# DIABATIC FLOW OF A COMPRESSIBLE FLUID*

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## I. Introduction

The principal concern of theoretical aerodynamics in the past has been with the theory of adiabatic flow. The development of a corresponding theory of diabatic[1] flow is needed in order to provide proper theoretical interpretations of present basic experimental studies of combustion aerodynamics and to make possible the most general application of these studies to the design of combustion and ignition apparatus.

Quite general discussions have been given of several features of the diabatic flow of a compressible viscous fluid. Thus Kiebel [K-1]** has classified such flow into a number of dynamically permissible types with applications to meteorology, and Bateman [B-1] and others have proposed several variational principles. Unsteady diabatic flows less general than those studied by Kiebel are considered in meteorology but the heat addition function apparently does not usually enter explicitly. (See for example the development of the Bjerknes theorem in Ch. VI of Haurwitz [H-7].) In the one-dimensional or hydraulic approximation, steady, frictional, continuous (i.e., shockless) diabatic flow ([H-5] and references cited there) and discontinuous steady flow (deflagrations and detonations—see [B-1], [B-2], [H-1], and references in [B-2], [H-1], and [H-5]) have both been treated. However, there appears to be no published analysis which is designed to apply specifically to steady, diabatic flow of an inviscid, compressible fluid in two and three dimensions, although some transformations of the equations are useful in both adiabatic and diabatic flow.

Our purpose in the present series of papers is to describe the principal characteristics of steady diabatic compressible flow. The treatment will apply to both thin and thick burning regions and to subsonic and supersonic flow conditions if it is understood that algebraic equations should be added whenever necessary to express conservation of energy, mass, and momentum across flow discontinuities. The theory will not apply generally where viscous effects may be important, as in the interior of detonation fronts, ignition within boundary layers, etc. Our discussion depends upon the use of vectors other than the velocity vector to represent the flow, these vectors being chosen in order that the complexity of the basic differential equations can be reduced. A general discussion of these vectors will be given later (reference H-6). In the present paper, we shall derive new basic equations in terms of the velocity vector and in two other vector languages. We shall exhibit the physical content and implications of these equations for rotational and irrotational flow, sub- and supersonic flow and compare them with their counterparts in adiabatic flow.[2]

No effort is made to solve specific technical problems in this paper because it is believed desirable first to survey the whole field in coherent fashion. Formal manipula-

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[1]The word diabatic in place of nonadiabatic was suggested by Dr. D. J. Montgomery.

**The number in square brackets refers to the list of references at the end of the paper.

[2]Some of these results were presented first in [H-2], [H-3].

tion of the equations is based upon simple physical ideas. Boundary and initial-value problems are not discussed. Although only a few types of quasi-linear partial differential equations occur in this diabatic theory, and all these types have appeared in adiabatic theory, the boundary-value problems may be more difficult than in adiabatic flow because of the interdependence of the flow pattern and the position of an extended burning region.

Projected applications of the diabatic theory in the field of combustion aerodynamics provided the initial impetus for our research. It is hoped that the theory may prove useful in other fields as well.

## II. Three-Dimensional Flow

**1. Fundamental equations.** Formulation of the diabatic flow equations will be based upon the ordinary hydrodynamic equations of continuity and motion for an inviscid fluid

$$\nabla \cdot \rho \mathbf{V} = 0, \tag{1.1}$$

$$\nabla p + \rho \mathbf{V} \cdot \nabla \mathbf{V} = 0, \tag{1.2}$$

(where $p$, $\rho$ and $\mathbf{V}$ represent static pressure, density, and the velocity vector), the first law in the form given by Vazsonyi [V-2]

$$Q = c_p \mathbf{V} \cdot \nabla T_t , \tag{1.3}$$

(where $Q$ is the heat added to the fluid per unit mass and time and $c_p T_t$ is the stagnation or total enthalpy), and the equation of state for a perfect gas

$$p = R\rho T, \tag{1.4}$$

(where $T$ is the static temperature and $R$, the gas constant).

These equations can be modified to advantage if a vector $\mathbf{W}$ is introduced which was found to be appropriate by Crocco [C-1] (compare [T-1]) for adiabatic, iso-energetic flow and in reference [H-4] for the most general type of adiabatic compressible flow. This vector $\mathbf{W}$ is defined as

$$\mathbf{W} = \mathbf{V}/V_t , \tag{1.5}$$

where

$$V_t = (2c_p T_t)^{1/2} \tag{1.6}$$

is the "limiting velocity" at any point in the flow. The quantity $W^2$ thus represents the ratio of the kinetic energy of a fluid particle to its total energy (cf. ref. [H-4] and appendix of [H-5]). It is noted that $W$ and $M$, the local Mach number, are related by the equation

$$(1 - W^2)\left(1 + \frac{\gamma - 1}{2} M^2\right) = 1. \tag{1.7}$$

A second vector transformation that will be used is the substitution $\mathbf{V} = a\mathbf{M}$, where

$$a = (\gamma RT)^{1/2} \tag{1.8}$$

is the local velocity of sound.

In terms of $W$, the total or stagnation temperature $T_t$ and total pressure $p_t$ are given by

$$T_t = T(1 - W^2)^{-1} = T\left(1 + \frac{\gamma - 1}{2} M^2\right), \tag{1.9}$$

$$p_t = p(1 - W^2)^{-\gamma/\gamma-1} = p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/\gamma-1}. \tag{1.10}$$

The symbols $\omega_V$, $\omega_W$, and $\omega_M$ will be used for the vorticity functions $\nabla \times \mathbf{V}$, $\nabla \times \mathbf{W}$ and $\nabla \times \mathbf{M}$.

**2. Transformed equations.** The basic dynamical equations, (1.1) to (1.4), can be reduced to a smaller number of equations containing only $\mathbf{W}$, $\omega_W$, $p_t$, and a parameter $q$ proportional to $Q$. We first express the continuity Eq. (1.1) in terms of $\mathbf{W}$, $p_t$, and $T_t$ with the help of equations (1.4), (1.9), (1.10).

$$\nabla \cdot \mathbf{W} + \mathbf{W} \cdot [\nabla \log (p_t T_t^{-1/2}) + (\gamma - 1)^{-1} \nabla \log (1 - W^2)] = 0. \tag{2.1}$$

Similarly, using Eq. (1.10), the definition of $\omega_W$ and the identity $\mathbf{W} \cdot \nabla \mathbf{W} = \frac{1}{2} \nabla W^2 - \mathbf{W} \times \omega_W$, we find that the equation of motion (1.2) becomes

$$\nabla \log p_t = \frac{2\gamma}{\gamma - 1} (1 - W^2)^{-1} [\mathbf{W} \times \omega_W - \tfrac{1}{2} \mathbf{W}(\mathbf{W} \cdot \nabla \log T_t)]. \tag{2.2}$$

We now introduce the factor $q$ which is defined as

$$q = Q/V_t^3 \tag{2.3}$$

and which therefore satisfies, according to Eqs. (1.3), (1.5), the relation

$$q = \tfrac{1}{2} \mathbf{W} \cdot \nabla \log T_t = \mathbf{W} \cdot \nabla \log V_t . \tag{2.4}$$

(The quantity $q_W = q/(1 - W^2)$ will also be used later; $q_W$ corresponds to the notation of [H-6].)

