4. In the process of numerical integration, to reach a point P_3 from a knowledge of the solution along the curve P_1P_2 (Fig. 1) one may use the set of equations (6) to calculate x_3, t_3, u_3, and p_3, since q^* and A are supposed to be known functions. The

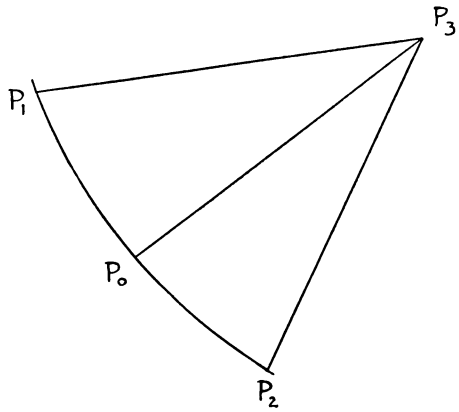

FIG. 1.

path line P_0P_3 can then be drawn in accordance with the second equation of (5); the first equation of (5) will give the value of s_3. One can then calculate ρ_3 from (4) and obtain all of the dynamic and thermodynamic variables at P_3. Iteration processes can be carried out in the usual manner.

A NEW SUPERPOSITION PRINCIPLE FOR STEADY GAS FLOWS*

BY R. C. PRIM** (*Naval Ordnance Laboratory*)

This paper is concerned with steady flows in the absence of extraneous fields of force of a frictionless, thermally-nonconducting gas having a product equation of state, i.e., an equation of state connecting density, pressure, and specific entropy in the form $\rho = P(p)S(s)$.

H. Poritsky [1]† has discussed the construction of steady, spatial gas flow solutions from steady plane flow solutions by the superposition of a uniform velocity field normal to the given plane flow field. In particular, he pointed out that if

$$\mathbf{V}_p = \mathbf{i}u(x, y) + \mathbf{j}v(x, y) \tag{1}$$

is a plane velocity field (referred to ordinary rectangular Cartesian coordinates x, y, z with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) satisfying the equations of steady-state gas dynamics, then the spatial velocity field

$$\mathbf{V} = \mathbf{V}_p + \mathbf{k}V_n, \tag{2}$$

where V_n is a *constant,* also satisfies those equations.

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†Numbers in square brackets refer to the bibliography at the end of the paper.

The validity of this superposition principle follows at once from (Newtonian) relativity considerations. In reference to an observer having a uniform velocity $-\mathbf{k}V_n$ with respect to the flow field (1), this field has the velocity vector (2). It should be noted that it is essential that the given field (1) not depend on z, otherwise the flow field with respect to the moving observer will be unsteady.

The basic superposition method of deriving spatial flow fields from a given plane flow solution will now be given a new order of power by making use of the Substitution Principle for steady gas flow.

This Substitution Principle was first established by Munk and Prim [2] for the case of gases having constant specific heats, and was extended by the present author [3] to the broader class of gases having a product equation of state, $\rho = P(p)S(s)$. It can be stated briefly as follows. *If* $\mathbf{V}$, ρ *and* p *are, respectively, the velocity, density, and pressure of a possible flow of a given gas, then* $m\mathbf{V}$, ρ/m^2 *and* p *are the corresponding quantities of another possible flow of the same gas, provided only that* $\mathbf{V}\cdot\text{grad } m = 0$, *that is, that* m *is constant along each individual streamline.*

The members of a family of flows related by this Substitution Principle clearly share the same streamline pattern and pressure field and, as may easily be verified, they have also a common reduced velocity field $\mathbf{W}$. ($\mathbf{W}$ is defined as $\mathbf{V}/a$, where a is the ultimate velocity magnitude attainable on a given streamline by expansion to zero pressure, $a^2 = V^2 + 2h$, with h the specific enthalpy.) In terms of a given reduced velocity field, the Substitution Principle manifests itself in the arbitrary assignability of the ultimate velocity a for each streamline. Hence, it is the reduced velocity field $\mathbf{W}$ rather than the actual velocity field $\mathbf{V}$ which plays the basic role in the theory of rotational gas flows. We, therefore, focus our attention on the problem of obtaining spatial reduced velocity fields satisfying the equations of gas dynamics (see [4]) from plane fields that do so.

