A PROBLEM CONCERNING ORTHOGONAL TRAJECTORIES*

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1. Introduction. The purpose of this note is to discuss a class of conjugate harmonic functions u, v which possess a unique geometric property. This property will be defined precisely in Sec. 2. For the present we shall describe it in terms relative to a two-dimensional steady state heat flow problem.

Let S be a given region in a plane which we choose to call the uv-plane, and let C be the boundary of the region. Consider a family of heat flow lines $f(u, v) = c_1$ together with the isothermal lines $g(u, v) = c_2$ in the region S. Suppose we desire to make a plot of a set of curves $f_1, f_2, f_3, \cdots$ from the family $f(u, v) = c_1$ and a set $g_1, g_2, g_3, \cdots$ from the family $g(u, v) = c_2$. For this purpose it is often convenient to choose one of the heat flow lines (or isothermal lines), whose position can be determined in advance from some symmetry property of the boundary C, as a reference line for construction of the remaining curves. This heat flow reference line is divided into segments and the end points of the segments taken as origins for the isothermal lines. Then by a process of numerical integration the isothermal lines and the remaining heat flow lines are plotted for the entire region.[1]

The following problem then arises: Let f_1 be the reference heat flow line and suppose three isothermal lines g_1, g_2, g_3 are chosen to intersect f_1 at segments of equal arc length measured along f_1. Under what conditions will the isothermal lines g_1, g_2, g_3 intersect all remaining heat flow lines in S at segments of equal arc length?

In the following sections we shall determine necessary and sufficient conditions for a family of isothermals to have this property and, moreover, determine all such isothermals.

2. Notations and definitions. We shall assume throughout this paper that we deal with functions having continuous derivatives of the second order; whenever a function occurs as a denominator we exclude points at which this function vanishes. A function $\omega(x, y)$ will be called *separable* if $\omega(x, y) = h(x)k(y)$. It is easily shown that $\omega(x, y)$ is separable if and only if $\omega\omega_{xy} - \omega_x\omega_y = 0$, where here, and in the following, subscripts x and y denote partial derivatives.

It seems advantageous to replace the letters c_1 and c_2 of Sec. 1 by x and y, respectively, and to deal with the inverse functions of $f(u, v) = x$ and $g(u, v) = y$. Thus let

$$u = u(x, y), \qquad v = v(x, y) \tag{1}$$

be defined in a region R of the xy-plane. The relations (1) represent a mapping of R on a region S of the uv-plane. We shall denote the curves in S obtained by putting $y =$ const in (1) as the family $\mathfrak{X}$, and the curves obtained by putting $x =$ const as the family $\mathfrak{Y}$. In other words, (1) maps the lines parallel to the x-axis into the family $\mathfrak{X}$ and the lines parallel to the y-axis into the family $\mathfrak{Y}$.

We shall use the usual notation for the three fundamental coefficients $E = u_x^2 + v_x^2$, $F = u_xu_y + v_xv_y$, $G = u_y^2 + v_y^2$. It is obvious that the families $\mathfrak{X}$ and $\mathfrak{Y}$ remain invariant under any substitution $x = \phi(x_1)$, $y = \psi(y_1)$ in (1); moreover, in this manner

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[1]See, for example, W. H. McAdams, *Heat transmission*, 2nd ed., McGraw-Hill, New York, 1942, pp. 16-17.

all possible representations of the form (1) of the families $\mathfrak{X}$ and $\mathfrak{Y}$ are obtained. If E_1, F_1, G_1 are the three fundamental coefficients in terms of the new variables x_1, y_1 then

$$E_1 = [\phi'(x_1)]^2 E, \qquad F_1 = \phi'(x_1)\psi'(y_1)F, \qquad G_1 = [\psi'(y_1)]^2 G. \tag{2}$$

The necessary and sufficient condition for orthogonality of $\mathfrak{X}$ and $\mathfrak{Y}$ is $F = 0$. Two families of orthogonal trajectories $\mathfrak{X}$ and $\mathfrak{Y}$ are called *isothermic trajectories* or *isothermals* if it is possible to make substitutions $x = \phi(x_1)$ and $y = \psi(y_1)$ in (1) such that u and v become conjugate harmonic functions in x_1 and y_1, i.e., such that $w = u + iv$ becomes an analytic function of $z_1 = x_1 + iy_1$. As is well-known, two families of orthogonal trajectories $\mathfrak{X}$ and $\mathfrak{Y}$, given by (1), are isothermic if and only if E/G is separable.[2]