Equation (2.2) now becomes

$$\nabla \log p_t = \frac{2\gamma}{\gamma - 1} (1 - W^2)^{-1} [\mathbf{W} \times \omega_W - q\mathbf{W}] \tag{2.5}$$

which we will call the $\mathbf{W}$ equation of motion. This equation is a generalization of the Bjerknes type equation (cf. [V-2], Eq. (6.1))

$$-T \nabla S + c_p \nabla T_t = \mathbf{V} \times \omega \tag{2.6}$$

in $\mathbf{V}$ language and of the $\mathbf{W}$ equation previously given in [H-4].

Final reduction of the continuity equation is effected by calculating $\mathbf{W} \cdot \nabla \log (p_t T_t^{-1/2})$ from Eqs. (2.4), (2.5) and substituting in Eq. (2.1):

$$\nabla \cdot \mathbf{W} + \frac{1}{\gamma - 1} \mathbf{W} \cdot \nabla \log (1 - W^2) = q\left(1 + \frac{\gamma + 1}{\gamma - 1} W^2\right)/(1 - W^2) \tag{2.7}$$

or

$$\nabla \cdot (1 - W^2)^{1/\gamma-1} \mathbf{W} = q\left(1 + \frac{\gamma + 1}{\gamma - 1} W^2\right)(1 - W^2)^{(2-\gamma)/(\gamma-1)}. \tag{2.8}$$

When $q = 0$, the equation

$$\nabla \cdot (1 - W^2)^{1/\gamma - 1} \mathbf{W} = 0 \tag{2.9}$$

is obtained (cf. [C-1] and [H-4]).

Similar development in the **M** language leads to two equations analogous to (2.5) and (2.8) (cf. [H-4]):

$$\frac{1}{\gamma} \nabla \log (pe^{\gamma M^2/2}) - \frac{\gamma - 1}{\gamma + 1} \mathbf{M} \nabla \cdot \mathbf{M} = \frac{2}{\gamma + 1} q_M \mathbf{M} + \mathbf{M} \times \boldsymbol{\omega}_M , \tag{2.10}$$

$$\nabla \cdot \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-(\gamma+1)/2(\gamma-1)} \mathbf{M} = (1 + \gamma M^2)\left(1 + \frac{\gamma - 1}{2} M^2\right)^{-(3\gamma-1)/2(\gamma-1)} q_M , \tag{2.11}$$

where $q_M = (\gamma - 1)Q/2a^3$ in accordance with the convention to be used in [H-6]. It is important to note that the dimensionless quantities $q_W$ and $q_M$ rather than $Q$ appear in these transformed equations. As will be shown in H-6, reductions of the form employed here, (i.e., $\mathbf{V} = [g(N)RT]^{1/2} \mathbf{N}$), never yield equations in which $Q$ is the most appropriate heating factor.

**3. Comparison with adiabatic flow.** In adiabatic flow, $T_t$ and $p_t$ are constant on streamlines though they can vary between streamlines [H-4]. In diabatic flow, however, the variation of $T_t$ on streamlines is prescribed by Eq. (1.3) or (2.4), its variation normal to streamlines still not being restricted. The rate of change of $p_t$ along a streamline is proportional to $qW$ and normal to streamlines, to the appropriate component of $\mathbf{W} \times \boldsymbol{\omega}_W$ as in adiabatic flow [H-4]. The **W** language is thus particularly useful because it permits simple expression of the variations in stagnation pressure. In the $M$ language, the quantity $pe^{\gamma M^2/2}$ is a more appropriate variable according to Eq. (2.10) than $p$ or $p_t$ , but its variation along streamlines depends both upon $q_M$ and upon $\nabla \cdot M$.

We note that the new equations of motion (2.5), of continuity (2.7), and the first law (2.4) are more compact than the conventional equations (1.1), (1.2), (1.3) having been stripped of a non-essential variable in that now only $p_t$ , $q$, and $W$ appear in place of $p$, $\rho$, $Q$ and **V**. The new equations also connect directly the changes in $p_t$ and $q$ with variation throughout the **W** field. By taking the curl of Eq. (2.5) we can eliminate $p_t$ and find another differential equation in addition to Eq. (2.7) which describes the variation of the **W** field. Thus,

$$\nabla \times [\mathbf{W} \times \boldsymbol{\omega}_W/(1 - W^2)] = \nabla \times [q\mathbf{W}/(1 - W^2)] \tag{3.1}$$

which is a generalization of Crocco's equation [C-1] and also of an equation in reference [H-4]. (Cf. later discussion of $\boldsymbol{\omega}_V = 0$ and of $\omega_W = 0$.) It is noted that $q$ in Eq. (3.1) can be eliminated by use of Eq. (2.8) resulting in a differential equation involving **W** only. The analog of (3.1) in **M** language will contain terms in $\nabla \times (\mathbf{M}\ \nabla \cdot \mathbf{M})$ which have no counterpart in the **W** language. Elimination of $q_M$ with the help of (2.11) is possible however.

The behavior of the fluid when it reaches sonic velocity in a diabatic flow is also worth examination. Let us use the convention suggested in [H-4] that $\nabla \cdot (\mathbf{W}/W) = \nabla \cdot \mathbf{s}$ is a measure of the fractional rate of variation along a streamline of stream-tube area. Then from Eq. (2.7)

$$\nabla \cdot \mathbf{s} + (1 - W^2)^{-1}\left(1 - \frac{\gamma + 1}{\gamma - 1} W^2\right)\mathbf{s} \cdot \nabla \log W = qW^{-1}(1 - W^2)^{-1}\left(1 + \frac{\gamma + 1}{\gamma - 1} W^2\right). \tag{3.2}$$

For local velocity equal to the local velocity of sound, $W = [(\gamma - 1)/(\gamma + 1)]^{1/2}$ and if the flow is continuous,

$$\nabla \cdot \mathbf{s} = q(\gamma + 1)^{3/2}(\gamma - 1)^{-1/2} \tag{3.3}$$

Thus the stream-tube diverges at sonic velocity if $q > 0$, converges if $q < 0$ and has minimum area for the adiabatic case $q = 0$ (cf. discussion in H-4). "One-dimensional" analogs of Eq. (3.3) are described in H-5.

