THE B.W.K. APPROXIMATION AND HILL'S EQUATION, II.*

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1. The B.W.K. Method. The B.W.K. procedure[1] was discovered in connection with problems of wave mechanics, where Planck's constant h was treated as a very small quantity, and functions could be expanded in series of powers of h. The first terms of the expansion correspond to classical mechanics (completed with some quantum conditions) and higher terms represent typical wave-mechanical effects. Optical problems can be discussed along similar lines, starting with "geometrical optics," and obtaining "physical optics" (diffraction, for instance) as higher order corrections.

The B.W.K. method was discussed recently[2] by the present author in connection with its possible use in a purely mathematical problem, the discussion of Hill's equation. The method sketched on this occasion proved very valuable and showed the need for a more complete discussion of the whole question. It will be shown that the B.W.K. procedure can yield a very good approximation in a great many mathematical problems, and leads directly to asymptotic expressions similar to those obtained by P. Debye in the case of Bessel functions.

There is a variety of problems leading to equations of the general type

$$\frac{d^2 f}{dx^2} + F(x)f = 0, \tag{1}$$

where $F(x)$ is a real function. We may, for instance, discuss wave propagation along a line† whose properties vary from place to place. If $W(x)$ is the velocity of propagation and ω the frequency, the solution is

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[1]L. Brillouin, J. de Phys. **7**, 353 (1926); G. Wentzel, Z. Phys. **38**, 518 (1926); H. A. Kramers, Z. Phys. **39**, 828 (1926). For a more complete discussion and literature, see: E. C. Kemble, *The fundamental principles of quantum mechanics*, McGraw Hill, New York, 1937, Ch. III; W. H. Furry, Phys. Rev. 71, 360 (1947). The first indications about a similar procedure are found in J. Liouville's papers (1837) and H. Jeffreys' publications (1923).

[2]L. Brillouin, Q. Appl. Math. **6**, 167 (1948); this paper will be quoted as L.B.H. 1.

†The propagation of waves along a dissipationless electric line leads to the following equations

$$\frac{\partial V}{\partial x} = -l\frac{\partial I}{\partial t}, \qquad \frac{\partial I}{\partial x} = -c\frac{\partial V}{\partial t}, \tag{A.1}$$

where in general $l(x)$ and $c(x)$ are two functions of x. One may introduce a new variable y defined by

$$l_0\, dy = l\, dx, \tag{A.2}$$

where l_0 is a constant that may represent the average value of $l(x)$. This yields

$$\frac{\partial V}{\partial y} = -l_0\frac{\partial I}{\partial t} \qquad \frac{\partial I}{\partial y} = -c\frac{l_0}{l}\frac{\partial V}{\partial t} \tag{A.3}$$

and

$$\frac{\partial^2 V}{\partial y^2} - l_0^2\frac{c}{l}\frac{\partial^2 V}{\partial t^2} = 0, \tag{A.4}$$

an equation which corresponds to our general type (1). It should be noticed that $(l/c)^{1/2}$ represents the

$$fe^{i\omega t} \tag{2}$$

with

$$F = \frac{\omega^2}{W^2},$$

f satisfying Eq. (1). This representation will be found very useful and will enable us to give a simple physical translation of our mathematical formulas.

Let us now consider a function

$$u = G^{-1/2}e^{-iS}, \qquad S = \int_a^x G\,dx \tag{3}$$

where $G(x)$ is any given function of x. Straightforward computation shows that u satisfies the following equation:

$$u'' + u\left[G^2 - \frac{3}{4}\left(\frac{G'}{G}\right)^2 + \frac{1}{2}\frac{G''}{G}\right] = 0, \tag{4}$$

and a comparison between (4) and (1) leads to the condition

$$F = G^2 - \frac{3}{4}\left(\frac{G'}{G}\right)^2 + \frac{1}{2}\frac{G''}{G}. \tag{5}$$

If this equation can be solved to a certain approximation, the function u of Eq. (3) represents an approximate solution of Eq. (1). On many occasions, terms in G' and G'' can be omitted in Eq. (5), and a reasonably good approximation is obtained by taking

$$G = G_0 = F^{1/2}. \tag{6}$$

This is the zero order approximation in the B.W.K. procedure, and the question is to discuss its limits of validity.