We shall now define for orthogonal trajectories the property which is described in Sec. 1 and which is the fundamental concept of this paper. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two families of orthogonal trajectories. We shall say that one of them, say $\mathfrak{X}$, is *of proportional arc length* if the following condition is satisfied: Let Y_1, Y_2, Y_3 be any three curves of $\mathfrak{Y}$, and let s_1 be the arc length of any curve of $\mathfrak{X}$ between Y_1 and Y_2 and s_2 be the arc length of the same curve between Y_2 and Y_3. Then the ratio s_1/s_2 is to be constant for all curves of $\mathfrak{X}$, i.e., dependent only on the choice of Y_1, Y_2, Y_3. It is obvious that we might express this property of $\mathfrak{X}$ also in the following way: Any three trajectories of $\mathfrak{Y}$ which cut out segments of equal arc length from one curve of $\mathfrak{X}$ do the same for any other curve of $\mathfrak{X}$. In an analogous manner we define what is meant for $\mathfrak{Y}$ to be of proportional arc length.

3. A condition for proportional arc length. We shall prove the following theorem, which gives a necessary and sufficient condition for proportional arc length.

THEOREM 1. *If $\mathfrak{X}$ and $\mathfrak{Y}$, defined by* (1), *are orthogonal trajectories, then $\mathfrak{X}$ is of proportional arc length if and only if $E = u_x^2 + v_x^2$ is separable.*[3] *Similarly, $\mathfrak{Y}$ is of proportional arc length if and only if $G = u_y^2 + v_y^2$ is separable.*

Let three trajectories of $\mathfrak{Y}$, corresponding to the values $x = x_0$, $x = x_1$, $x = x_2$, be given and consider any two trajectories of $\mathfrak{X}$, corresponding to the values $y = y_1$ and $y = y_2$. The element of arc length along any curve of $\mathfrak{X}$ is given by $ds = q(x, y)\, dx$, where we write $q(x, y) = [E(x, y)]^{1/2} = (u_x^2 + v_x^2)^{1/2}$. If $\mathfrak{X}$ is of proportional arc length we have, by the definition given above (see Sec. 2):

$$\int_{x_0}^{x_1} q(x, y_1)\, dx : \int_{x_0}^{x_2} q(x, y_1)\, dx = \int_{x_0}^{x_1} q(x, y_2)\, dx : \int_{x_0}^{x_2} q(x, y_2)\, dx; \tag{3}$$

this has to hold for all values of x_0, x_1, x_2 and y_1, y_2, restricted only by the region of definition R. We conclude that the function

$$\int_{x_0}^{x_1} q(x, y)\, dx \Big/ \int_{x_0}^{x_2} q(x, y)\, dx \tag{4}$$

is independent of y. If we consider x_0 and x_2 as fixed, then (4) becomes a function of x_1 alone, which we shall call $h(x_1)$. The denominator of (4) is a function of y alone, say $k(y)$. Hence we have

[2]See G. Scheffers, *Anwendung der Differential- und Integralrechnung auf Geometrie,* Walter de Gruyter, Berlin und Leipzig, 1923, vol. 1, 3rd ed., p. 170.

[3]The separability of E is independent of the particular representation (1) of $\mathfrak{X}$ and $\mathfrak{Y}$. This can be seen directly from (2).

$$\int_{x_0}^{x_1} q(x, y)\, dx = h(x_1)k(y). \tag{5}$$

Differentiating with respect to x_1 and replacing x_1 by x, we obtain $q(x, y) = h'(x)k(y)$. Thus $E = q^2$ is separable and the necessity of the condition is proved. To prove its sufficiency we note that (5) follows at once from the separability of E and that (5) immediately implies the proportion (3). This completes the proof of Th. 1.

The question at once arises as to whether there are families of orthogonal trajectories in which $\mathfrak{X}$ is of proportional arc length but $\mathfrak{Y}$ is not. We shall see below (Th. 2) that this can only occur if $\mathfrak{X}$ and $\mathfrak{Y}$ are non-isothermic. In this case the conditions

$$E \text{ separable}, \qquad F = 0, \qquad G \text{ non-separable}, \tag{6}$$

are compatible. We shall give two examples of such trajectories.

