2f}{d\psi^2} + \frac{4}{\psi}\frac{df}{d\psi} + \frac{5\psi}{2}\frac{df}{d\psi} + 5f\right) + \frac{\lambda^2}{2\nu R_\lambda}\frac{dR_\lambda}{dt}\left(\psi\frac{df}{d\psi}\right) + \frac{1}{2}R_\lambda\left(\frac{dk}{d\psi} + \frac{4k}{\psi}\right) = 0. \tag{3.2}$$

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[6] G. I. Taylor, *The spectrum of turbulence*, Proc. Roy. Soc. (A) **164**, 476-490 (1937).

Each of the expressions within circular brackets is a function of ψ only, while the coefficients outside the brackets depend only on t.

It is worth noting that the second coefficient, viz. $\lambda^2\, dR_\lambda/2\nu R_\lambda\, dt$, is a constant for a very general class of energy decay law. If u'^2 decays as some power of t, the energy equation (1.4) shows that the decay laws will be

$$u'^{-2} \sim t^n, \qquad \lambda^2 \doteq \frac{10\nu}{n}\, t, \qquad R_\lambda^2 \sim t^{1-n} \tag{3.3}$$

and hence

$$\frac{\lambda^2}{2\nu R_\lambda}\frac{dR_\lambda}{dt} = \frac{5(1-n)}{2n}. \tag{3.4}$$

An exponential decay of u'^2 also makes this factor constant. When the second coefficient is constant, the first two groups of terms in (3.2) are functions of ψ only and the equation can only be satisfied by

$$R_\lambda = \text{constant}.$$

This leads to $n = 1$ provided the constant value of R_λ is not zero (in which case we should have $n > 1$, $t = \infty$). When the energy decay law is not such as to make $\lambda^2\, dR_\lambda/2\nu R_\lambda\, dt$ constant, there is another way in which Eq. (3.2) can be satisfied. This alternative method is suggested by the work of Sedov,[7] and will be discussed in section 5.

Now the law of energy decay is already fully determined by the assumption of self-preservation of the correlation function f for *all* values of r. For in this case the condition (2.1) becomes

$$u'^2\lambda^5 \int_0^\infty \psi^4 f(\psi)\, d\psi = \Lambda$$

provided that the integral converges, and thence $u'^2\lambda^5$ is constant during the decay. The energy equation then gives the decay laws as

$$u'^{-2} = At^{5/2}, \qquad \lambda^2 = 4\nu t, \qquad R_\lambda^2 = \frac{4}{A\nu}\, t^{-3/2}, \tag{3.5}$$

where A is a constant and t is measured from the instant at which $1/u' = \lambda = 0$. This is a power law of energy decay (with $n = 5/2$) and as shown above, Eq. (3.2) can only be satisfied when R_λ is constant. Since R_λ also varies with t according to (3.5), the two requirements can only be consistent if

$$t = \infty, \qquad R_\lambda = 0.$$

Under these circumstances Eq. (3.2) becomes

$$\frac{d^2 f}{d\psi^2} + \frac{df}{d\psi}\left(\frac{4}{\psi} + \psi\right) + 5f = 0 \tag{3.6}$$

and the solution which makes $f = 1$ when $r = 0$ is

$$f(\psi) = e^{-\psi^2/2}. \tag{3.7}$$

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[7] L. I. Sedov, *Decay of isotropic turbulent motions of an incompressible fluid*, C. R. Acad. Sci. U. R. S. S. **42**, 116-119 (1944).

The requirement $R_\lambda = 0$ shows that the triple correlations have no influence on the double correlations under the conditions for which a completely self-preserving solution is possible.

Thus it has been shown that a solution in which the function f is completely self-preserving is only possible at large decay times and is described by (3.5) and (3.7). This suggests that we should examine conditions as $t \to \infty$ in order to see if such a self-preserving solution does in fact exist there. It is not difficult to see at once that the answer is likely to be in the affirmative. The correlation function $f(\psi)$ certainly has the same (parabolic) form for all values of t for $\psi \ll 1$, and the only alternative to an approach to a definite shape as $t \to \infty$ is an oscillation of the remaining part of the curve. Such an oscillation does not seem appropriate to the problem.

However, more definite evidence that $f(\psi)$ is independent of t when t is large can be obtained from the basic equation. For consider (2.3) in the form

$$\frac{\partial}{\partial t}\left(R_\lambda^2 \int_0^\infty \frac{r}{\lambda} f \, d\frac{r}{\lambda}\right) = \frac{3u'^2}{\nu} R_\lambda \int_0^\infty k \, d\frac{r}{\lambda} - 6\frac{u'^2}{\nu}.$$

All the measurements of $f(r)$ which have hitherto been made have shown it to be everywhere positive and we may safely assume the expression within brackets on the left side to be positive and, in view of the existence of a parabolic variation for $r/\lambda \ll 1$, non-zero, for all values of t. Also, as discussed in section 2, the evidence is that k is everywhere negative (or zero), so that

$$\frac{\partial}{\partial t}\left(R_\lambda^2 \int_0^\infty \frac{r}{\lambda} f \, d\frac{r}{\lambda}\right) \leq -6\frac{u'^2}{\nu} \tag{3.8}$$

Suppose now that when t is large, the decay of u' and λ conforms to the general laws (3.3). In the first place, n cannot be less than or equal to one, for (3.8) then requires $R_\lambda^2 \int_0^\infty r/\lambda \, f \, d \, r/\lambda$ to decrease indefinitely and to become negative as $t \to \infty$. Secondly, the inequality (3.8) requires the expression in brackets on the left side to vary as some power of t not greater than $1 - n$, whereas the factors in this expression show that it varies as some power of t not less than $1 - n$. Hence the power can only be $1 - n$ and $\int_0^\infty r/\lambda \, f \, d \, r/\lambda$ tends to a constant when t is large, which shows that f tends to become a function only of r/λ. Notice that the assumption $k \leq 0$ is over-sufficient; the deduction is only rendered invalid if $\int_0^\infty k \, d \, r/\lambda$ is positive and increases as some power of t not less than $(n - 1)/2$.