A special case of importance was presented in the preceding paper (L.B.H. 1, p. 172). With

$$G = \frac{A}{(a + x)^2}, \tag{7}$$

terms in G', G'' cancel out in Eq. (5) making (3) with (6) a rigorous solution of Eq. (1).

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characteristic impedance of the line, hence the result: if $l(x)$ and $c(x)$ vary by the same amount, and c/l remains a constant, Eq. (A.4) reduces to the standard type of wave equation. There is no reflection and no perturbation to wave propagation.

The change of variables (A.2) means that we measure the length of the line in "henrys" and not in yards. We could also measure the length in "farads," by taking

$$c_0\,dz = c\,dx \tag{A.5}$$

and thus obtain another equation:

$$\frac{\partial^2 I}{\partial z^2} - c_0^2 \frac{l}{c}\frac{\partial^2 I}{\partial t^2} = 0 \tag{A.6}$$

When l and c vary in such a way as to keep a constant ratio (constant characteristic impedance) Eq. (A.6) is again a standard wave equation, and the variables y and z are proportional to each other.

These remarks show the connection between our discussion and the one given by Schelkunoff (5) who considers the case of two independent functions $l(x)$ and $c(x)$.

2. Systematic successive approximations. A systematic set of successive approximations can be worked out under the following assumptions:

$$\left|\frac{G_0'}{G_0^2}\right| < \epsilon, \qquad \left|\frac{G_0''}{G_0^3}\right| < \epsilon^2, \qquad \epsilon^2 \ll 1. \tag{8}$$

We may look for an expansion

$$G = G_0 + G_1 + G_2 + \cdots \tag{9}$$

where the successive terms would be of order ϵ^2, ϵ^4, $\cdots$, and substitute into Eq. (5):

$$G_0 = F^{1/2},$$

$$2G_0G_1 = \frac{3}{4}\left(\frac{G_0'}{G_0}\right)^2 - \frac{1}{2}\frac{G_0''}{G_0},$$

$$2G_0G_2 + G_1^2 = \frac{3}{2}\left(\frac{G_0'}{G_0}\right)^2\left(\frac{G_1'}{G_0'} - \frac{G_1}{G_0}\right) - \frac{1}{2}\frac{G_0''}{G_0}\left(\frac{G_1''}{G_0''} - \frac{G_1}{G_0}\right). \tag{10}$$

The expansion G of Eq. (9) would then yield, according to Eq. (3), a first solution of Eq. (1),

$$u = G^{-1/2}e^{-iS}, \qquad S = \int_a^x G\,dx. \tag{3}$$

The next step is to obtain another independent solution v of Eq. (1). Two cases must be distinguished here: in the first case,

(A) $F > 0$, G real and positive, S real; (11)

$$v = u^* = G^{-1/2}e^{+iS},$$

the complex conjugate of u, gives a second independent solution. If we think of a problem of propagating waves (Eq. 2), u corresponds to a wave propagating to the right, and v to a wave propagating to the left. The second case arises when

(B) $F < 0$, G and S purely imaginary, $i^{1/2}u$ real. (12)

In that case, we may use the general relation (12) of the preceding paper (L.B.H. 1, Eq. 12),

$$uv' - vu' = C, \tag{13}$$

which must hold between two independent solutions of Eq. (1). The constant C can be taken as unity for convenient normalization.* When u is given, Eq. (13) represents a first order differential equation for v and its solution is easily found as

$$v = Cu\int_0^x u^{-2}\,dx. \tag{14}$$

The first case (11) corresponds to propagating waves; the second one (12) represents attenuated waves without propagation.

*In the preceding example, where $v = u^*$ the constant C equals $2i$.