I. Let $u = y + e^{x-y}$, $v = e^{(x-y)/2}(1 - e^{x-y})^{1/2} + \sin^{-1} e^{(x-y)/2}$. The region R is taken as the half-plane above the line $y = x$, so that $0 < e^{x-y} < 1$ for (x, y) in R. It is easily verified that $E = e^{x-y}$, $F = 0$, $G = 1 - e^{x-y}$, and hence (6) is satisfied.

II. Let $p(x)$, $q(y)$ and $r(y)$, defined for $a < x < b$ and $c < y < d$, respectively, be such that neither $p(x)$ nor $q(y)$ nor $r(y)/q'(y)$ is constant. Let

$$u = p(x) \cos q(y) - \int r(y) \sin q(y)\, dy,$$

$$v = p(x) \sin q(y) + \int r(y) \cos q(y)\, dy.$$

The region R is the rectangle $a < x < b$, $c < y < d$. We obtain $E = [p'(x)]^2$, $F = 0$, and $G = [p(x)q'(y) + r(y)]^2$. In order to show that (6) is satisfied, we only have to convince ourselves that G is not separable. We apply the separability test (see Sec. 2) to $\omega(x, y) = p(x)q'(y) + r(y)$ and obtain $\omega\omega_{xy} - \omega_x\omega_y = -p'(x)[q'(y)]^2\,(d/dy)[r(y)/q'(y)]$, which is different from zero by the conditions mentioned at the beginning of this example.

We remark that in the second example E is actually a function of x alone. The geometric interpretation of this special situation is that the family $\mathfrak{X}$ is not only of proportional arc length but, as we might say, of *equal* arc length, i.e., any two fixed curves of $\mathfrak{Y}$ cut out a segment of the same arc length from all curves of $\mathfrak{X}$.

4. Isothermals. Before taking up the main problem of this paper (see Sec. 1), we prove the following theorem.

THEOREM 2. *If two families of orthogonal trajectories are both of proportional arc length then they are isothermals. Conversely, if one of two families of isothermals is of proportional arc length so is the other.*

If two of the three functions E, G, and E/G are separable, so is the third. Hence Th. 2 follows at once from Th. 1 and from the fact (see Sec. 2) that the separability of E/G is a sufficient and necessary condition for isothermals.

The second part of Th. 2 allows us to speak of "isothermals of proportional arc length", without our having to mention the families $\mathfrak{X}$ or $\mathfrak{Y}$ specifically. Our aim is to find the totality of all isothermals of proportional arc length. In preparation of the solution of this problem we shall first make the following remarks.

In the first place, we may assume right from the start that the functions u and v in (1), which define our isothermals, are real and imaginary parts, respectively, of an analytic function $w = F(z)$, where $z = x + iy$ (see Sec. 2). Secondly, it is obvious that a substitution of the form $z = \alpha z_1 + \beta$, $\alpha \neq 0$, will leave the two families of isothermals invariant if α is real, and that it will merely interchange the two families with one another if α is purely imaginary. Thirdly, we shall say that one pair of families of isothermals is *of the same type* as another pair if the second pair can be obtained from the first one by a rotation, expansion (contraction) and translation of the w-plane, i.e., by a transformation $w = aw_1 + b$, where a is any complex number different from zero. To summarize, we shall consider two pairs of families of isothermals to be of the same type if the analytic functions $w = F(z)$ and $w_1 = G(z_1)$ defining them are related to one another in the following manner:

$$\begin{aligned} w &= aw_1 + b, \qquad a \neq 0, \\ z &= \alpha z_1 + \beta, \qquad \alpha \neq 0, \qquad \alpha \text{ real or purely imaginary.} \end{aligned} \tag{7}$$

We are now ready to state the principle result of our paper.

THEOREM 3. *The totality of different types of isothermals of proportional arc length are those obtained from one of the following functions*: (i) $w = z$, (ii) $w = e^z$, (iii) $w = \exp(ze^{i\gamma})$, *where* $0 < \gamma < \pi/2$, *and* (iv)

$$w = \int_0^z \exp(-\zeta^2)\, d\zeta.$$

Let $w = F(z) = u(x, y) + iv(x, y)$ be such that the isothermals defined by $F(z)$ are of proportional arc length. Since $E = G = | F'(z) |^2$ we conclude from Th. 1 that $| F'(z) |$ is separable and hence $\log | F'(z) | = M(x) + N(y)$. Now $\log | F'(z) |$ is a harmonic function and, therefore, $M''(x) + N''(y) = 0$ so that $M''(x) = -N''(y) = 2A$, where A is a real constant. We thus obtain $M(x) = Ax^2 + B_1x + C_1$ and $N(y) = -Ay^2 + B_2y + C_2$ with real B_1, B_2, C_1, C_2. Thus $\log | F'(z) | = A(x^2 - y^2) + B_1x + B_2y + C_3$, where $C_3 = C_1 + C_2$, and hence $\log F'(z) = Az^2 + Bz + C$, where $B = B_1 - iB_2$ and C is any complex number whose real part equals C_3. We thus obtain $F'(z) = \exp(Az^2 + Bz + C)$ with real A and distinguish between the following cases.