It is thus possible to make quite definite predictions about the turbulence when t is large. Making assumptions about f and k which are well supported by experiment, and assuming that the energy decay follows a power law (3.3), it can be shown that $f(r)$ preserves its shape when t is large. Then Loitsiansky's invariant relation shows that the decay laws must be as in (3.5). Finally the fundamental equation for f gives the solution (3.7). The dependence of these deductions on a decay law of the type specified by (3.3) when t is large does not seem likely to be critical.

4. Solutions obtained by neglecting the triple correlation. Two Russian authors, Loitsiansky[4] and Millionshtchikov,[8] have each discussed the solutions of Eq. (1.3) which are obtained when the term describing the effect of the triple velocity correlation

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[8]M. Millionshtchikov, *Decay of homogeneous isotropic turbulence in a viscous incompressible fluid*, C. R. Acad. Sci. U. R. S. S. 22, 231-237 (1939).

is ignored. Their work is an extension of the "small Reynolds number" solution first put forward by v. Kármán and Howarth.[1] There is considerable indirect evidence for the belief that neglect of the triple correlations is only permissible at a late stage in the decay of turbulence so that the resulting solutions ought to be compared with that deduced in the previous section for t large.

The equation to be solved is

$$\frac{\partial u'^2 f}{\partial t} = 2\nu\left(\frac{\partial^2 u'^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial u'^2 f}{\partial r}\right), \tag{4.1}$$

which is similar to the equation for the propagation of heat in a spherically-symmetrical five-dimensional field in which there is no convection. Using this analogy the solution is known to be

$$u'^2 f = \frac{1}{(8\pi\nu t)^{5/2}} \iiiint\!\!\int_{-\infty}^{\infty} F(s, 0) e^{-\rho^2/8\nu t}\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5 \tag{4.2}$$

where

$$F(r, t) = u'^2 f,$$

$$\rho^2 = \sum_{n=1}^{5} (\xi_n - x_n)^2, \qquad r^2 = \sum_1^5 \xi_n^2, \qquad s^2 = \sum_1^5 x_n^2.$$

The law of energy decay is obtained by putting $r = 0$ in (4.2), i.e.

$$u'^2 = \frac{1}{(8\pi\nu t)^{5/2}} \iiiint\!\!\int_{-\infty}^{\infty} F(s, 0)\, e^{-s^2/8\nu t}\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5$$

$$= \frac{1}{48(2\pi)^{1/2}(\nu t)^{5/2}} \int_0^\infty F(s, 0)\, e^{-s^2/8\nu t} s^4\, ds. \tag{4.3}$$

Thus, in general, the solution depends upon, and is uniquely determined by, the function $F(r, 0)$.

If the initial state of the turbulence is such that

$$\left.\begin{aligned} F(r, 0) &= 0, \quad \text{when} \quad r > 0, \\ &= \infty, \quad \text{when} \quad r = 0, \\ \int_0^\infty F(r, 0) r^4\, dr &= \Lambda \end{aligned}\right\} \tag{4.4}$$

where Λ is finite, then the integrals can be evaluated giving

$$u'^2 = \frac{\Lambda}{48(2\pi)^{1/2}} (\nu t)^{-5/2}, \tag{4.5}$$

$$f(r, t) = e^{-r^2/8\nu t} = e^{-r^2/2\lambda^2}. \tag{4.6}$$

Loitsiansky describes these initial conditions as referring to a "point source of strength Λ" since the analogous problem in the five-dimensional field is simply the spread of heat from an initial point source. Millionshtchikov has also obtained this special solution and remarks that it describes the turbulence which exists subsequent to an initial random distribution of concentrated line eddies, provided that the effect of triple correlations is ignored. Since the triple velocity correlations cannot be neglected in the early stages of the decay when u' is not small compared with the characteristic velocity of the turbulence-producing device, Millionshtchikov's interpretation of the initial conditions (4.4) cannot be regarded as having physical reality.

v. Kármán and Howarth considered the particular set of solutions of (4.1) which are functions of $r/(\nu t)^{1/2}$ only, i.e. a self-preserving solution was assumed. The solution (4.6) is the only solution of this kind if certain conditions concerning the behaviour of $f(r)$ are accepted. v. Kármán and Howarth gave a family of self-preserving solutions of (4.1) with the quantity α, $= \nu t/\lambda^2$, as parameter, viz.

$$f(x) = 2^{15/4}\chi^{-5/2}e^{-\chi^2/16}M_{10\alpha-5/4,3/4}(\chi^2/8), \tag{4.7}$$

where $\chi = r/(\nu t)^{1/2}$, and $M_{k,m}(z)$ is the same solution of the confluent hypergeometric equation as that defined by Whittaker and Watson (*Modern analysis*, 1927, p. 337) and denoted by this symbol. The solution (4.6) corresponds to the particular value $\alpha = \frac{1}{4}$. Now when $\alpha < \frac{1}{4}$, the expression (4.7) is proportional to $\chi^{-20\alpha}$ when χ is large. If the restriction that the various moments of $f(r)$ should all be finite is accepted on intuitive physical grounds, then certain values of α can be rejected. In particular, if the fourth moment $\int_0^\infty r^4 f\,dr$ (which occurs in the invariant of Sec. 2) is required to be finite then all values of α less than $\frac{1}{4}$ must be rejected.

