# THE RADIATION AND TRANSMISSION PROPERTIES OF A PAIR OF PARALLEL PLATES-II*

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**1. Introduction.** In Part I, we discussed the excitation of a pair of semi-infinite parallel plates by an electromagnetic plane wave which had only one component of the electric field. We now reverse our discussion in order to find the effect of a parallel plate mode on the excitation. In Fig. 1 we assume that the structure has been excited for $z \ll 0$ by a mode of the form $e^{i\kappa z} \sin (\pi x/a)$, that is a parallel plate mode traveling to the right. A reflected mode is excited at the mouth of the parallel plate region and has the form $e^{-i\kappa z} \sin (\pi x/a)$ for $z \ll 0$. We are now interested in calculating the reflection coefficient, that is the ratio of the amplitude of the reflected parallel plate mode $e^{-i\kappa z}$ $\sin (\pi x/a)$ to the amplitude of the incident one, $e^{i\kappa z} \sin (\pi x/a)$. Because of the presence of certain symmetries we shall find that this problem may be formulated as a single integral equation. We shall find that much of the mathematical technique which we developed in Part I carries over, so that we shall not have to discuss this part in great detail. The author wishes to thank Dr. J. F. Carlson and Dr. J. S. Schwinger for several stimulating discussions on this problem.

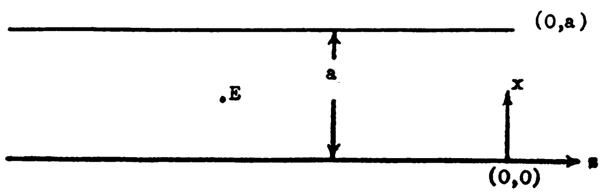


FIG. 1.

**2. The formulation of the problem.** The structure in Fig. 1 has been excited symmetrically about the line $x = a/2$ by an electric field of the form $E_y(x, z) = \rho_1 e^{+i\kappa z}$ $\sin (\pi x/a)$. As such, no even modes containing factors of the form $\exp\{z[(4m^2\pi^2/a^2) - k^2]^{1/2}\} \sin (2m\pi x/a)$, $z \ll 0$ will be excited in the parallel plate region.[1] The line $x = a/2$ is then a line of maximum electric intensity or zero tangential magnetic intensity. Figure 1 may therefore be replaced by Fig. 2. The condition for $0 \leq x \leq a/2$, $z \ll 0$

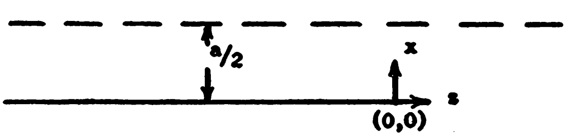


FIG. 2.

still holds. That is, $E_y(x, z)$ is asymptotic to $\sin (\pi x/a)(\rho_1 e^{i\kappa z} + \rho_2 e^{-i\kappa z})$. On the plane

*Received March 29, 1947. Presented to the American Mathematical Society on Dec. 27, 1946. Part I of this paper appeared in this Quarterly 5, 157-166 (1948).

[1]It is assumed as in Part I that only the lowest mode propagates.

$x = 0, z < 0$, $E_v(x, z)$ vanishes, while on the plane $x = a/2$, $-\infty < z < \infty$, $\partial E_v/\partial x$ vanishes identically in $x$ and $z$. Since there is no plane wave field external to the parallel plate region, $E_v(x, z)$ is at most of the form of a radiated field and is asymptotic to $e^{ikr}/r^{1/2}$ for $r \gg 0$, where $r = (x^2 + z^2)^{1/2}$.

We first divide the half plane $x < a/2$, $-\infty < z < \infty$ into two regions, the strip $-\infty < z < \infty, 0 \leq x \leq a/2$, and the half plane $-\infty < z < \infty, x < 0$. It is now possible to express $E_v(x, z)$ in each region in terms of an appropriate Green's function and $E_v(0, z)$. Let us first look at the half plane $x \leq 0$, $-\infty < z < \infty$. We have here

$$E_v(x, z) = -\int_0^\infty E_v(0, z') \frac{\partial G^{(1)}}{\partial x'}(x, z, 0, z')\, dz', \tag{2.1}$$

where

$$G^{(1)}(x, z, x', z') = \frac{i}{4}\left[H_0^{(1)}\{k[(x - x')^2 + (z - z')^2]^{1/2}\} - H_0^{(1)}\{k[(x + x')^2 + (z - z')^2]^{1/2}\}\right]$$

and $H_0^{(1)}$ is the usual notation for the Hankel function. This Green's function is quite similar to the one which we employed in Part I. However, we note that we have subtracted a source free term from the first term and this has the effect of making $G^{(1)}$ vanish for $x = 0$. Thus, noting boundary conditions for $E_v(x, z)$ and $G^{(1)}$ on the line $x = 0$, we are left with (2.1).