(i) $A = 0$, $B = 0$. We obtain $w = F(z) = mz + p$ with $m = e^C$. The substitutions (7) with $w = mw_1 + p$, $z = z_1$ give $w_1 = z_1$. Thus all isothermals of Case (i) are of the same type, namely, $w = z$. The families $\mathfrak{X}$ and $\mathfrak{Y}$ consist of lines parallel to the real and imaginary axes, respectively. Obviously, both families are trajectories of equal arc length, in the sense defined above at the end of Sec. 3.

(ii) $A = 0$, $B \neq 0$, B real or purely imaginary. We obtain $w = F(z) = me^{Bz} + p$, where $m = B^{-1}e^C$. The substitutions (7) with $w = mw_1 + p$, $z_1 = Bz$ give $w_1 = \exp(z_1)$. Thus all isothermals of Case (ii) are of the same type, namely, $w = e^z$. The family $\mathfrak{X}$ consists of rays from the origin to the point at infinity, the family $\mathfrak{Y}$ of circles about the origin. The family $\mathfrak{X}$ is again of equal arc length but the family $\mathfrak{Y}$ is not.

(iii) $A = 0$, B neither real nor purely imaginary. As in Case (ii) we obtain $w = F(z) = me^{Bz} + p$ with $m = B^{-1}e^C$. We now write $B = | B | \exp[i(\gamma + \pi n/2)]$, where n is an integer and $0 < \gamma < \pi/2$. The substitutions (7) with $w = mw_1 + p$, $z_1 = | B | z \exp(i\pi n/2)$ give $w_1 = \exp(z_1 e^{i\gamma})$. Therefore, every pair of isothermals of Case (iii) is of the same type as $w = \exp(ze^{i\gamma})$ for an appropriate value of γ, $0 < \gamma < \pi/2$. The families $\mathfrak{X}$ and $\mathfrak{Y}$ may be written in polar coordinates r, θ as follows:

$$\mathfrak{X}: r = \exp(\theta \cot \gamma - y \csc \gamma), \qquad y = \text{const},$$

$$\mathfrak{Y}: r = \exp(-\theta \tan \gamma + x \sec \gamma), \qquad x = \text{const}.$$

Each of these families consists of logarithmic spirals about the origin; the curves of $\mathfrak{X}$ intersect the rays through the origin at the angle γ and those of $\mathfrak{Y}$ at the angle $\gamma - \pi/2$. Thus for two different values of γ, $0 < \gamma < \pi/2$, two different types of isothermals are obtained.[4]

(iv) $A \neq 0$. We put $A = -\nu^2$, so that ν is real or purely imaginary since A is real. We write

$$w = F(z) = \int_{B/2\nu^2}^{z} \exp(-\nu^2\zeta^2 + B\zeta + C)\, d\zeta + p.$$

The substitutions (7) with $w = p + (w_1/\nu) \exp(\Delta/4\nu^2)$, $z_1 = \nu z - B/2\nu$, where $\Delta = B^2 + 4\nu^2 C$, give

$$w_1 = \int_0^{z_1} \exp(-\zeta^2)\, d\zeta.$$

Thus all isothermals of Case (iv) are of the same type, namely,

$$w = \int_0^{z} \exp(-\zeta^2)\, d\zeta = \operatorname{Erf}(z).$$

We shall not attempt here to describe these isothermals in detail. We merely restrict ourselves to stating that the curves of $\mathfrak{X}$ approach $\pm(\pi)^{1/2}/2$ as $x \to \pm\infty$, while the curves of $\mathfrak{Y}$ approach the point at infinity as $y \to \pm\infty$. A brief discussion of the complex error function is given by Whittaker and Watson;[5] a more detailed discussion can be found in a recent monograph by Rosser.[6]