On the other hand, when $\alpha > \frac{1}{4}$, the expression (4.7) becomes negative for certain values of r since it has the expansion

$$f(\chi) = e^{-\chi^2/8}\left\{1 + \sum_{n=1}^{\infty} \frac{(2.5 - 10\alpha)(3.5 - 10\alpha)\cdots(2.5 - 10\alpha + n - 1)}{n!\,(2.5)(3.5)\cdots(2.5 + n - 1)}\left(\frac{\chi^2}{8}\right)^n\right\} \tag{4.8}$$

valid for all finite values of χ (Whittaker and Watson, p. 337). (Note that although the fourth moment $\int_0^\infty r^4 f\,dr$ converges when $\alpha > \frac{1}{4}$, the invariant relation of section 2 cannot be employed to obtain a definite value for α and thence a definite decay law, because the solution makes the fourth moment vanish and Λ is constant whatever the energy decay law). Negative values of f have never been measured, so that there are some physical grounds for rejecting values of α greater than $\frac{1}{4}$. Thus when these restrictions are applied, v. Kármán and Howarth's family of self-preserving solutions reduces to the single solution (4.6). Both restrictions are of course implicit in Loitsiansky's analogy with the propagation of heat in a spherically-symmetrical five-dimensional field.

Let us return now to the general solution (4.2), with its associated decay law (4.3), which satisfies the Eq. (4.1) obtained by neglecting the triple correlation term. When t is so large that $\nu t \gg s^2$ for all values of s for which $F(s, 0)$ is large enough to contribute to the integrals, the integrals in (4.3) and (4.2) simplify. Thus when $t \to \infty$,

$$u'^2 \to \frac{1}{48(2\pi)^{1/2}(\nu t)^{5/2}}\int_0^\infty F(s, 0)s^4\,ds$$

and

$$u'^2 f \rightarrow \frac{1}{48(2\pi)^{1/2}(\nu t)^{5/2}} \int_0^\infty F(s, 0)\, e^{-r^2/8\nu t} s^4\, ds,$$

$$\text{i.e.,} \qquad f \rightarrow e^{-r^2/8\nu t}.$$

These limiting forms show that if the correlation $f(r, t)$ and turbulence intensity u'^2 are calculated on the assumption that the triple correlation is without effect, the solutions obtained tend to the forms (4.5) and (4.6) as $t \rightarrow \infty$ whatever the choice of conditions at $t = 0$. The common limiting form is a self-preserving solution, so that there is here further support for the contention of the previous section that a self-preserving solution does exist when t is large, and that it is given by Eqs. (4.5) and (4.6).

In a later work[9] Millionshtchikov made an attempt to determine the effect of triple correlations at decay times which are not large, using a reiterative method. However his attempt is based upon the existence of the solution (4.6) as a first approximation and leads only to solutions which have the same self-preserving character. Since a self-preserving solution has been shown to be possible only when t is large, it is questionable whether Millionshtchikov's approximate solutions at decay times which are not large have any significance.*

5. Partially self-preserving solutions at Reynolds number which are not large. It has been seen that a correlation function which is completely self-preserving can only occur when the decay time is large. On the other hand, it is known that some measure of self-preservation exists when t is not large. The function f is always parabolic near $r = 0$. Recent experiments[5] have indicated that the expansion of f in powers of r/λ as far as the term of fourth degree, and of k as far as the term of third degree, are independent of t at decay times which are not large. This suggests that we should explore the consequences of assuming partially self-preserving solutions for the correlation functions. We therefore write

$$f(r) \equiv f(\psi), \qquad k(\psi) \equiv k(\psi), \qquad \psi = \frac{r}{\lambda},$$

for a range of values of r, $0 \leq r < l$, where l is an unknown length. From the above evidence l must be at least as great as the maximum value of r for which a fourth degree polynominal gives a good representation of $f(r)$. The fundamental Eq. (1.3) again reduces to the form (3.2) for the restricted range of r, viz.

$$\left(f'' + \frac{4}{\psi} f' + \frac{5\psi}{2} f' + 5f\right) + \frac{1}{2} \frac{\lambda^2}{\nu R_\lambda} \frac{dR_\lambda}{dt} (\psi f') + \frac{1}{2} R_\lambda\left(k' + \frac{4}{\psi} k\right) = 0. \qquad (3.2)$$

Since the hypothesis leaves arbitrary the behaviour of the correlation functions at large values of r, we cannot make use of the invariant relation of Sec. 2. The energy

[9]M. Millionshtchikov, *On the theory of homogeneous isotropic turbulence*, C. R. Acad. Sci. U. R. S. S. **32**, 615-621 (1941).

*A further criticism of Millionshtchikov's work is that he has omitted to take into consideration the correlation between two velocity components and the pressure when determining the relation between triple and quadruple velocity correlations. His idea of determining quadruple velocity correlations from the approximation that the relation between double and quadruple correlations is as it is for a Gaussian distribution of the sums of velocity components at two points seems to be useful and to warrant further exploitation.

decay laws are therefore not prescribed as they were in the case of completely self-preserving solutions and there will be a degree of indeterminacy in the deductions.

It was mentioned in Sec. 3 that there are in general two methods of satisfying Eq. (3.2) for non-zero ranges of ψ and t. According to the first method, in which the energy decay follows a power law, R_λ is constant and for non-zero values of this constant we have $n = 1$, i.e.

$$u'^{-2} = Bt, \qquad \lambda^2 = 10\nu t, \qquad R_\lambda = \frac{10}{B\nu} \tag{5.1}$$

where B is a constant. The functions $f(\psi)$ and $k(\psi)$ are in this case connected by the equation

$$f'' + f'\left(\frac{4}{\psi} + \frac{5\psi}{2}\right) + 5f + \frac{1}{2}R_\lambda\left(k' + \frac{4}{\psi}k\right) = 0 \tag{5.2}$$

provided $0 \leq r < l$. The relations (5.1) and (5.2) become identical with Dryden's deductions[10] from the postulate of self-preservation when his scale factor L is replaced by the length used in the present analysis, viz. λ; Dryden took Eq. (5.2) to be valid for all values of r but we have already seen that the assumption of complete self-preservation leads to a quite different set of results.