**4. Types of Irrotational flow:** $\boldsymbol{\omega}_V = 0$. We first observe the consequences of irrotationality in the **V** field. Just as in adiabatic flow we find at once from Eq. (1.2) (cf. [H-4]) that for $\boldsymbol{\omega}_V = 0$, the flow must be barytropic, i.e., that $p = F(\rho)$. The continuity equation (1.1) then leads us to a partial differential equation for a velocity potential $\varphi_V$ of familiar form. Thus from Eq. (1.2)

$$\mathbf{V} \cdot \nabla \log \rho = -a_b^{-2}\mathbf{V} \cdot \nabla \tfrac{1}{2}V^2, \tag{4.1}$$

where $a_b^2 = dp/d\rho$ and Eq. (1.1) can be written in summation notation as follows:

$$\sum_i (a_b^2 - V_i^2)\frac{\partial V_i}{\partial x_i} - 2\sum_{i>j} V_i V_j \frac{\partial V_i}{\partial x_j} = 0, \tag{4.2}$$

where $V_r$ and $x_r$ are the components of velocity and position vectors, respectively. Placing $V_i = \partial\varphi_v/\partial x_i$ we find

$$\sum_i \left[a_b^2 - \left(\frac{\partial \varphi_V}{\partial x_i}\right)^2\right]\frac{\partial^2 \varphi_V}{\partial x_i^2} - 2\sum_{i>j}\frac{\partial \varphi_V}{\partial x_i}\frac{\partial \varphi_V}{\partial x_j}\frac{\partial^2 \varphi_V}{\partial x_i\,\partial x_j} = 0. \tag{4.3}$$

Equation (4.3) changes type from elliptic to hyperbolic as $V$ increases and passes through the value $a_b$. When $M \neq 1$, $a_b$ can be greater or less than $a$, but for $M = 1$, $a_b = a$ and change of type still corresponds to passage between subsonic and supersonic flow. This correspondence no longer holds for some types of irrotational diabatic flow [cf. Eqs. (4.11), (4.13); Sec. 6; and Eq. (5.1)].

Further development yields relations between $a_b$ and $Q$. Vazsonyi [V-2] gives the equation governing the entropy variation, $\nabla S$, in the form

$$\mathbf{V} \cdot \nabla S = Q/T \tag{4.4}$$

which, together with the definition of $S$

$$dS = c_v d \log (\rho^{-\gamma} p), \tag{4.5}$$

leads to

$$\left(\frac{1}{\gamma}\frac{d \log p}{d \log \rho} - 1\right)\mathbf{V} \cdot \nabla \log \rho = Q/c_pT \tag{4.6}$$

and finally, with introduction of $\varphi_v$ to

$$\left(\frac{1}{\gamma}\frac{d \log p}{d \log \rho} - 1\right)\nabla^2\varphi_V = -Q/c_pT. \tag{4.7}$$

We see that $Q/T$ is a measure of the divergence of the **V** field and of the departure of the $(p, \rho)$ relation from the adiabatic one.

This result can be thrown in a form that is directly comparable with the one-dimensional theory. From Eqs. (1.1), (1.2) for $\omega_V = 0$

$$\nabla \cdot \mathbf{s} = -\frac{\partial \log \rho}{\partial s} - \frac{\partial \log V}{\partial s}, \tag{4.8}$$

$$\frac{\partial p}{\partial s} = -\rho V^2 \frac{\partial \log V}{\partial s}, \tag{4.9}$$

where $\partial/\partial s = \mathbf{s} \cdot \nabla$. Hence

$$\nabla \cdot \mathbf{s} = \left( \frac{1}{\gamma M^2} \frac{d \log p}{d \log \rho} - 1 \right) \frac{\partial \log \rho}{\partial s}. \tag{4.10}$$

We can then use Eq. (4.6) to eliminate $\partial \log \rho/\partial s$ obtaining

$$\frac{dp}{d\rho} = a_b^2 = a^2 M^2 (\nabla \cdot \mathbf{s} - Q/c_p TV)/(M^2 \nabla \cdot \mathbf{s} - Q/c_p TV). \tag{4.11}$$

When $Q = 0$, we obtain the usual adiabatic formula $dp/d\rho = a^2$. We now see that, depending upon the relative magnitudes of $M$, $\nabla \cdot \mathbf{s}$ and $Q/c_p TV$, $a_b^2$ may differ from $a^2$ by a factor of either sign as well as any magnitude (cf. S-1). In comparing with one-dimensional flow in the direction of $x$ in a duct of (variable) area, we place $\nabla \cdot \mathbf{s} = -d\alpha/dx$ and $Q/(c_p TV) = d\theta^*/dx$ (see reference [H-5]) and obtain

$$\frac{dp}{d\rho} = a^2 M^2 (d\alpha + d\theta^*)(M^2\, d\alpha + d\theta^*)^{-1} \tag{4.12}$$

which is what could also be computed from Eqs. (15) and (17) of this reference.

It is noted that the analog of (3.2) in **V** language is

$$\nabla \cdot \mathbf{s} = (Q/c_p TV) - (1 - M^2) \frac{\partial \log V}{\partial s} \tag{4.13}$$

which reduces to the known expression for adiabatic flow $\nabla \cdot \mathbf{s} = -(1 - M^2)$ $(\partial \log V)/\partial s$ when $Q = 0$ and to expression (3.3) for $M = 1$.

We now show that given $F(\rho)$ and appropriate boundary conditions $Q$ can be determined as a function of position. Integration of Bernoulli's equation gives the functional relationship between $V$ and $\rho$.

$$\frac{1}{2} V^2 = \frac{1}{2} V_0^2 - \int \rho^{-1} F'(\rho)\, d\rho \tag{4.14}$$

from which $a_b^2 = F'(\rho)$ is also expressible implicitly as a function of $V = |\nabla \varphi_V|$. Accordingly, the continuity Eq. (4.3) is a quasi-linear partial differential equation which can be solved with appropriate boundary conditions for $\varphi_V$ as a function of position. Subsequent calculation of $\nabla^2 \varphi_V$ and $(d \log p)/(d \log \rho)$ yields $Q/T$ as a function of $V$ along any streamline. The energy equation (1.3) can be written for any one streamline $s_1$ as follows:

$$\frac{dT}{ds_1} - (Q/c_p TV)T = -\frac{1}{2c_p} \frac{dV^2}{ds_1}, \tag{4.15}$$

where $V^2$ is a known function of $s_1$. Accordingly, $T$ as a function of $s_1$, and therefore also $Q$ can be found along each streamline and therefore throughout the field of flow. This process, although complex, is direct; it would be essentially more difficult to proceed in the opposite direction, that is, with knowledge of $Q$ (or some combination involving $Q$) because compatible solutions of (4.3) and (4.7) would not be obtainable for arbitrary $Q$ functions.