Specifically, we suppose given a plane flow field with reduced velocity $\mathbf{W}_p$ and inquire what spatial reduced velocity fields $\mathbf{W}$ can be obtained by a combined application of the Substitution Principle and the Newtonian relativity considerations discussed above. We write $\mathbf{W}$ in the form

$$\mathbf{W} = \alpha\mathbf{W}_p + \mathbf{k}W_n\,. \tag{3}$$

Denoting the ultimate velocity functions for the plane flow field and the spatial flow field by a_p and a, respectively, we have:

$$\mathbf{V}_p = a_p\mathbf{W}_p\,, \tag{4}$$

$$\mathbf{V} = a\mathbf{W}, \tag{5}$$

and

$$V_n = aW_n\,. \tag{6}$$

The ultimate velocity functions a_p and a are defined by the following relations:

$$a_p^2 = V_p^2 + 2h_p\,, \tag{7}$$

$$a^2 = V^2 + 2h. \tag{8}$$

Now the specific enthalpy of the plane flow is not affected by the superposition of a

constant V_n. Hence $h_p = h$. Furthermore, from (2), $V^2 = V_p^2 + V_n^2$. Equations (7) and (8), therefore, yield the following relation between a_p, a, and V_n:

$$a_p^2 = a^2 - V_n^2,$$

whence, making use of (6),

$$\frac{a_p}{a} = (1 - W_n^2)^{1/2} \tag{9}$$

and

$$W_n = \frac{V_n}{(V_n^2 + a_p^2)^{1/2}}. \tag{10}$$

From (2) and (5) we obtain

$$\begin{aligned} \mathbf{W} &= \frac{\mathbf{V}_p}{a} + \mathbf{k}\frac{V_n}{a} \\ &= \frac{\mathbf{V}_p}{a_p}\frac{a_p}{a} + \mathbf{k}\frac{V_n}{a}. \end{aligned} \tag{11}$$

Substitution of (4), (6), and (9) into (11) yields

$$\mathbf{W} = (1 - W_n^2)^{1/2}\mathbf{W}_p + \mathbf{k}W_n. \tag{12}$$

Equation (12) is a formula for constructing spatial reduced velocity fields from a given plane reduced velocity field and a function W_n. The nature of this function W_n is clarified by (10) and the Substitution Principle. V_n can be any (positive or negative) constant, while, by the substitution principle, the reduced velocity function a_p can be assigned any (positive) values constant along each individual streamline without affecting $\mathbf{W}_p$. Therefore, letting $\psi_p(x, y)$ be a streamfunction of the given plane field, it is only required that in the spatial field the surfaces of constant ψ_p coincide with the surfaces of constant W_n and that W_n be of one sign throughout. That is,

$$W_n = W_n[\psi_p(x, y)], \tag{13}$$

where W_n is either a non-negative or a non-positive function everywhere. In vector form, the restriction (13) can be written

$$\mathbf{W}_p \cdot \text{grad } W_n = \mathbf{k} \cdot \text{grad } W_n = \mathbf{W} \cdot \text{grad } W_n = 0, \tag{14}$$

or,

$$\mathbf{W}_p \cdot \text{grad } W_n = \frac{\partial W_n}{\partial z} = 0. \tag{15}$$

Of course, physical meaningfulness requires also that $W_n^2 \leq 1$.

In the formulas (12) and (13) or (12) and (15) we have a means for generating a vast variety of spatial reduced velocity fields of possible gas flows from the reduced velocity field of any given plane flow. The power of this generating method stems, of course, from the essentially arbitrary nature of the function $W_n[\psi_p]$.