As for the strip, we choose a Green's function $G^{(2)}(x, z, x', z')$ which vanishes for $x = 0$ and whose normal derivative vanishes for $x = a/2$. Such a Green's function has the form

$$G^{(2)}(x, z, x', z') = \frac{2}{\pi}\sum_{n=1}^{\infty} \sin\frac{(2n + 1)\pi x}{a} \sin\frac{(2n + 1)\pi x'}{a} \exp\{-\,|z - z'|\pi\kappa_n/a\}/\kappa_n + \frac{2i}{\kappa a}\sin\frac{\pi x}{a}\sin\frac{\pi x'}{a}\exp\{i\kappa\,|z - z'|\}$$

where

$$\kappa_n = \{(2n + 1)^2 - (ak)^2/\pi^2\}^{1/2}.$$

Upon applying Green's theorem, with $G^{(2)}$ as a kernel, to the strip $-\infty < z < \infty$, $0 \leq x \leq a/2$ and noting the form of $E_v(x, z)$ for $|z| \to \infty$, we get

$$E_v(x, z) = \int_0^\infty E_v(0, z') \frac{\partial G^{(2)}}{\partial x'}\, dz' + \rho_1 \sin\frac{\pi x}{a} e^{i\kappa z}, \tag{2.2}$$

where $G^{(2)}$ has been evaluated at $x' = 0$.

Equation (2.2) contains the definition of the reflection coefficient which we desire. For $z$ large and negative $E_v(x, z)$ is asymptotic to $(\rho_1 e^{i\kappa z} + \rho_2 e^{-i\kappa z}) \sin(\pi x/a)$. But from Eq. (2.2), we see that this is equal to

$$\rho_1 \sin\frac{\pi x}{a} e^{i\kappa z} + \frac{2i\pi}{a^2\kappa}\int_0^\infty E_v(0, z') \sin\frac{\pi x}{a} e^{i\kappa(z' - z)}\, dz'.$$

From this it follows that

$$\rho_2 = \frac{2i\pi}{a^2\kappa}\int_0^\infty E_v(0, z')e^{i\kappa z'}\, dz'.$$

We note that $\rho_2$ is the unilateral Fourier transform of $E_v(0, z)$ evaluated at $-\kappa$. In view of the fact that we shall solve our problem by Fourier methods, this quantity will appear directly.

We can now form the desired integral equation by observing that the $z$ component of the magnetic field is continuous on the surface $x = 0, z > 0$. On this surface we have then

$$\int_0^\infty E_v(0, z')\left[\frac{\partial^2 G^{(1)}}{\partial x\, \partial x'} + \frac{\partial^2 G^{(2)}}{\partial x\, \partial x'}\right] dz' + \frac{\rho_1 \pi}{a} e^{i\kappa z} = 0, \tag{2.3}$$

where $x$ and $x'$ are now evaluated at zero. Eq. (2.3) is of the Wiener-Hopf type because of the limits on the integral and the particular $z$ dependence of the Green's function. For analytical convenience, we assume that $k$ has a small positive imaginary part.

**3. The Fourier transform solution of Eq. (2.3).** Following Wiener and Hopf we extend Eq. (2.3) for all $z$ to read

$$\phi(z) = \int_{-\infty}^\infty E_v(0, z')\left[\frac{\partial^2 G^{(1)}}{\partial x\, \partial x'} + \frac{\partial^2 G^{(2)}}{\partial x\, \partial x'}\right] dz' + \phi_0(z), \tag{3.1}$$