A note on the place of the Earnshaw pressure-density relation in the theory is in order. When $\omega_v = 0$ in diabatic flow, a hodograph transformation yields an additional term proportional to $Q$ in the hodograph partial differential equation for $\psi$, the stream function. Assumption of a linearized (i.e., Earnshaw) $p$ vs. $\rho^{-1}$ relation corresponds to a restriction upon the type of $Q$ function that can be specified. In order to obtain the partial differential equation for $\psi$ used by Chaplygin and by von Kármán, the additional term proportional to $Q$ must be neglected. Therefore, treatments which utilize *any* linearized $p$ vs. $\rho^{-1}$ relation in the Chaplygin equation can be regarded as approximate treatments of a special type of diabatic flow. (Compare also [V-1], p. 348.)

The previous discussion also indicates that a Glauert-Prandtl treatment (cf. [T-2]) should be feasible. In the series of approximations leading to the expansion or contraction factor $(1 - M_b^2)^{1/2}$ where $M_b = V_0/a_{b0}$, $a_b$ will now replace $a$. The method is especially simple when the $p$, $\rho$ relation is polytropic, $p \propto \rho^k$. Then

$$\left(1 - \frac{k}{\gamma}\right)\nabla^2\varphi_V = Q/c_pT \tag{4.16}$$

and for small distortions of the flow

$$\nabla^2\varphi_V = M_b^2\frac{\partial^2\varphi_V}{\partial x^2}. \tag{4.17}$$

Combination of these equations yields

$$\frac{\partial^2\varphi_V}{\partial x^2} = M_b^{-2}\left(1 - \frac{k}{\gamma}\right)^{-1}Q/c_pT. \tag{4.18}$$

Now transform the variables according to the scheme

$$x = (1 - M_b^2)^{1/2}\xi, \qquad y = \eta, \qquad z = \zeta, \qquad \varphi_V(x, y, z) = \varphi_V^*(\xi, \eta, \zeta)$$

The equations (4.17), (4.18) become

$$\nabla^2_{\xi,\eta,\zeta}\varphi_V^* = 0 \tag{4.19}$$

$$\frac{\partial^2\varphi_V^*}{\partial\xi^2} = (1 - M_b^2)Q/c_pTM_b^2\left(1 - \frac{k}{\gamma}\right) \tag{4.20}$$

According to the second of these equations,

$$\nabla^2_{\xi,\eta,\zeta}\frac{Q}{c_pT} = 0 \tag{4.21}$$

Thus in the Glauert-Prandtl approximation in a diabatic, polytropic field of flow, the function $Q/c_pT$ must be harmonic in the variables $(\xi, \eta, \zeta)$. As $M_b \to 0$, $Q$ and $\nabla^2\varphi_V$ likewise approach zero.

Other assumed $(p, \rho)$ relations than polytropic or isentropic have not so far led to simple results of interest.

**5. Types of irrotational flow (cont'd.):** $\omega_W = 0$. In an irrotational **W** field, $\omega_W = 0$, and the equation of motion (2.5) becomes

$$\nabla \log p_t = -\frac{2\gamma}{\gamma - 1} q\mathbf{W}/(1 - W^2). \tag{5.1}$$

Thus the only change in stagnation pressure is along streamlines and is of "momentum pressure drop" type due to the heating. If this equation is integrable

$$\log p_t = -\frac{2\gamma}{\gamma - 1} \int q_W(\varphi_W)\, d\varphi_W ,$$
$$q = (1 - W^2) q_W(\varphi_W), \tag{5.2}$$

in accordance with the notation to be used in [H-6], which restricts the possible modes of variation of $q$ or $q_W$ in that now $q_W$ only varies along streamlines and must therefore be a function of $\varphi_W$. Accordingly, the $W$ continuity equation can be written as follows:

$$\nabla \cdot (1 - W^2)^{1/\gamma - 1}\mathbf{W} = \left(1 + \frac{\gamma + 1}{\gamma - 1} W^2\right)(1 - W^2)^{1/\gamma - 1} q_W(\varphi_W). \tag{5.3}$$

In terms of $\varphi_W$ this becomes

$$\sum_i \left[1 - W^2 - \frac{2}{\gamma - 1}\left(\frac{\partial \varphi_W}{\partial x_i}\right)^2\right] \frac{\partial^2 \varphi_W}{\partial x_i^2} - \frac{4}{\gamma - 1} \sum_{i > j} \frac{\partial \varphi_W}{\partial x_i} \frac{\partial \varphi_W}{\partial x_j} \frac{\partial^2 \varphi_W}{\partial x_i\, \partial x_j}$$
$$= \left(1 + \frac{\gamma + 1}{\gamma - 1} W^2\right)(1 - W^2) q_W(\varphi_W). \tag{5.4}$$

This quasi-linear partial differential equation changes type when $W^2 = \sum_i (\partial \varphi_W/\partial x_i)^2 = (\gamma - 1)/(\gamma + 1)$ (i.e., for $M = 1$) from elliptic to hyperbolic as $W$ increases. Equation (5.4) is reminiscent of Crocco's equation (number 11 in [C.1]) in the stream function for two-dimensional rotational flow in that a function (arbitrary) of the dependent variable appears on the R H S. Other similar equations will be derived later in generalizing the Crocco theory. It appears that equation (5.4) might be easier to solve than its analog for the irrotational **V** field, Eq. (4.3). The key functions in the two cases are $q_W(\varphi_W)$ and $a_b(V^2)$ where $V^2 = \sum_i (\partial \varphi_V/\partial x_i)^2$ and the second function involves higher order derivatives of the dependent variable than does the first. In both cases solution of the partial differential equation for $\varphi$ is the central problem, for the equations of motion are integrated by Eqs. (4.14) and (5.2).