In any region x where our approximate solutions u, v can be obtained, the general solution of Eq. (1) reads

$$f = Au + Bv. \tag{15}$$

The difficulty begins with the problem of connecting these different regions and choosing in each of them coefficients A, B which will correspond to approximations to one and the same solution of Eq. (1). This problem was first clearly stated by Kramers and discussed by Kemble, Furry and other authors. We are going to discuss that problem again and show that many important cases have been overlooked by previous authors.

3. Cases of exception—Connection of different regions. The method developed in Sec. 2 is based upon assumptions (8) and stops working in the following three cases:

$$G_0 \to 0, \tag{16}$$

$$G_0' \to \infty, \tag{17}$$

$$G_0'' \to \infty. \tag{18}$$

We must now discuss what happens when such critical conditions are realized.

Case (16) is apparently the only one which has been discussed by previous authors. In the neighborhood of a point where the function F (Eq. 1) is zero, one may expand

$$F = b(x - x_0)^n, \qquad n > 0 \tag{19}$$

and discuss the corresponding equation. Taking $x_0 = 0$ brings the zero point to the origin, and Eq. (1) reduces to

$$f'' + bx^n f = 0 \tag{20}$$

whose solution is (Jahnke-Emde tables, Dover ed. p. 147)

$$f = x^{1/2} Z_{1/(n+2)} \left(\frac{2(b)^{1/2}}{n+2} x^{(n+2)/2} \right), \tag{21}$$

where Z is a solution of Bessel's equation of order $1/(n + 2)$. This covers the case originally discussed by Kramers, when the F function has a single root at the origin

$$n = 1, \qquad F = bx$$

$$f = x^{1/2} Z_{1/3} \left(\frac{2(b)^{1/2}}{3} x^{3/2} \right). \tag{22}$$

Assuming $b > 0$, we consider both sides of the origin:

(a) $$x > 0, \qquad F > 0, \qquad Z_{1/3} = H_{1/3}^{(1)} \text{ or } H_{1/3}^{(2)} . \tag{23}$$

Here the functions corresponding to the two B.W.K. solutions u_r and u_r^* (Eq. 11) are the two Hankel functions. On the other side of the origin,

(b) $$x < 0, \qquad F < 0, \qquad Z_{1/3} = J_{1/3} \text{ or } Y_{1/3} . \tag{24}$$

Here the functions corresponding to the B.W.K. solution u_l and to the second solution v_l (Eq. 14) are the Bessel and Neumann functions.

Hence the B.W.K. approximate solution u_r to the right is not the analytic continuation of the B.W.K. solution u_l to the left, but

$$u_l \text{ goes over into } 1/2\ (u_r + u_r^*). \tag{25}$$

A more complete discussion of this case is found in Kemble's book[1] and in Furry's paper. A refinement of that discussion was recently given by Isao Imai in a very interesting letter to the Phys. Rev. **74,** 113 (1948). If we take the case when Eq. (1) results from a problem of wave propagation, as stated in Eq. (2), condition (16) corresponds to the *total reflection* at the end of the line. This is apparent since u_r and u_r^* represent two waves propagating in opposite directions, the superposition of which is needed to match the u_l attenuated wave in the region where propagation is no longer possible. This point of total reflection is often called a "turning point."

Cases (17) and (18) correspond to rather sudden changes in the properties of the line, with propagation on both sides of the discontinuity (provided $F(x)$ remains positive). This is a case of mismatch at a junction and results in *partial reflection* as we shall see.