where the $x$ and $x'$ in the Green's functions are evaluated at zero. We define

$$E_v(0, z) \equiv 0 \qquad z < 0,$$

$$\phi_0(z) \equiv \frac{\pi\rho_1}{a} e^{i\kappa z} \qquad z > 0,$$

$$\equiv 0 \qquad z < 0,$$

and

$$\phi(z) \equiv 0 \qquad z > 0.$$

It is a simple matter to find the growth of $E_v(0, z)$ for $z \gg 0$, and $\phi(z)$ for $z \ll 0$. We shall verify with the solution of the problem that they are both integrable for finite $z$. $E_v(0, z)$ is asymptotic to $e^{ikz}/z^{1/2}$ for $z \gg 0$, while $\phi(z)$ is asymptotic to either $e^{-ikz}/z^{1/2}$ or $e^{-i\kappa z}$ depending upon which goes to zero more slowly for $z \ll 0$.[2] Furthermore $G^{(1)}(0, z, 0, z')$ is asymptotic to $e^{ikz}$ for $z \gg z'$ and $e^{-ikz}$ for $z \ll z'$, at least insofar as the exponential growth is concerned, while $G^{(2)}(0, z, 0, z')$ is asymptotic to $e^{i\kappa z}$ or $e^{-i\kappa z}$ depending on whether $z \gg z'$ or $z \ll z'$. With this information at hand, we can define the regions of regularity of the various Fourier transforms we shall encounter.

Consider first, the unilateral Fourier transform of $E_v(0, z)$,

$$\psi_1(w) = \int_0^\infty e^{-iwz} E_v(0, z)\, dz.$$

$\psi_1(w)$ is regular in the lower half plane $\Im m w < \Im m k$ because of the growth of $E_v(0, z)$ for $z \gg 0$. As for $\phi(z)$, we note that

[2]With appropriate restrictions on the imaginary part of $k$, we may determine the magnitude of $\Im m k$ relative to $\Im m\, \kappa$. However, as we shall see, the information is unessential.

$$\psi_2(w) = \int_{-\infty}^{0} e^{-iwz}\phi(z)\,dz$$

is regular in the upper half plane $\Im m w > -\Im m k$ (or $-\Im m \kappa$). The right side of this inequality is the smaller of the two quantities ($\Im m k$, $\Im m \kappa$). The expression $G^{(1)}(0, z, 0, z')$ will have the bilateral Fourier transform

$$g_1(w) = \int_{-\infty}^{\infty} e^{-iwz}G^{(1)}(0, z, 0, z')\,dz$$

which is regular in the strip $-\Im m k < \Im m w < \Im m k$, while the bilateral transform of $G^{(2)}(0, z, 0, z')$,

$$g_2(w) = \int_{-\infty}^{\infty} e^{-iwz}G^{(2)}(0, z, 0, z')\,dz$$

is regular in the strip $-\Im m \kappa < \Im m w < \Im m \kappa$. Upon noting the transform of $\phi_0(z)$, we see that all transforms involved in Eq. (3.1) have a common strip of regularity, $-\Im m k$ (or $-\Im m \kappa$) $< \Im m w < \Im m k$ (or $\Im m \kappa$) and it is thus permissible to apply the Fourier transform to Eq. (3.1).

We have given elsewhere the Fourier transforms of the Green's functions. For example, with $\omega = (k^2 - w^2)^{1/2}$, we have

$$\int_{-\infty}^{\infty} e^{-iwz}G^{(1)}(x, z, x', z')\,dz = -\frac{1}{\omega}\exp\{-iwz' - i\omega x'\}\sin \omega x, \qquad \text{for } x > x',$$

$$= -\frac{1}{\omega}\exp\{-iwz' - i\omega x'\}\sin \omega x', \quad \text{for } x < x',$$

and this is clearly regular in the strip $-\Im m k < \Im m w < \Im m k$. For the transform of $G^{(2)}(x, z, x', z')$, we have

$$\int_{-\infty}^{\infty} e^{-iwz}G^{(2)}(x, z, x', z')\,dz = \frac{e^{-iwz'}\sin \omega x' \cos \omega(x - a/2)}{\omega \cos(a\omega/2)} \qquad \text{for } x > x',$$

$$= \frac{e^{-iwz'}\sin \omega x \cos \omega(x' - a/2)}{\omega \cos(a\omega/2)} \qquad \text{for } x < x',$$

and this is regular in the strip $-\Im m \kappa < \Im m w < \Im m \kappa$.