For the incompressible case ($W \ll 1$) Eq. (5.4) reduces to

$$\nabla^2 \varphi_W = q_W(\varphi_W) \tag{5.5}$$

which is of the same form as an equation for the potential function $\varphi_M$ of the Mach vector for adiabatic flow (cf. Sec. 6; also H-4). A Glauert-Prandtl treatment of (5.4) leads to an equation similar to (5.5):

$$\nabla^2_{\xi,\eta,\zeta} \varphi_W^* = \left(1 + \frac{\gamma + 1}{\gamma - 1} W_0^2\right) q_W^*(\varphi_W^*) \tag{5.6}$$

in which now

$$x = (1 - W_0^2)^{-1/2}\left(1 - \frac{\gamma + 1}{\gamma - 1} W_0^2\right)^{1/2} \xi, \qquad y = \eta, \qquad z = \zeta,$$

$$\varphi_W(x, y, z) = \varphi_W^*(\xi, \eta, \zeta).$$

**6. Types of irrotational flow (cont'd.):** $\omega_M = 0$. If the **M** field is irrotational, (2.10) shows that

$$\log (pe^{\gamma M^2/2}) + \int F_M(\varphi_M)\, d\varphi_M = 0 \tag{6.1}$$

and

$$\nabla^2 \varphi_M = \frac{2\gamma}{\gamma - 1} q_M - \frac{\gamma + 1}{\gamma(\gamma - 1)} F_M(\varphi_M) \tag{6.2}$$

where $F_M(\varphi_M)$ is arbitrary.

The function $pe^{\gamma M^2/2}$ is thus constant on potential surfaces as in adiabatic irrotational flow in the **M** field [H-4].

Elimination of $q_M$ from (2.11) and (6.2) yields the partial differential equation for $\varphi_M$:

$$\sum_i \left[1 - \gamma\left(\frac{\partial \varphi_M}{\partial x_i}\right)^2\right] \frac{\partial^2 \varphi_M}{\partial x_i^2} - 2\gamma \sum_{i > j} \frac{\partial \varphi_M}{\partial x_i} \frac{\partial \varphi_M}{\partial x_j} \frac{\partial^2 \varphi_M}{\partial x_i\, \partial x_j} = \frac{1}{\gamma}(1 + \gamma M^2) F_M(\varphi_M). \tag{6.3}$$

This equation is of the same form as (5.4), but changes type for $M = \gamma^{-1/2}$ rather than for $M = 1$. This behavior is reminiscent of a property of diabatic flow in one dimension that was described in [H-1] and [H-5]. It was there found that a compressible fluid which is subjected to heating and is flowing in a duct of constant area attained its maximum static temperature $T$ for $M = (\gamma)^{-1/2}$. A connection between this result and the situation described by the irrotational **M** field is shown as follows. From our previous equations,

$$\mathbf{M} \cdot \nabla \log a = \frac{1}{\gamma} F_M(\varphi_M). \tag{6.4}$$

Comparison with Eq. (6.3) for the one-dimensional case $\mathbf{M} = M\mathbf{i}$ gives

$$M \frac{d \log a}{dx} = (1 - \gamma M^2)(1 + \gamma M^2)^{-1} \frac{dM}{dx} \tag{6.5}$$

and $a$ (or $T$) has a maximum for $M = 1$. This maximum in the one-dimensional case occurs, of course, no matter what language is used to replace the $M$ language. It is possible that an extremum of the temperature might also occur in three-dimensional irrotational **M** flow at $M = \gamma^{-1/2}$ if $F_M(\varphi_M)$ is chosen properly.

If throughout the flow $M \ll 1$ ("incompressible approximation") then

$$\nabla^2 \varphi_M = q_M = \frac{1}{\gamma} F_M(\varphi_M) \tag{6.6}$$

which is similar to the **W** case (5.5). If the Glauert-Prandtl type of approximation is used ($M_x = M = M_0$, $M_y \ll M_0$, $M_z \ll M_0$ where $M_0$ is the uniform Mach number at infinity),

$$\nabla^2_{\xi,\eta,\zeta}\varphi^*_M = \frac{1}{\gamma}(1 + \gamma M_0^2)F_M(\varphi^*_M), \tag{6.7}$$

where $x = \xi(1 - \gamma M_0^2)^{1/2}$, $y = \eta$, $z = \zeta$, $\varphi_M(x, y, z) = \varphi^*_M(\xi, \eta, \zeta)$ which is of the same form as (5.6). Note that $q_M$ is not simply related to $F_M(\varphi_M)$ in the Glauert-Prandtl approximation for irrotational **M** flow, because $\nabla^2_{x,y,z}\varphi_M$ occurs in the second integral (6.2) of the **M** equation of motion. If $\underline{\xi}$ is put equal to $x(1 + (\gamma - 1)/2\, M_0^2)^{1/2}(1 - M_0^2)^{-1/2}$ then $q_M$ satisfies the equation similar to

$$\nabla^2_{\underline{\xi},\eta,\zeta}\varphi^*_M = (1 + \gamma M_0^2)\left(1 + \frac{\gamma - 1}{2} M_0^2\right)^{-1} q_M\,. \tag{6.8}$$

As $M_0$ decreases, $\underline{\xi} \rightarrow \xi$ and (6.8) approaches Eq. (6.6) in form. For larger $M_0$ than corresponds to use of (6.6), $\underline{\xi}$ and $\xi$ are still approximately equal and $q_M$ is approximately proportional to $F(\varphi_M)$. It will be recalled that in the irrotational **W** field, $q_W = q_W(\varphi_W)$ exactly.

## III. Uniplanar Flow

When we pass to the two-dimensional case of uniplanar flow we find that there are a number of systems of variables in the three languages (**V**, **W**, **M**), each of which may be useful in the proper circumstances. On the one hand, in the "physical plane" we may choose $W$ (or $M$, $V$) and $\theta$, the direction angle of a streamline, or alternatively, potential and stream functions $\varphi$, $\psi$ to be the dependent variables (see [V-1]). Cartesian or curvilinear coordinates may appear as independent variables. On the other hand, in the "hodograph plane" either the Cartesian components $u$, $v$ or the polar components $W$ (or $V$, $M$), $\theta$ will enter as independent variables and usually $\varphi$, $\psi$ as dependent variables. We will now derive or state the differential equations for uniplanar diabatic flow for a number of these cases. Their most general form, that is, when neither $q$ nor $\omega$ vanishes, will always be given first. Because in some ways the **W** formulation appears at present to be the most convenient we will give the derivations in **W** language only.