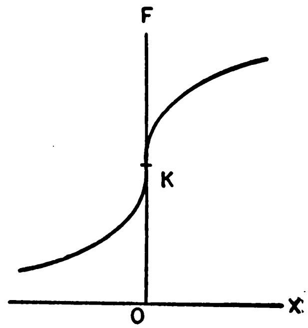

FIG. 1

Let us consider first the case of condition (17), assuming that F' becomes very large at $x = 0$. This corresponds to Fig. 1 with an inflection point at K. Simplified examples are shown in Figs. 2 and 3, when the function F has a discontinuity $2H$ at the origin,

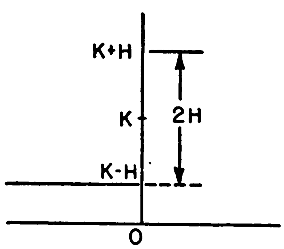

FIG. 2

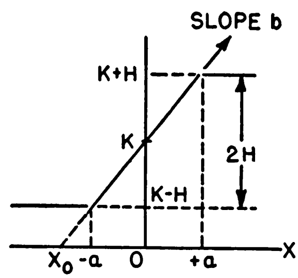

FIG. 3

or a very rapid change from $K - H$ to $K + H$ on a small interval $-a < x < a$. In the case of Fig. 2, with a discontinuity in F, we have a solution on the left

$$f_l = A_l u_l + B_l v_l \tag{26.a}$$

to match a similar solution on the right

$$f_r = A_r u_r + B_r v_r \,. \tag{26.b}$$

We must write the continuity of f and f', and this determines A_r , B_r when A_l , B_l are given, or vice versa. This was the procedure followed in the preceding paper (L.B.H. 1, p.169). Assuming the function $F(x)$ to remain positive on both sides of the origin, u and $v = u^*$ correspond to waves traveling in opposite directions. If we have a single wave on the right ($B_r = 0$) we need two waves on the left. Hence an incident wave from the left generates a reflected wave traveling backwards and a transmitted wave penetrating into the region to the right. This obviously means partial reflection at the junction $x = 0$.

In the example of Fig. 3, we would use a solution (26.a) for $x < -a$ and (26.b) for $x > a$, together with solutions of the type (22) in the interval $-a < x < a$:

$$f = x^{1/2}[AH^{(1)}_{1/3}(y) + BH^{(2)}_{1/3}(y)], \qquad y = \frac{2(b)^{1/2}}{3}(x - x_0)^{3/2}. \tag{27}$$

We must write two continuity conditions at $x = -a$ and two more at $x = a$, thus enabling us to compute the 4 coefficients A, B, A_r , B_r when A_l , B_l are given. The net result is again partial reflection, and this is what should be expected to happen in the more difficult problem of Fig. 1.

A similar situation is obtained in the case of condition (18), which corresponds to a discontinuity in the derivative $F'(x)$ and to a sharp angle in the curve $F(x)$.

4. Example. Bessel functions.[3] The differential equation for Bessel functions of order n can be written (L.B.H. 1, Eq. 33)

$$\frac{d^2f}{dx^2} + (e^{2x} - n^2)f = 0, \tag{28}$$

$$e^x = z, \qquad x = \log z,$$

where z is the original variable. This is an equation of type (1) with

$$F = e^{2x} - n^2, \qquad G_0 = \pm(e^{2x} - n^2)^{1/2}. \tag{29}$$

The B.W.K. method can be used for

$$\begin{aligned} &\text{I.} \qquad z \ll n, \qquad x \ll \log n \qquad n \gg 1, \qquad \text{or} \\ &\text{II.} \qquad n \ll z, \qquad x \gg \log n \end{aligned} \tag{30}$$

and fails in the neighborhood of $z = n$, which represents a "turning point" (16).

In the first interval we expand

$$G_0 = +in\left(1 - \frac{1}{2n^2}e^{2x} + \cdots\right) \tag{31}$$

The expansion (9) for G reads, with (10),

$$G = G_0 + G_1 + \cdots = G_0 + \frac{3}{8}G_0^{-3}G_0'^2 - \frac{1}{4}G_0^{-2}G_0'' + \cdots \,. \tag{32}$$

[3]See J. C. Slater and N. H. Frank, *Introduction to theoretical physics*, McGraw Hill, New York, 1933, pp. 148 and 347.

In our present case (31) it is easy to check that all correction terms can be ignored, since they are of order n^{-3}, n^{-5}, etc.