If we now apply the Fourier transform theorem to Eq. (3.1) we get immediately

$$\psi_2(w) = \psi_1(w)\omega[\tan(a\omega/2) + i] + \pi\rho_1/ai(w - \kappa)$$

$$= i\omega\psi_1(w)\frac{\exp(-ia\omega/2)}{\cos a\omega/2} + \pi\rho_1/ai(w - \kappa).$$

Eq. (3.2) may now be split into two parts, one of which is regular in the appropriate lower half plane, while the other of which is regular in the appropriate upper half plane. We have already described these details in Part I, so without any further discussion we can write

$$\pi\rho_1 K_+(\kappa)/ai(w - \kappa) + i\psi_1(w)K_-(w) = E(w) \tag{3.3}$$

and

$$\pi\rho_1[K_+(w) - K_+(\kappa)]/ai(w - \kappa) - \psi_2(w)K_+(w) = E(w) \tag{3.4}$$

$E(w)$ is an integral function and

$$K_-(w) = N_-(w)/D_-(w),$$

where

$$N_-(w) = \{k - w\}^{1/2} \exp\left[-\frac{a\omega i}{\pi} \arctan\left\{\frac{k + w}{k - w}\right\}^{1/2} + \chi(w)\right]$$

and

$$D_-(w) = (w - \kappa) \prod_{n=1}^{\infty} [\{1 - (ak)^2/\pi^2(2n + 1)^2\}^{1/2} + iaw/\pi(2n + 1)] \exp\{-iaw/\pi(2n + 1)\}$$

while

$$\frac{1}{K_+(w)} = -\frac{\pi^2}{a^2} K_-(-w).$$

$\chi(w)$ is an integral function which is chosen so as to render $K_-(w)$ and $K_+(w)$ of algebraic growth in the appropriate half plane as $|w|$ becomes infinite. We have already seen that

$$\chi(w) = \frac{iaw}{2\pi}\left[\log\frac{i\pi}{ak} + 3 - \gamma\right]$$

where $\gamma$ is the Euler-Mascheroni constant. With $\chi(w)$ so chosen, $\mathfrak{Im}\, w$ in the appropriate lower half plane and $|w| \gg 0$, $K_-(w) = O(w^{1/2})$. Upon employing these asymptotic forms and noting that $\psi_1(w)$ and $\psi_2(w)$ approach zero for $|w| \gg 0$ with $w$ in the appropriate half plane, we find by a direct application of Liouville's theorem, that $E(w)$ is identically zero. Thus we have

$$\psi_1(w) = \int_0^\infty e^{-iwz} E_y(0, z)\, dz = \frac{\pi\rho_1 K_+(\kappa)}{a(w - \kappa)K_-(w)}.$$

This tells us, incidently, that $\psi_1(w) = O(w^{-3/2})$ for $|w| \gg 0$, $\mathfrak{Im}\, w$ in the appropriate lower half plane, and hence that $E_y(0, z) = O(z^{1/2})$, $z \to 0^+$.

It is now a simple task to compute the reflection coefficient. We have that

$$\rho_2 = \frac{2i\pi}{a^2\kappa} \psi_1(-\kappa) = \frac{2i\pi}{a^2\kappa} \int_0^\infty e^{i\kappa z} E_y(0, z)\, dz.$$

Hence $\rho_2/\rho_1 = R$, the reflection coefficient, is

$$R = -\frac{i\pi^2}{a^3\kappa^2} \frac{K_+(\kappa)}{K_-(-\kappa)}.$$

Let us now dispense with the assumption that $k$ has an imaginary component. We find that

$$R = i\left\{\frac{k - \kappa}{k + \kappa}\right\}^{1/2} \exp\left[2i\Theta_1 + 2i \arctan\left\{\frac{k - \kappa}{k + \kappa}\right\}^{1/2} + 2\chi(\kappa)\right]$$

where $\Theta_1$ has been defined in Part I. Here $\pi < ak < 2\pi$, and $a^2\kappa^2 = a^2k^2 - \pi^2$.

**4. Some remarks on the method employed.** We have discussed and solved in this series of papers, a group of free space problems in electromagnetic theory. Because of the peculiar geometry, it was always possible to formulate this set of problems as integral equations which are closely related to the Wiener-Hopf type. Despite the subtle difference which exists between the integral equation we have treated and the original Wiener-Hopf theory, the mathematical machinery carries over. It is worth noting here that the class of integral equations we solve here, for example Eq. (3.1), belongs to the inversion formula type. We shall pursue this remark elsewhere.

It is clear that in order to apply Fourier methods to this class of integral equations we must be certain that the various functions involved have proper growths at infinity, as well as in any finite interval. An examination of the integral equations we have solved reveals that integrability over any finite interval is demanded if the equations have been properly formulated. The same remark holds true for conditions at infinity. Indeed, on the basis of the integrals we use to calculate certain physical parameters, the integrability condition at the origin would enter immediately. As for the asymptotic form of the various field quantities, the physics of the situation dictates the precise form that we require. Every Fourier transform solution we have obtained can be readily shown to have appropriate properties and therefore the field quantities also do.