**7. The unit vectors s, n.** The basic equations (2.5), (2.8) in the case of uniplanar flow are made easier to handle if the mutually orthogonal unit vectors (**s**, **n**, **k**) are introduced. The plane of the flow is normal to **k**, **s** lies in the direction of the streamline at each point and **n** in the direction normal to the streamline. The sense of the vectors is such that

$$\mathbf{k} = \mathbf{s} \times \mathbf{n}. \tag{7.1}$$

Then it follows that

$$\boldsymbol{\omega} = \omega\mathbf{k}, \tag{7.2}$$

$$\mathbf{W} \times \boldsymbol{\omega} = -W\omega\mathbf{n}. \tag{7.3}$$

Accordingly, the **W** equation of motion (2.5) becomes

$$\nabla \log p_t = -\frac{2\gamma W}{\gamma - 1}(1 - W^2)^{-1}(\omega\mathbf{M} + q\mathbf{s}). \tag{7.4}$$

With the convention

$$\frac{\partial}{\partial s} = \mathbf{s}\cdot\nabla, \qquad \frac{\partial}{\partial n} = \mathbf{n}\cdot\nabla, \tag{7.5}$$

Eq. (7.4) may be rewritten as follows:

$$\frac{\partial \log p_t}{\partial s} = -\frac{2\gamma W}{\gamma - 1}(1 - W^2)^{-1}q,$$
$$\frac{\partial \log p_t}{\partial n} = -\frac{2\gamma W}{\gamma - 1}(1 - W^2)^{-1}\omega. \tag{7.6}$$

The continuity equation is now

$$W(1 - W^2)\nabla\cdot\mathbf{s} + \left(1 - \frac{\gamma + 1}{\gamma - 1}W^2\right)\frac{\partial W}{\partial s} = \left(1 + \frac{\gamma + 1}{\gamma - 1}W^2\right)q. \tag{7.7}$$

It should be noted that $\partial/\partial s$, $\partial/\partial n$ cannot be treated like partial derivatives in general because these symbols are shorthand for the longer expressions, $\partial/h_{s'}\ \partial s'$, $\partial/h_{n'}\ \partial n'$ where $h_{s'}$ and $h_{n'}$ are elements of the metric for the orthogonal curvilinear coordinate system $s'$, $n'$ and $h_{s'}$ , $h_{n'}$ can be functions of both $s'$, $n'$. Thus $p_t$ cannot be eliminated from (7.6) by simple cross differentiation but elimination can be effected by vector operations on Eq. (7.4) as shown in (3.1) which now can be written as

$$W(\omega\mathbf{n} + q\mathbf{s}) \times \nabla \log (1 - W^2) + \nabla \times (W\omega\mathbf{n} + q\mathbf{W}) = 0. \tag{7.8}$$

After dotting in $\mathbf{k}$ and using the unit-vector formulas in the appendix we find that

$$W\left(-\omega\frac{\partial}{\partial s} + q\frac{\partial}{\partial n}\right)\log (1 - W^2) + \omega\nabla\cdot\mathbf{W} + W\frac{\partial \omega}{\partial s} + q\omega - W\frac{\partial q}{\partial n} = 0. \tag{7.9}$$

This may be combined with (2.7) and rearranged to read

$$2W^{-1}(1 - W^2)^{-1}\left(1 + \frac{1}{\gamma - 1}W^2\right)q\omega + \omega\frac{\partial}{\partial s}\log [\omega/(1 - W^2)^{1/\gamma-1}]$$
$$= q\frac{\partial}{\partial n}\log [q/(1 - W^2)]. \tag{7.10}$$

It is noted that this can also be written in the form

$$2(1 - W^2)^{-1}\left(1 + \frac{1}{\gamma - 1}W^2\right)q\omega + \omega\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\log [\omega/(1 - W^2)^{1/\gamma-1}]$$
$$= q\left(u\frac{\partial}{\partial y} - v\frac{\partial}{\partial x}\right)\log [q/(1 - W^2)], \tag{7.11}$$

where $\mathbf{W} = u\mathbf{i} + v\mathbf{j}$.
With the help of the expression for $\mathbf{W}\cdot\nabla \log p_t$ we can find another form which can be directly compared with Crocco's results. This form is

$$2W^{-1}q\omega + \omega\frac{\partial}{\partial s}\log (\omega/p) = q\frac{\partial}{\partial n}\log [q/(1 - W^2)]. \tag{7.12}$$

Equations (7.10), (7.12) exhibit the intimate relationship between the heating factor $q$ and the vorticity $\omega$ in steady flow. For example, if $W$ and therefore $\omega$ are specified throughout the field of flow, then the variation of $q/(1 - W^2)$ and also of $q$

along directions normal to the streamlines cannot be arbitrarily specified although variation of this quantity along streamlines is still not restricted by (7.10).

Crocco's equation follows at once from (7.12) when $q = 0$, i.e.,

$$\frac{\partial}{\partial s}(\omega/p) = 0. \tag{7.13}$$

When instead, $\omega = 0$, then $q$ is subject to the restriction (reference [H-3])

$$\frac{\partial}{\partial n}[q/(1 - W^2)] = 0. \tag{7.14}$$

This can also be written as

$$\frac{\partial q_W}{\partial n} = \frac{\partial}{\partial n}(Q/2c_pTV_t) = 0. \tag{7.15}$$

Both equations imply a degree of uniformity of the heating on curves normal to the streamlines if the flow is to be irrotational in the **W** field.

We will now suppose that $W$ can be neglected compared to one, that is, that the flow is "incompressible." Equation (7.10) then reduces to [H-3]

$$2W^{-1}q_W\omega + \frac{\partial \omega}{\partial s} = \frac{\partial q_W}{\partial n} \tag{7.16}$$

Now when $q_W = 0$, then $\partial\omega/\partial s = 0$, a known result for rotational incompressible flow, and when $\omega = 0$, $\partial q_W/\partial n = 0$, or the factor $q = q_W$ does not vary on curves normal to streamlines. Finally, if $\partial\omega/\partial s$ and $\partial q_W/\partial n$ are to remain finite as $W \to 0$, either $q_W$ or $\omega$ (or their product) must likewise approach zero. Very low speed continuous flow therefore cannot locally be both diabatic and rotational with arbitrarily high values of $q$ and $\omega$.

It is shown in the appendix that $\nabla \cdot \mathbf{s} = \partial\theta/\partial n$ where $\theta$ is the angle of inclination of a streamline. The continuity equation (7.7) accordingly becomes

$$W(1 - W^2)\frac{\partial \theta}{\partial n} + \left(1 - \frac{\gamma + 1}{\gamma - 1}W^2\right)\frac{\partial W}{\partial s} = \left(1 + \frac{\gamma + 1}{\gamma - 1}W^2\right)q \tag{7.17}$$

For $W \ll 1$,

$$\frac{\partial \theta}{\partial n} + \frac{\partial \log W}{\partial s} = q/W$$

We can complete our set of equations for uniplanar flow by adding the definition of $\omega$, using the expression $\mathbf{k} \cdot \nabla \times \mathbf{s} = \partial\theta/\partial s$, developed in the appendix,

$$\frac{\partial \theta}{\partial s} - \frac{\partial \log W}{\partial n} = \omega/W \tag{7.19}$$

and the definition of $q$ (energy equation)

$$\frac{\partial \log V_t}{\partial s} = q/W \tag{7.20}$$

both of which hold also when $W$ is not $\ll 1$. These equations (7.16), (7.17), (7.19), (7.20) reduce to the conventional ones when $q = \omega = 0$.