It does not seem possible to check Eq. (5.2) relating f and k, since measurements of k are difficult to make and no results have yet been published. However, measurements of u' at different stages of decay have frequently been made, and the validity of (5.1) can be assessed. The evidence for the law of energy decay has been discussed by Dryden.[11] The data from different sources are not wholly consistent, but inasmuch as any one law does describe the experimental relations, it is $u'^{-2} \sim t^n$, where n lies between one and two. In more recent experiments at the Cavendish Laboratory, Cambridge,[6] the decay of λ has been measured simultaneously with that of u' and the energy equation has in this case provided a check on the consistency of the two sets of measurements. It was in fact found that the relations (5.1) were obeyed to a quite high degree of accuracy for the decay range $40\ M/U < t < 120\ M/U$, where M and U are the periodic length and velocity associated with the turbulence-producing grid and were varied over the range $5.5 \times 10^3 < UM/\nu < 2.2 \times 10^4$. Under these same conditions it was found, as mentioned above, that $f(r)$ and $k(r)$ were self-preserving during decay at least as far as the terms in r^4 and r^3 respectively. The above experiments are therefore quite consistent with the postulate of limited self-preservation and with the deductions obtained by using the first method of satisfying Eq. (3.2). A point left open is the value of l, which presumably can be determined by the region of validity of the relation (5.2).

According to the second method of satisfying Eq. (3.2) (see Sedov[7]), the coefficients of bracketed terms therein are related by

$$\frac{\lambda^2}{\nu R_\lambda}\frac{dR_\lambda}{dt} = aR_\lambda + b \tag{5.3}$$

where a and b are constants. For solutions of this equation which do not make R_λ constant, f and k must then satisfy the equations

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[10] H. Dryden, *Isotropic turbulence in theory and experiment*, v. Kármán Anniversary Volume on Applied Mechanics, 1941, pp. 85-102.

[11] H. Dryden, *A review of the statistical theory of turbulence*, Q. Appl. Math. 1, 7-42 (1943).

$$f'' + \frac{4}{\psi} f' + \left(\frac{5+b}{2}\right)\psi f + 5f = 0, \tag{5.4}$$

$$k' + \frac{4}{\psi} k = -a\psi f', \tag{5.5}$$

provided $0 \leq r < l$. Apart from the constants a and b, this method of solution provides unique determinations of the correlations f and k within the restricted range of r.*

Consider the decay law described by (5.3). This equation is to be solved with the aid of the energy equation in the form

$$\frac{2\lambda^2}{\nu R_\lambda}\frac{dR_\lambda}{dt} = \frac{d\lambda^2/\nu}{dt} - 10. \tag{5.6}$$

The solution cannot be obtained explicitly, and is given by

$$\frac{\lambda^2}{\nu} = \frac{R_\lambda^2}{K(a + b/R_\lambda)^{10/b}} = \frac{\nu R_\lambda^2}{u'^2} \tag{5.7}$$

where K is a constant of integration, and

$$\frac{dR_\lambda}{dt} = K\left(a + \frac{b}{R_\lambda}\right)^{1+10/b} \tag{5.8}$$

It will be seen in a moment that a is necessarily negative and it is then evident from (5.7) that b must be positive if u' is to become large for some values of R_λ. The nature of the decay relations specified by (5.3) is as indicated in Fig. 1. The variation of u'

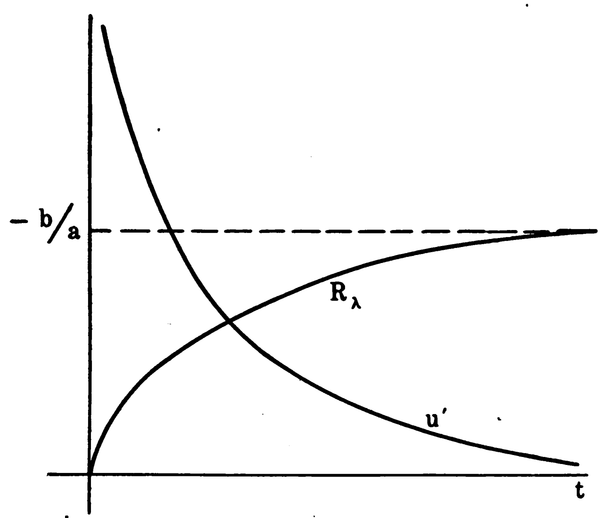

FIG. 1.

has the general features of measured energy decay curves. R_λ increases with decay time and asymptotes to a constant value $-b/a$. It follows that u'^{-2} and λ^2 asymptote to the values $(-a/b)\ 10t/\nu$ and $10\ \nu t$ respectively as t increases.

*This might be regarded as a hint that the solution is physically impossible. However I have not been able to find any definite anomalies.

It might be argued that this method of satisfying Eq. (3.2) has thus far led to predictions which are not inconsistent with experiment, since we cannot be sure that the asymptotic variation (R_λ = constant) does not occur over the range of decay times used in the experiments mentioned earlier. But the Eqs. (5.4) and (5.5) determining the functions $f(\psi)$ and $k(\psi)$ make possible a further comparison with experiment. There is available sufficient experimental evidence to determine the value of the constant a in (5.5). It has been found[5] that when r is sufficiently small, the function $k(r)$ has the form

$$k(r) = -\frac{1}{6} S\left(\frac{r}{\lambda}\right)^3, \tag{5.9}$$

where S is an absolute constant (of value about 0.39) for the ranges of decay times and mesh Reynolds numbers used in the experiments. There are also theoretical reasons,[12] derived from Kolmogoroff's theory of locally isotropic turbulence,[13] for believing that S is an absolute constant whenever the Reynolds number is sufficiently high. Thus comparison of the coefficients of powers of r^2 in (5.5) shows