One question that naturally presents itself is whether the generating formula (12)

is rotation-preserving; that is, does curl $\mathbf{W}_p \neq 0$ imply curl $\mathbf{W} \neq 0$? To answer this question we compute from (12):

$$\text{curl } \mathbf{W} = (1 - W_n^2)^{1/2} \text{ curl } \mathbf{W}_p - \frac{W_n}{(1 - W_n^2)^{1/2}} \text{ grad } W_n \times \mathbf{W}_p + \text{grad } W_n \times \mathbf{k}. \tag{16}$$

From (16) we find that

$$\mathbf{k} \times \text{curl } \mathbf{W} = \text{grad } W_n ,$$

from which it follows that curl $\mathbf{W} = 0$ implies grad $W_n = 0$. But, together, curl $\mathbf{W} = 0$ and grad $W_n = 0$ imply (from (16)) that curl $\mathbf{W}_p = 0$ (for $W_n^2 < 1$). Therefore, curl $\mathbf{W}$ is zero if, and only if, both curl $\mathbf{W}_p$ and grad W_n are zero. Hence rotational $\mathbf{W}$ fields can be derived from irrotational $\mathbf{W}_p$ fields, but the converse is not possible.

Another class of spatial flow fields of particular interest are the "generalized Beltrami flows" investigated by Nemenyi and Prim [5, 6]. These are the flow fields for which

$$\mathbf{W} \times \text{curl } \mathbf{W} = 0 \tag{17}$$

throughout. For plane and axially-symmetric flow fields this condition is evidently equivalent to the condition, curl $\mathbf{W} = 0$; however, for more general spatial flows, the class of flows satisfying (17) is much larger than that satisfying curl $\mathbf{W} = 0$. Aside from their interesting geometric and kinematic properties, these generalized Beltrami flows have a special physical significance, shown by the relation

$$\text{grad } \log H(p_0) = 2 \frac{\mathbf{W} \times \text{curl } \mathbf{W}}{1 - W^2} , \tag{18}$$

where $H(p) \equiv \int_0^p [1/P(p)]\, dp$ and p_0 denotes the stagnation pressure (pressure on a given streamline for zero velocity). From (18) it is seen that (for $W^2 < 1$) the class of generalized Beltrami flows is identical with the class of flows having a uniform stagnation pressure. The given general form of (18) is due to the present writer. The particular form valid for gases having constant specific heats (and hence a constant adiabatic exponent γ) was published earlier by B. Hicks and his colleagues [6]:

$$\frac{\gamma - 1}{2\gamma} \text{ grad } \log p_0 = \frac{\mathbf{W} \times \text{curl } \mathbf{W}}{1 - W^2} . \tag{19}$$

(For the case of gases having constant specific heats, the function $P(p)$ can be taken as $p^{1/\gamma}$ whence $H(p) = \int_0^p p^{-1/\gamma}\, dp = \gamma/(\gamma - 1)p^{(\gamma-1)/\gamma}$.)

We shall now make use of this property of generalized Beltrami flows to establish a method of generating them through the Superposition Principle of the present paper. In addition, we shall need the following two relations:

$$H(p) = H(p_0)(1 - W^2), \tag{20}$$

and

$$(1 - W^2) = (1 - W_n^2)(1 - W_p^2). \tag{21}$$

The latter follows at once from (12). The former is the general form (cf. [3]) for gases with a product equation of state of the familiar formula

$$p = p_0(1 - W^2)^{\gamma(\gamma-1)} \tag{22}$$

assuming constant specific heats. Now the application of either the Substitution Principle or the Newtonian Relativity Principle does not affect the pressure field of the given plane flow field. Therefore, denoting the stagnation pressure of the given plane field by p_{0p}, we have from (20) that

$$H(p) = H(p_0)(1 - W^2) = H(p_{0p})(1 - W_p^2);$$

making use of (21), we obtain

$$H(p_0)(1 - W_n^2) = H(p_{0p}). \tag{23}$$

(It should be remarked here that p_{0p}, like W_n, is a function of ψ_p.) Now $H(p_{0p})$ is a known function of x and y once the plane field is given. It can be computed from a given $\mathbf{W}_p$ by integration from (18). Therefore, in order to obtain from (12) a spatial field which is a generalized Beltrami field, it is only necessary to choose the function W_n so that in (23) $H(p_0)$ is a constant. Since $0 \leq W_n^2 \leq 1$, the constant must be chosen so that throughout the region of the given plane field considered

$$H(p_{0p}) \leq H(p_0). \tag{24}$$

This implies that any curves in the plane flow field on which $H(p_{0p}) = \infty$ must be excluded from the region considered.