**8. Introduction of the stream and potential functions.** We will follow Crocco and define a stream function in terms of $(1 - W^2)^{1/\gamma-1}\,\mathbf{W}$ rather than in terms of $\rho\mathbf{V}$ (cf., however, [T-1]), and the potential function $\varphi$ will be introduced similarly. Thus, we set [H-3]

$$(1 - W^2)^{1/\gamma-1}\mathbf{W} = \nabla\varphi + \nabla \times \mathbf{k}\psi, \tag{8.1}$$

where now, for $q \neq 0$, both $\varphi$ and $\psi$ must be used. It follows that the relations

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\psi}{\partial s}, \tag{8.2}$$

$$(1 - W^2)^{1/\gamma-1}W = \frac{\partial\varphi}{\partial s} + \frac{\partial\psi}{\partial n} \tag{8.3}$$

hold. The continuity equation becomes

$$\nabla^2\varphi = \left(1 + \frac{\gamma+1}{\gamma-1}W^2\right)(1 - W^2)^{(2-\gamma)/(\gamma-1)}q. \tag{8.4}$$

It is not advantageous to re-express the equation of motion and its consequence, Eq. (7.10), in terms of $\varphi$, $\psi$. An expression for $\omega$ can be found however. Our development is similar to that of Crocco and of Vazsonyi for the adiabatic case.

We first write (8.1) in component form

$$(1 - W^2)^{1/\gamma-1}u = \varphi_x + \psi_y\,, \tag{8.5}$$

$$(1 - W^2)^{1/\gamma-1}v = \varphi_y - \psi_x\,, \tag{8.6}$$

where $W = u\mathbf{i} + v\mathbf{j}$ and the subscripts indicate partial differentiation. We differentiate (8.5) with respect to $y$ and let

$$(a^2/V_t^2) = c^2 = \frac{\gamma-1}{2}(1 - W^2). \tag{8.7}$$

Then,

$$(1 - W^2)^{1/\gamma-1}u_y = (1 - W^2)^{1/\gamma-1}\frac{Wu}{c^2}W_y + \varphi_{xy} + \psi_{yy}\,. \tag{8.8}$$

By symmetry also

$$(1 - W^2)^{1/\gamma-1}v_x = (1 - W^2)^{1/\gamma-1}\frac{Wv}{c^2}W_x + \varphi_{xy} - \psi_{xx} \tag{8.9}$$

and therefore

$$\begin{aligned} -(1 - W^2)^{1/\gamma-1}(v_x - u_y) &= -(1 - W^2)^{1/\gamma-1}\omega \\ &= (1 - W^2)^{1/\gamma-1}\frac{W}{c^2}(-vW_x + uW_y) + \nabla^2\psi. \end{aligned} \tag{8.10}$$

This equation is the same as the equation on p. 7 of Crocco's paper and therefore is of the same form whether the flow is diabatic or not. The terms $W_x$ and $W_y$ can be eliminated. From (8.5), (8.6)

$$(1 - W^2)^{2/\gamma-1}W^2 = \varphi_x^2 + \varphi_y^2 + \psi_x^2 + \psi_y^2 + 2\varphi_x\psi_y - 2\varphi_y\psi_x . \tag{8.11}$$

The gradient of this expression can be reduced to the form

$$W(1 - W^2)^{1/\gamma-1}\left(1 - \frac{W^2}{c^2}\right)\nabla W = u\nabla(\varphi_x + \psi_y) + v\nabla(\varphi_y - \psi_x). \tag{8.12}$$

Now taking the cross product with (8.1), we find

$$\begin{aligned} W\left(1 - \frac{W^2}{c^2}\right)(1 - W^2)^{1/\gamma-1}(uW_y - vW_x) &= -uv(\varphi_{xx} + \psi_{yx}) - v^2(\varphi_{xy} - \psi_{xx}) \\ &\quad + u^2(\varphi_{xy} + \psi_{yy}) + uv(\varphi_{yy} - \psi_{xy}). \end{aligned} \tag{8.13}$$

Combination of (8.10) and (8.13) yields the desired formula for $\omega$.

$$\begin{aligned} -(1 - W^2)^{1/\gamma-1}\omega = \left(1 - \frac{W^2}{c^2}\right)^{-1}\Bigg[\frac{uv}{c^2}(\varphi_{yy} - \varphi_{xx}) + \frac{u^2 - v^2}{c^2}\varphi_{xy} \\ + \left(1 - \frac{u^2}{c^2}\right)\psi_{xx} + \left(1 - \frac{v^2}{c^2}\right)\psi_{yy} - 2\frac{uv}{c^2}\psi_{xy}\Bigg] \end{aligned} \tag{8.14}$$

in which $u$, $v$, $W$, $c$ are to be evaluated from (8.5), (8.6), (8.7).

This equation is still quasi-linear in $\varphi$ and $\psi$. It reduces to Crocco's equation (10″) if $\varphi = 0$. For the incompressible case, $W \ll 1$, it becomes simply

$$\nabla^2\psi = -\omega \tag{8.15}$$

as in adiabatic flow whereas (8.4) becomes

$$\nabla^2\varphi = q_W \tag{8.16}$$

The symmetry in $q_W$, $\omega$ of Eqs. (7.16), (8.15), and (8.16) has been remarked upon previously [H-3].

When $q = 0$, $\varphi$ may be taken to vanish and (8.14) then is a partial differential equation for $\psi$ which changes type, as $W$ increases through the value $c$, from elliptic to hyperbolic. When $\omega = 0$, it is appropriate to use $\varphi_W$ and Eq. (5.4) gives similar behavior at $W = c = [(\gamma - 1)/(\gamma + 1)]^{1/2}$. In general, with neither $q$ nor $\omega = 0$, $\varphi$ and $\psi$ are to satisfy (7.11), (8.4), and (8.14). Since there are three equations for the four dependent variables $\varphi$, $\psi$, $q$, $\omega$, the system is underdetermined until one of them is specified independently.

**9. The generation of vorticity by nonuniform heating.** Let us suppose that in a field of compressible flow described in **W** language that $\omega_W$ is equal to zero upstream of a region $R$ (see figure) and that only within $R$ is $q$ different from zero. Consider the curve $N$ passing through $R$ and intersecting all streamlines orthogonally. At $A$ and $B$, outside $R$ and on $N$, $q = 0$ but within $R$, $q$ is different from zero. Then the rate of variation of $q$ along $N$, represented by $\partial q/\partial n$, must also be different from zero somewhere between $A$ and $B$, say at $C$ among other places. Along the streamline through $C$, $\omega$ must then be changing at $C$, according to (7.12) for, in general, the terms $2q\omega/W - \omega\partial \log p/\partial s + q\partial \log (1 - W^2)/\partial n$ will not cancel one another. Accordingly, on each streamline which passes through a point on $N$ (or other such curves) where $\partial q/\partial n \neq$

0, $\partial\omega/\partial s$ will also not equal zero, and the value of $\omega$ on some streamlines issuing from $R$ will differ from zero unless the $q$ distribution is of such a character that $\int_R (\partial\omega/\partial s)\, ds = 0$ for every streamline passing through $R$.