$$a = -\frac{7}{6} S. \tag{5.10}$$

Then if R'_λ be written for the asymptotic value of the Reynolds number R_λ, $= u'\lambda/\nu$, the value of b is

$$b = -aR'_\lambda = \frac{7}{6} SR'_\lambda . \tag{5.11}$$

Changing the variable of (5.4) to χ, $= \psi/(\alpha)^{1/2}$, where

$$\alpha = \frac{1}{10 + 2b} = \frac{1}{10 + 7SR'_\lambda/3},$$

the equation for f becomes

$$f'' + f'\left(\frac{4}{\chi} + \frac{\chi}{4}\right) + 5\alpha f = 0. \tag{5.12}$$

v. Kármán and Howarth[1] have pointed out, in a slightly different context, that the solution of this equation is related to the hypergeometric function and that the solution which satisfies $f = 1$ and $f' = 0$ when $r = 0$, can be written

$$f(\chi) = \frac{\Gamma(5/2)}{\Gamma(10\alpha)\Gamma(5/2 - 10\alpha)} \int_0^1 \tau^{10\alpha - 1}(1 - \tau)^{3/2 - 10\alpha} e^{-(\chi^2\tau)/8}\, d\tau \tag{5.13}$$

when, as is here the case, $\alpha < \frac{1}{4}$. Knowing the value to which $u'\lambda/\nu$ tends according to this method of satisfying the requirement of limited self-preservation of the correlation functions, it is thus possible to determine f. Figure 2 shows not f, but the related double correlation function g, which, in the notation of Sec. 1, is given by

$$u'^2 g(r) = \overline{(u_i u'_j)}_{\substack{i=j \\ \xi_i = 0}} .$$

[12]G. K. Batchelor, *Kolmogoroff's theory of locally isotropic turbulence*, Proc. Camb. Phil. Soc. (to be published).

[13]A. N. Kolmogoroff, *The local structure of turbulence in an incompressible viscous fluid for very large Reynolds numbers*, C. R. Acad. Sci. U. R. S. S. **30**, 301-5 (1941) and **32**, 16-18 (1941).

$g(r)$ is easier to measure in practice, and is related to $f(r)$ by the continuity equation

$$g(r) = f(r) + \frac{r}{2}\frac{\partial f(r)}{\partial r}. \tag{5.14}$$

g has been calculated from (5.13) and (5.14), in part by numerical integration and in part by the use of an asymptotic expansion of the integral, for a number of values of R'_λ comparable with those found in the experiments at Cambridge. The largest value, viz. 44, is the value of $u'\lambda/\nu$ measured at intermediate decay times with a mesh of circular rods at 1″ spacing and a wind speed of 42 ft/sec.

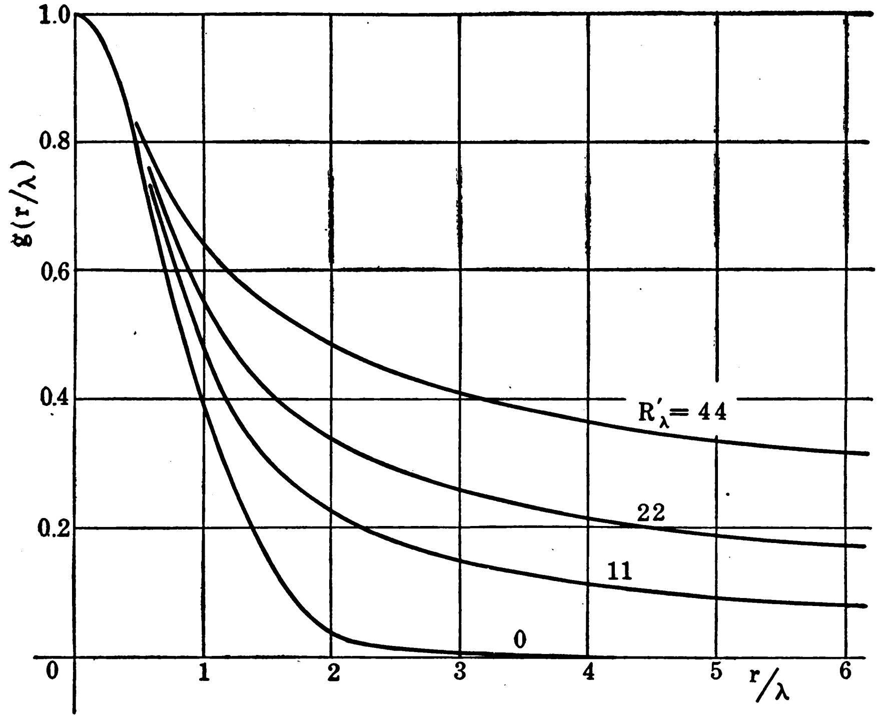

FIG. 2. Solutions of $d^2f/d\psi^2 + (4/\psi + \frac{1}{2}(5 + b)\psi)\, df/d\psi + 5f = 0$ and $g = f + \frac{1}{2}\psi\, df/d\psi$, where $\psi = r/\lambda$, $b = 7SR'_\lambda/6$, $S = 0.39$.