5. Physical interpretations. We return to the notation of Sec. 1 and assume that $f(u, v) = c_1$ are the heat flow lines and $g(u, v) = c_2$ are the isothermal lines of a two-dimensional steady state heat flow problem. Let F be the flux of heat across any isothermal line g_i. The flux will in general vary from point to point along g_i. Let F_1 be the flux across g_i in the direction of f_1, F_2 in the direction of f_2, etc. Then for two neighboring isothermal lines g_1 and g_2, defined by $g(u, v) = c$ and $g(u, v) = c + \Delta c$, respectively, where Δc is small, the value of the flux is given approximately by

$$F_1 = -k\frac{\Delta c}{\Delta n_1}, \qquad F_2 = -k\frac{\Delta c}{\Delta n_2},$$

where k is the thermal conductivity and where Δn_i represents the distance between the curves g_1 and g_2, measured along the curves f_i, $i = 1, 2$. Now suppose the isothermals are of proportional arc length. Then the ratio F_1/F_2 will be the same for all isothermals g_i.

[4]For an example in fluid dynamics involving these isothermals, see G. Hamel, *Spiralförmige Bewegungen zäher Flüssigkeiten*, Jber. Deutschen Math. Verein. **25**, 34-60 (1917).

[5]E. T. Whittaker and G. N. Watson, *Modern analysis*, Macmillan and Co., New York, 1947, p. 341.

[6]J. B. Rosser, *Theory and Application of*

$$\int_0^z \exp(-x^2)\, dx \qquad \textit{and} \qquad \int_0^z \exp(-p^2y^2)\, dy \int_0^y \exp(-x^2)\, dx,$$

Mapleton House, Brooklyn, 1948.

Obviously, the problem as described above in terms of flux of heat can be restated in terms of other physical quantities such as potential in an electrostatic field, velocity potential in fluid flow, gravitational potential, etc.

For example, consider a potential function $f(u, v) = c$ and a vector force function $\mathbf{F}(u, v)$. Let $f(u, v) = c$ and $f(u, v) = c + \Delta c$ be two neighboring equipotential curves. Then $\mathbf{F}(u, v)$ acts in a direction orthogonal to the equipotential curves and its magnitude is given by $F = \Delta c/\Delta n$, where Δn denotes the distance between the two curves. That is, the magnitude of $\mathbf{F}$ is inversely proportional to the distance between the two curves. Hence if $f_1, f_2, f_3, \cdots$ are a set of equipotential curves, and g_1, g_2 any two lines of force and if $\mathbf{F}_1$ acts along g_1, $\mathbf{F}_2$ along g_2 the property of proportional arc length leads to $F_1/F_2 = \text{const.}$

BOOK REVIEWS

Mathematics our great heritage. Selected and Edited by William L. Schaaf, Ph.D. Harper & Brothers, New York. xi + 291 pp. $3.50.

This book is a collection of essays on mathematics and its relationships to other fields of endeavor. including the fine arts, philosophy, experimental science and technology. The authors include both eminent mathematicians and those whose primary fields are not pure mathematics, but whose work brings them in close contact with mathematics, and has induced them to examine the relations between mathematics and other activities. The essays are on a high level, and should be of particular interest to applied mathematicians.

P. S. Symonds

The principles of quantum mechanics. By P. A. M. Dirac. Oxford, at the Clarendon Press, 1947. xii + 311 pp. $9.00.

This work, which has always been a cornerstone in the theoretical foundation of Quantum Mechanics, retains all of its stature in this new edition. Substantially the same subject matter is covered as in the two earlier editions. However, it is apparent that the new edition has been very largely rewritten. The author has changed his notation, but the new notation is almost self-explanatory and should cause little difficulty. The new "bra" and "ket" vector notation for states allows a more direct connection to be made between the abstract algebra of states and observables, and the theory of representations. At the same time, the author has put the formal algebra of states and observables into neater and more elegant form.

Other significant changes are in the presentation of the theory of systems containing identical particles, which is in a somewhat simpler form in this new edition, and in the treatment of quantum electrodynamics, which has been carried a little further than in the previous edition. However, the problem of divergent solutions remains, and the theory cannot be carried to a satisfactory conclusion.

D. F. Hornig

Mechanical behavior of high polymers. By Turner Alfrey, Jr. Interscience Publishers, Inc., New York, 1948. xiv + 580 pp. $9.50.

While the number of books dealing with deformation and strength of materials considered as an elastic continuum is growing like mushrooms, no book has so far been written in which the mechanical behavior of real materials (which, being formed by aggregation of particles into groups, are neither