Next we compute

$$S = \int_a^x G dx = C + inx - \frac{i}{4n} e^{2x} = C + in \log z - \frac{iz^2}{4n},$$

$$G^{-1/2} = \left(\frac{-i}{n}\right)^{1/2}\left(1 + \frac{z^2}{4n^2} + \cdots\right)$$

and our general expression (3) now yields

$$u = K\left(\frac{-i}{n}\right)^{1/2} z^n\left(1 - \frac{z^2}{4n} + \frac{z^2}{4n^2} + \cdots\right) \tag{33}$$

where $K(-i/n)^{1/2}$ represents just an arbitrary constant. This solution (33) obviously corresponds to the J_n function

$$J_n = K'z^n\left(1 - \frac{z^2}{4(n+1)} + \cdots\right) = K'z^n\left(1 - \frac{z^2}{4n}\left(1 - \frac{1}{n}\right) + \cdots\right),$$

$$K' = \frac{1}{2^n n!}. \tag{34}$$

Both expansions (33), (34) coincide when $n \gg 1$. We thus prove that in the first interval (30) the B.W.K. method leads directly to the J_n solution.

The situation is different in the second interval, where z is larger than n:

$$G_0 = (e^{2x} - n^2)^{1/2} \approx e^x - \frac{n^2}{2} e^{-x},$$

$$G_0' = e^x + \frac{n^2}{2} e^{-x}, \qquad G_0'' = e^x - \frac{n^2}{2} e^{-x} = G_0 . \tag{35}$$

Correction terms in the G formula (32) can no longer be neglected and

$$G = e^x + e^{-x}\left(-\frac{n^2}{2} + \frac{3}{8} - \frac{1}{4}\right) + \cdots = e^x - \frac{1}{2} e^{-x}\left(n^2 - \frac{1}{4}\right),$$

$$S = \int_a^x G dx = C + e^x + \frac{1}{2} e^{-x}\left(n^2 - \frac{1}{4}\right) = C + z + \frac{1}{2z}\left(n^2 - \frac{1}{4}\right), \tag{36}$$

$$e^{-iS} = Ke^{-iz}\left(1 - \frac{i}{2z}\left(n^2 - \frac{1}{4}\right) + \cdots\right),$$

$$G^{-1/2} = z^{-1/2}\left[1 - \frac{1}{2z^2}\left(n^2 - \frac{1}{4}\right)\right]^{-1/2},$$

$$u = G^{-1/2}e^{-iS} = \frac{K}{z^{1/2}} e^{-iz}\left[1 + \left(n^2 - \frac{1}{4}\right)\left(\frac{-i}{2z} + \frac{1}{4z^2} + \cdots\right)\right]. \tag{37}$$

This reminds us immediately of the famous Debye[4] approximation for Hankel's functions when x is large, or of the rather similar expressions given by Hankel:

$$H_n^{(2)}(z) = \exp\left\{-i\left(z - \frac{1}{4}(2n + 1)\pi\right)\right\}\left(\frac{2}{\pi z}\right)^{1/2}[P_n(z) - iQ_n(z)], \tag{38}$$

$$P_n = 1 - \frac{(4n^2 - 1)(4n^2 - 9)}{2(8z)^2}, \qquad Q_n = \frac{4n^2 - 1}{8z}.$$

Except for a different K factor, Eqs. (37) and (38) check up to terms in $1/z^2$, but we did not push our approximation far enough in Eqs. (36) and (37) to be sure to obtain all terms in $1/z^2$ correctly. The preceding discussion exemplifies the problems discussed in Secs. 2 and 3 about the fact that the B.W.K. solutions obtained in different regions do not represent an analytic continuation of each other, and that their junction across the borders of these regions needs special care. Debye also developed formulas for the junction of his asymptotic solutions across a zero of the $F(x)$ function; this intermediate solution uses Bessel functions of order 1/3 just as the Kramer solution discussed in Sec. 3.

This comparison teaches us furthermore:

1) that the series obtained by the B.W.K. method must be only semi-convergent in general;

2) that they should represent a very good approximation to the actual function, since this is the case with the Debye series. There are, however, some fundamental limitations to the method of successive approximations developed in Sec. 2, Eq. (10), and a very striking example will be found in the discussion of the Mathieu-Hill equation (Secs. 7 and 8).