Mr. A. A. Townsend and the author hope that it will soon be possible to present measurements of $g(r)$ during decay under different conditions in order to determine the validity of the solution (5.13). Assuming that the measurements show $u'\lambda/\nu$ to be constant at decay times which are not too large and that self-preservation of the function $f(r)$ during decay occurs over a limited range $0 \leq r < l$, then agreement between experiment and the curves of Fig. 2 over this same range* of values of r would show that

*A hint concerning the possible range of validity of the curves in Fig. 2 is provided by the requirement, deduced from (5.14), that the *complete* function $g(r)$ obeys the relation

$$\int_0^\infty rg(r)\, dr = 0.$$

the second method of satisfying Eq. (3.2) is here valid. Necessary consequences are that k is given by (5.5), i.e.

$$k(\psi) = \frac{7}{6} S\psi^{-4} \int_0^{\psi} \psi^5 \frac{df}{d\psi}\, d\psi \qquad (0 \leq r < l) \tag{5.15}$$

where f is given by the expression (5.13), and that at decay times which are sufficiently small the decay of u' is such that $u'\lambda/\nu$ increases from a small value to its asymptotic value R'_λ. On the other hand, lack of confirmation of the curves of Fig. 2 would show that the first method of satisfying Eq. (3.2) is valid, i.e. that $u'\lambda/\nu$ is constant over the whole of the decay range for which limited self-preservation holds and that k is determined by (5.2) as

$$R_\lambda k(\psi) = -2\left(\frac{df}{d\psi} + \psi f\right) - 3\psi^{-4} \int_0^{\psi} \psi^5 \frac{df}{d\psi}\, d\psi. \tag{5.16}$$

The function $f(\psi)$ is not prescribed by this method of satisfying (3.2), except in the particular case mentioned below.

The case in which R_λ, or R'_λ in the second method, is very small (but constant) is particularly interesting, for the two methods of satisfying (3.2) then lead to identical approximate equations for $f(\psi)$, viz.

$$f'' + f'\left(\frac{4}{\psi} + \frac{5\psi}{2}\right) + 5f = 0. \tag{5.17}$$

The solution is shown in Fig. 2 and it can be predicted that the correlation function $f(r)$ will approximate to this shape at low Reynolds numbers of turbulence over the range of values of r for which this same correlation preserves its form during decay.

6. Solutions at high Reynolds number. (*a*) *No assumption of self-preservation.* Using the energy Eq. (1.4), the basic Eq. (1.3) can be written

$$u'\left(k' + \frac{4k}{r}\right) = \frac{\partial f}{\partial t} - \nu\left(\frac{10f}{\lambda^2} + f'' + \frac{4f'}{r}\right). \tag{6.1}$$

We compare the orders of magnitudes of the two terms

$$\frac{10f}{\lambda^2} \qquad \text{and} \qquad f'' + \frac{4f'}{r}.$$

When r is sufficiently small for f to be represented by the parabola $1 - r^2/2\lambda^2$, these terms are of equal order of magnitude, however small λ may be. Now when the Reynolds number of the turbulence is increased observation shows that the correlation curve does not change appreciably outside the parabolic region, the change being confined to a diminution in the value of λ. The first of the two terms will therefore become relatively large at high Reynolds numbers for values of r lying beyond the parabolic region, and the approximate form of (6.1) is

$$u'\left(k' + \frac{4k}{r}\right) = \frac{\partial f}{\partial t} - \frac{10\nu f}{\lambda^2}. \tag{6.2}$$

It is possible to integrate Eq. (6.2) if r is further restricted to be sufficiently small for the approximation $f = 1$ to be valid. In this case,

$$k' + \frac{4k}{r} = -\frac{10\nu}{\lambda^2 u'} \qquad = -\frac{10}{\lambda R_\lambda} \tag{6.3}$$

and the solution which makes $k(r)$ vanish at the origin is

$$k(r) = -\frac{2}{R_\lambda}\frac{r}{\lambda}. \tag{6.4}$$

The range in which (6.4) can be expected to apply is rather restricted. In the first place, the Reynolds number must be large enough for (6.2) to apply. r must be large enough to lie outside the parabolic region of $f(r)$, but in view of the requirement of high Reynolds number this is not an important limitation. More important is the requirement that r should be small enough for the value of f to be close to unity. The meaning of (6.4) is evidently that at large Reynolds numbers, the region near the origin in which $k(r)$ is cubic becomes small and the curve tends to a straight line with a slope which is determined by the turbulence Reynolds number R_λ. This discussion is substantially that already given by Kolmogoroff.[13]

(*b*) *Self-preserving solutions.* Under the assumed conditions of high Reynolds number, the dissipation length parameter λ is small compared with other lengths associated with the turbulence. In the limit, λ is zero and the tangent to the correlation function $f(r)$ at $r = 0$ is not horizontal, but makes an angle with the abscissa which is probably 90°. In discussing self-preserving solutions at high Reynolds numbers it is no longer possible to use λ as the reference length and some other length L associated with $f(r)$ will be employed. We specifically permit λ/L to vary during decay, since we should otherwise merely repeat the analysis of Sec. 3.

The present hypothesis is thus

$$f(r) \equiv f(\chi), \qquad k(r) \equiv k(\chi), \qquad \chi = \frac{r}{L} \tag{6.5}$$

for a range of values of r to be specified later. The reference length L must of course be defined by values of the function $f(r)$ within the range of r for which (6.5) is to hold. The basic equation (1.3) becomes

$$-5\frac{L^2}{\lambda^2}f - \frac{L}{2\nu}\frac{dL}{dt}\chi\frac{df}{d\chi} = \frac{1}{2}R_L\left(\frac{dk}{d\chi} + \frac{4k}{\chi}\right) + \left(\frac{d^2f}{d\chi^2} + \frac{4}{\chi}\frac{df}{d\chi}\right) \tag{6.6}$$

where $R_L = u'L/\nu$.