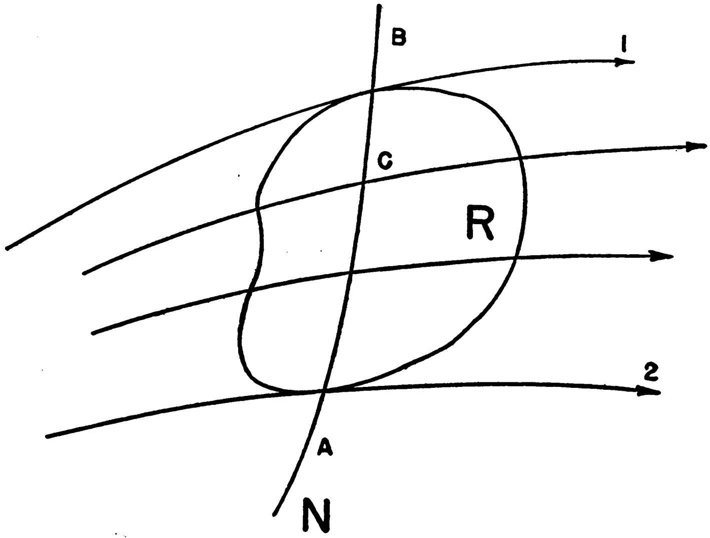


FIG. 1.

This situation also obtains in the incompressible case, as may be seen by referring to Eq. (7.16). When $q$ is constant within $R$ and changes suddenly at the boundaries of $R$ to zero, then only along the streamlines (1) and (2) tangent to the boundary will $\omega$ change in the incompressible case. The flow which has passed through $R$ will have been accelerated (or decelerated) leaving vortex lines (1) and (2) to effect the velocity jump to the flow outside (1) and (2). The heated region will therefore create a disturbance not unlike the wakes that have been studied in adiabatic flow.

## IV. APPENDIX

**10. Unit-vector formulas.** Let $\mathbf{s}$ be defined as the unit vector in the direction of flow.

$$\mathbf{s} = \mathbf{W}/W = \mathbf{V}/V \tag{10.1}$$

and $\mathbf{n}$ be a second unit vector in the plane of the flow and normal to $\mathbf{s}$ such that

$$\mathbf{n} = \mathbf{k} \times \mathbf{s} \tag{10.2}$$

from which it follows that

$$\mathbf{k} = \mathbf{s} \times \mathbf{n}$$
$$\mathbf{s} = \mathbf{n} \times \mathbf{k} \tag{10.3}$$

After the abbreviations

$$\mathbf{s}\cdot\nabla = \frac{\partial}{\partial s}, \qquad \mathbf{n}\cdot\nabla = \frac{\partial}{\partial n} \tag{10.4}$$

are introduced, it can be shown that

$$\mathbf{s} \times \nabla = \mathbf{k} \frac{\partial}{\partial n}, \tag{10.5}$$

$$\mathbf{n} \times \nabla = -\mathbf{k} \frac{\partial}{\partial s}, \tag{10.6}$$

$$\nabla \times (B\mathbf{n}) = \mathbf{k}\nabla \cdot (B\mathbf{s}), \tag{10.7}$$

where $B$ is an arbitrary scalar function. These are the equations used in the development of (7.11).

It can also be shown that

$$\nabla \cdot \mathbf{n} = -\mathbf{k} \cdot \nabla \times \mathbf{s}, \tag{10.8}$$

$$\nabla \times \mathbf{n} = \mathbf{k}\, \nabla \cdot \mathbf{s}, \tag{10.9}$$

and these quantities can be re-expressed in terms of derivatives of $\theta$, the angle of inclination of $\mathbf{s}$ to the $x$ axis. Thus

$$\mathbf{s} = \mathbf{i} \cos \theta + \mathbf{j} \sin \theta, \tag{10.10}$$

$$\mathbf{n} = -\mathbf{i} \sin \theta + \mathbf{j} \cos \theta, \tag{10.11}$$

$$\nabla \cdot \mathbf{s} = -\theta_x \sin \theta + \theta_y \cos \theta = \mathbf{n} \cdot \nabla \theta \tag{10.12}$$

or

$$\nabla \cdot \mathbf{s} = \frac{\partial \theta}{\partial n}. \tag{10.13}$$

Similarly

$$\nabla \times \mathbf{s} = \mathbf{k}\,(\theta_x \cos \theta + \theta_y \sin \theta) = \mathbf{k}\mathbf{s} \cdot \nabla \theta$$

and

$$\nabla \times \mathbf{s} = \mathbf{k} \frac{\partial \theta}{\partial s}. \tag{10.14}$$

## Concluding Remarks

Three representations have been studied of steady fields of flow of an inviscid compressible fluid that contains distributed heat sources. Of these representations, that afforded by the vector $\mathbf{W} = \mathbf{V}/V_t$, where $\mathbf{V}$ is the velocity vector and $V_t$ is the (variable) limiting velocity appears to be the most convenient for rotational flows. Thus the intimate connection between vorticity $\omega = |\nabla \times \mathbf{W}|$ and the heating factor $q = Q/V_t^3$ ($Q$ is the energy added per unit time and mass) is shown in $W$ language by the equation for uniplanar flow

$$2W^{-1}q\omega + \omega \frac{\partial}{\partial s} \log (\omega/p) = q \frac{\partial}{\partial n} \log [q/(1 - W^2)]$$

in which $\partial/\partial s$ and $\partial/\partial n$ denote spatial differentiation parallel and perpendicular to streamlines and $p$ is the (static) fluid pressure.

It has proved to be convenient to broaden the concept of irrotational flow in order to include diabatic flow fields in which $\mathbf{V}$, $\mathbf{W}$ or $\mathbf{M} = \mathbf{V}/a$ ($a$ is the local velocity of sound) is irrotational. Although the character of the corresponding heat source distributions is decidedly different in the three types of irrotational flow, the partial differential equations for the potential functions $\varphi_v$ , $\varphi_W$ and $\varphi_M$ are all quasilinear and, except in the case of $\varphi_v$ each contains an arbitrary function of the potential function, which is a consequence of the diabatic nature of the flow.

The question as to whether other representations than the three here studied may also be of interest in investigations of diabatic flow will be discussed in a subsequent paper describing the characterization of fields of diabatic flow.

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