We thus have a simple method of constructing a one-parameter family of generalized Beltrami flow fields from any given plane flow field: $H(p_0)$ is simply assigned any constant value satisfying (24), and the function W_n for use in (12) is then computed from

$$W_n = \left[1 - \frac{H(p_{0p})}{H(p_0)}\right]^{1/2}. \tag{25}$$

It should be noted that if curl $\mathbf{W}_p = 0$, (18) implies that $H(p_{0p})$ is a constant, so that (by (25)) setting $H(p_0)$ equal to a constant forces W_n to be constant also; hence by (16) curl $\mathbf{W} = 0$. Therefore, only rotational plane fields generate spatial fields for which $\mathbf{W} \times \text{curl } \mathbf{W} = 0$ while curl $\mathbf{W} \neq 0$.

This powerful method of generating generalized Beltrami flows produces, as quite special cases, all previously known examples of such fields.

As an example of an application of this method producing new Beltrami fields we consider the following one-parameter family of plane rotational flows, discovered by the present writer [7], for gases having constant specific heats:

$$\mathbf{W}_p = \mathbf{r}_1 \sin \frac{\theta}{\lambda v_0} + \boldsymbol{\theta}_1 v_0 \cos \frac{\theta}{\lambda v_0} + \mathbf{Z}_1 \cdot 0 \tag{26}$$

where $\mathbf{r}_1$, $\boldsymbol{\theta}_1$, $\mathbf{Z}_1$ are unit vectors in an ordinary cylindrical coordinate system (r, θ, Z), $\lambda \equiv (\gamma + 1)/(\gamma - 1)$ and v_0 is an arbitrary constant such that $0 < v_0^2 < 1$. For this set of fields we compute (from (19))

$$p_{0p} = \left(r \cos^\lambda \frac{\theta}{\lambda v_0}\right)^{(\lambda+1)(v^2_0 - 1/\lambda)/(1-v^2_0)} \tag{27}$$

and obtain the two-parameter (v_0, A) family of generalized Beltrami flows:

$$\mathbf{W} = \mathbf{r}_1 A \sin \frac{\theta}{\lambda v_0} \left(r \cos^\lambda \frac{\theta}{\lambda v_0} \right)^{(v^2_0 - 1/\lambda)/(1-v^2_0)}$$

$$+ \boldsymbol{\theta}_1 A v_0 \cos \frac{\theta}{\lambda v_0} \left(r \cos^\lambda \frac{\theta}{\lambda v_0} \right)^{(v^2_0 - 1/\lambda)/(1-v^2_0)} \tag{28}$$

$$+ \mathbf{Z}_1 \left[1 - A^2 \left(r \cos^\lambda \frac{\theta}{\lambda v_0} \right)^{2(v^2_0 - 1/\lambda)/(1-v^2_0)} \right]^{1/2},$$

where A is a constant such that

$$0 < A < \left(r \cos^\lambda \frac{\theta}{\lambda v_0} \right)^{-(v^2_0 - 1/\lambda)/(1-v^2_0)} \tag{29}$$

throughout the flow region considered. The lines in the r, θ plane on which $r \cos^\lambda (\theta/\lambda v_0)$ is equal to a constant (for fixed v_0) are the streamlines of the field (26). Therefore, for $v_0^2 < 1/\lambda$ the innermost streamline in the region considered fixes the upper bound on A, and it is clear that the line $\theta = \lambda v_0 \pi/2$ must be excluded. For $v_0^2 > 1/\lambda$ the point $r = \infty$ is the critical one in bounding A; only a finite part of the plane can be taken as the region considered.

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ON HEAT TRANSFER PROBLEMS IN VISCOUS FLOW*

By G. F. Carrier and J. A. Lewis (*Brown University*)

1. Summary. Many problems of physical interest which are associated with the flow of a viscous fluid through a narrow channel require the determination of the temperature distribution throughout the field of flow. In general, such problems may be separated into one of three classifications. The first of these is characterized by the existence of a thermal boundary layer, the second by a temperature distribution independent of the coordinate across the channel, and the third by an intermediate type

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