L/λ is large under the assumed conditions, and the term $(d^2f/d\chi^2 + 4/\chi\, df/d\chi)$ should be neglected by comparison with the term $5fL^2/\lambda^2$ in order to be consistent with the assumption that the non-self-preserving parabolic region near the origin is infinitesimal in extent. Hence (6.6) becomes

$$10\frac{L^2}{\lambda^2}(f) + \frac{L}{\nu}\frac{dL}{dt}(\chi f') + R_L\left(k' + \frac{4k}{\chi}\right) = 0. \tag{6.7}$$

Equation (6.7) will be satisfied if the three coefficients, which are functions only of t, are proportional, i.e. if

$$R_L = A\, L^2/\lambda^2, \tag{6.8}$$

$$\frac{L}{\nu}\frac{dL}{dt} = BR_L\,. \tag{6.9}$$

These are the equations derived by v. Kármán[1] in a discussion of self-preserving solutions at high Reynolds number. The equation connecting f and k becomes

$$10f + AB\chi f' + A\left(k' + \frac{4}{\chi}k\right) = 0. \tag{6.10}$$

As before, it is also possible to satisfy Eq. (6.7) in the manner suggested by Sedov,[7] but in this case there are reasons for rejecting this alternative. The appropriate relation between the coefficients is

$$10\frac{L^2}{\lambda^2} = a\frac{L}{\nu}\frac{dL}{dt} + bR_L , \tag{6.11}$$

where a and b are constants. For solutions of this equation which do not make $L\,dL/\nu\,dt$ proportional to R_L, the functions f and k must then satisfy

$$af + \chi f' = 0, \tag{6.12}$$

$$bf + k' + \frac{4}{\chi}k = 0. \tag{6.13}$$

However Eq. (6.12) has no solutions such that $f = 1$ when $\psi = 0$ and the method of satisfying (6.7) must be rejected.

v. Kármán's Eqs. (6.8) and (6.9), and Eq. (6.10), therefore appear as necessary consequences of self-preservation at high Reynolds numbers. The meaning of (6.8) becomes clear when we substitute for λ^2 in the energy equation to obtain

$$\frac{du'^2}{dt} = -\frac{10}{A}\frac{u'^3}{L}.$$

This equation shows that the quantities u' and L may be considered as characteristic of the whole turbulence in a calculation of the work done against Reynolds stresses, as indeed is to be expected when the correlation functions preserve their shape during decay. Perhaps less to be expected is that this expression for the energy decay remains valid (as will be seen later) when the correlation functions are only partially self-preserving.

When r is small and f may be replaced by unity, Eq. (6.10) becomes

$$\frac{dk}{d\chi} + \frac{4}{\chi}k = -\frac{10}{A} = -10\frac{L^2}{\lambda^2 R_L}$$

which is identical with Eq. (6.3) derived without any assumption of self-preservation. Clearly the general solution (6.4), viz.

$$k(r) = -\frac{2}{R_\lambda}\frac{r}{\lambda},$$

is self-preserving with L as the unit of length when the decay law (6.8) holds.

The decay equations (6.8) and (6.9) contain three variables, L, λ, u' and may be solved with the help of the energy equation (1.4). The differential equation for L is

$$L\frac{d^2L}{dt^2} = -\frac{5}{AB}\left(\frac{dL}{dt}\right)^2, \tag{6.14}$$

and has the solution*

$$L = L_0\left(\frac{t}{t_0}\right)^{(AB)/(5+AB)} \tag{6.15}$$

Then from (6.8) and (6.9)

$$\frac{1}{u'} = \frac{(5 + AB)t_0}{AL_0}\left(\frac{t}{t_0}\right)^{5/(5+AB)} = \frac{1}{u_0'}\left(\frac{t}{t_0'}\right)^{5/(5+AB)}, \tag{6.16}$$

$$\lambda^2 = \nu(5 + AB)t, \tag{6.17}$$

where L_0 and u_0' are the values of L and u' at $t = t_0$.

It is at this stage that we must be more specific about the range of values of r over which self-preservation is to be postulated. If the correlation functions are completely self-preserving, Loitsiansky's invariant relation (2.1) shows that

$$u'^2L^5 = \text{constant}, \tag{6.18}$$

and we must have

$$AB = 2,$$

as has been pointed out by Kolmogoroff.[14] The decay laws then become

$$\frac{L}{L_0} = \left(\frac{t}{t_0}\right)^{2/7}, \quad \frac{u_0'}{u'} = \left(\frac{t}{t_0}\right)^{5/7}, \quad \lambda^2 = 7\nu t. \tag{6.19}$$

The corresponding relation between f and k is, from (6.10),

$$10f + 2\chi f' + A\left(k' + \frac{4}{\chi}k\right) = 0 \tag{6.20}$$

which can be integrated to give

$$k = -\frac{2}{A}\chi f = -\left(\frac{2L_0}{7t_0u_0'}\right)\frac{r}{L}f. \tag{6.21}$$

At the present time, we have not got sufficient evidence at high Reynolds numbers of turbulence to determine the validity of the decay laws (6.19) and the correlation relation (6.21).

If, on the other hand, the hypothesis of partial self-preservation only is made, the value of AB remains arbitrary and deductions about the decay laws finish at the relations (6.15)–(6.17). We can consider one or two consequences of particular values of AB. The assumption of large Reynolds numbers entered the analysis via the postulate that L/λ is large. Since (6.15) and (6.17) show that

$$\frac{L}{\lambda} = \frac{L_0}{[(5 + AB)\nu t_0]^{1/2}}\left(\frac{t}{t_0}\right)^{(AB-5)/(2(5+AB))}, \tag{6.22}$$

the subsequent behaviour of L/λ depends critically on the sign of $AB - 5$.

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*v. Kármán[1] has written the exponent of the solution in error as $5/(5 + AB)$.

[14]A. N. Kolmogoroff, *On degeneration of isotropic turbulence in an incompressible viscous liquid*, C. R. Acad. Sci. U. R. S. S. **31**, 538-540 (1941).

If $AB > 5$, L/λ (and also $u'L/\nu$) increases and the assumed state of affairs applies with ever-increasing accuracy. A state of turbulence in which the Reynolds number $u'L/\nu$ increases indefinitely—although the energy decreases—does not seem possible, and an argument of the kind used in Sec. 3 does in fact confirm this impression. Equation (2.3) can be written

$$\frac{\partial}{\partial t}\left(R_L^2 \int_0^\infty \frac{r}{L} f \, d\frac{r}{L}\right) = \frac{3u'^2}{\nu} R_L \int_0^\infty k \, d\frac{r}{L} - \frac{6u'^2}{\nu}. \tag{6.23}$$

Suppose that when $t \to \infty$, we may write

$$\int_0^\infty \frac{r}{L} f \, d\frac{r}{L} \sim t^\alpha, \qquad \int_0^\infty k \, d\frac{r}{L} \sim t^\beta$$

and the decay laws (6.17)–(6.19) are valid. Replacing the terms of equation (6.23) by their orders of magnitude in t,

$$0(t^{(2AB-10)/(5+AB)+\alpha-1}) \sim 0(t^{(AB-15)/(5+AB)+\beta}) + 0(t^{(-10)/(5+AB)}).$$

In the present analysis the viscous term (i.e. the last on the right side) is assumed to be without effect, so that the remaining terms must be of equal order, i.e. $\alpha = \beta$. Moreover, with the assumption made previously that $\int_0^\infty k \, dr < 0$, the term on the left of (6.23) must be negative and consequently

$$\alpha < \frac{10 - 2AB}{5 + AB}.$$

The first term on the right side of (6.23) thus varies as a power of t which is less than (-1), and the viscous term will only be of smaller order if $AB < 5$. Since L/λ increases indefinitely with t when $AB > 5$, we have here an inconsistency which prohibits such values of AB except at small values of t.

The case $AB = 5$ leads to the decay laws

$$u'^{-2} \sim t, \qquad L^2 \sim \lambda^2 = 10\,\nu t.$$

Since L/λ is constant, there is no tendency for the approximation on which the solution is based to become invalid as t increases. Nevertheless such decay laws cannot persist indefinitely. When t is large, the first term on the right side of (6.23) must vary as some power of t not less than (-1) if the viscous term is not to become dominant; however such a variation with t requires $\int_0^\infty (r/L) f d(r/L)$ to approach $-\infty$, which is impossible.

Finally, when $AB < 5$, L/λ diminishes as t increases so that there comes a time when the postulate of high Reynolds number ceases to be valid; the case $AB = 2$ deduced from complete self-preservation is in this category. The system is here naturally unstable, whereas if the case $AB \geq 5$ occurs at all some secondary factor disturbs the system before t becomes large. Thus none of the regimes deduced for large Reynolds number are possible for indefinitely large times of decay.

7. Conclusion. The various hypotheses discussed in the last three sections, and the deductions made from them, are summarized in the following table.

Hypotheses or conditions	Reynolds number not large	Large Reynolds number
—	—	$k(r) = -\frac{2}{R_\lambda}\frac{r}{\lambda}$ for r so small that $f(r) \approx 1$
$t \to \infty$	$u'^{-2} \sim t^{5/2}$, $\lambda^2 = 4\nu t$, $f(\psi) = e^{-\psi^2/2}$ where $\psi = \frac{r}{\lambda}$	—
complete self-preservation of $f(r)$ and $k(r)$ during decay	Only possible when $t \to \infty$; see entry above.	$u'^{-2} \sim t^{10/7}$, $L^2 \sim t^{4/7}$, $\lambda^2 = 7\nu t$ $k\left(\frac{r}{L}\right) = -\frac{2}{7}\left(\frac{L_0}{t_0 u_0'}\right)\frac{r}{L} f\left(\frac{r}{L}\right)$
self-preservation of $f(r)$ and $k(r)$ for $0 \le r < l$	Either (1) $u'^{-2} \sim t$, $\lambda^2 = 10\nu t$ $f'' + f'\left(\frac{4}{\psi} + \frac{5\psi}{2}\right) + 5f + \frac{1}{2}R_\lambda\left(k' + \frac{4}{\psi}k\right) = 0$ $(0 \le r < l)$ or (2) $u'^2 = K\nu\left(a + \frac{b}{R_\lambda}\right)^{10/b}$, $t = \frac{1}{K}\int\left(a + \frac{b}{R_\lambda}\right)^{-1-10/b} dR_\lambda$ $f'' + f'\left(\frac{4}{\psi} + \frac{\psi}{4}\right) + \left(\frac{5}{10 + 0.91 R_\lambda'}\right)f = 0$, $k(\psi) = \frac{0.46}{\psi^4}\int_0^\psi \psi^5 f' \, d\psi$, where $R_\lambda' = -\frac{b}{a}$, $0 \le r < l$. In either case, $f'' + f'\left(\frac{4}{\psi} + \frac{5\psi}{2}\right) + 5f = 0$ $(0 \le r < l)$ when R_λ or $R_\lambda' = 0$	$u'^{-2} \sim t^{10/(5+AB)}$, $L^2 \sim t^{2AB/(5+AB)}$, $\lambda^2 = (5 + AB)\nu t$ $10f + AB\chi f' + A\left(k' + \frac{4}{\chi}k\right) = 0$ where $\chi = \frac{r}{L}$, $0 \le r < l$.

In conclusion, it should be noted that even if the deductions listed in the table are found to be in agreement with the appropriate measurements, the analysis of this paper does not by any means constitute a solution to the problem of the decay of isotropic turbulence. All that can be inferred from agreement with experiment is that the decay process takes place in a certain manner, for example, with self-preservation of the correlation functions for a certain range of r and all that this implies. (The deductions about the turbulence as $t \to \infty$, or when r is small and the Reynolds number is large, are exceptions to this statement since no hypotheses were required in these cases). In other words, the hypotheses of the above table are mathematical in origin and, so far, have no physical raison d'être. The task now is to find physical reasons why the decay process for t not large takes place as it does; indeed this might be said to have always been the chief problem of research on isotropic turbulence. Kolmogoroff's similarity hypotheses[13,12] are physical notions which promise to "explain" very successfully the structure of the turbulence (at high Reynolds number) at any instant. When we obtain equally convincing physical ideas about the way in which the governing parameters of this structure change from instant to instant, then our progress in the problem of isotropic turbulence will be considerable.

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