In such cases, it is necessary to have recourse to a different set of successive approximations which will now be presented.

5. A second method of successive approximations. We want to discuss Eq. (1) with a given function $F(x)$, assuming we have been able to obtain the solution of a similar equation

$$\frac{d^2u}{dx^2} + H(x)u = 0 \tag{39}$$

with a function $H(x)$ that does not differ very much from the original $F(x)$.[5] For instance, we may use a solution (3) of Eq. (4), where H represents the quantity in brackets of Eq. (4), and the G function is one of the steps of our former approximations (9), (10). We may try to use u as a first approximation and write

$$f = u + g, \tag{40}$$

substituting in (1) and using (39)

$$f'' + Ff = g'' + Fg + (F - H)u = 0. \tag{41}$$

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[4]Jahnke-Emde, *Tables of functions*, Dover, New York, 1943, pp. 137-139; *Smithsonian Mathematical Formulae*, Smiths. Inst., Washington, 1939, pp. 197-198.

[5]See S. A. Schelkunoff, Q. Appl. Math. **3**, 348 (1946); M. C. Gray and S. A. Schelkunoff, Bell System Tech. J. **27**, 350 (1948).

We now expand

$$g = g_1 + g_2 + g_3 + \cdots ,$$

$$g_1'' + Hg_1 = (H - F)u, \tag{42}$$

$$g_2'' + Hg_2 = (H - F)g_1 .$$

The physical meaning of the procedure can be explained in the following way: we start from the wave equation (39), which obtains a solution (3) exhibiting pure propagation and no reflections whatsoever. We use this equation (39) to build a set of equations with right-hand terms representing a continuous distribution of sources along the line, hence an emission of secondary wavelets, propagating in both directions and starting from all points along the line, especially from points where $(H - F)$ is not negligibly small.

We are thus able to obtain a solution taking into account all the reflections on possible irregularities of the line; this should lead to a better approximation than our first method of Sec. 2. Schelkunoff[5] has discussed practical methods for the integration of Eqs. (42). These equations read

$$\frac{d^2}{dx^2} g_n + Hg_n = \frac{d}{dx} h_n(x), \tag{43}$$

$$h_n' = \frac{d}{dx} h_n(x) = (H - F)g_{n-1} .$$

We know, from the discussion of Sec. 2, of two independent solutions u, v of the homogeneous equation (39), with the condition (13). With the help of u and v, the solution of (43) becomes

$$g_n(x_1x_0) = \frac{u(x)}{c} \int_{y=x_0}^{y=x} h_n(y)v'(y)\, dy - \frac{v(x)}{c} \int_{y=x_0}^{y=x} h_n(y)u'(y)\, dy \tag{44.a}$$

assuming g and g' to be zero at x_0 . We thus have a method for working out the successive approximations (42). The h_n functions represent the fictitious sources distributed along the line, resulting in additional waves propagating to the right (u function) or to the left (v function) as explained before.

The solution (44) can be written in a different way after an integration by parts,

$$g_n(x_1x_0) = - \frac{u(x)}{c} \int_{x_0}^{x} h_n'(y)v(y)\, dy + \frac{v(x)}{c} \int_{x_0}^{x} h_n'(y)u(y)\, dy, \tag{44.b}$$

and one should not forget that g_n may always contain additional arbitrary terms $(au + bv)$ satisfying the homogeneous equation.

6. The Mathieu-Hill equation, Floquet's theorem. The preceding discussion will enable us to investigate successfully some types of Hill equations that were left out of consideration in our previous paper (L.B.H. 1). But first of all, we must restate carefully Floquet's theorem; it was not stated quite accurately in the preceding paper.

Hill's equation corresponds to our Eq. (1) when the function $F(x)$ is periodic. Let us call d the period (instead of π in L.B.H. 1):

$$F(x + d) = F(x). \tag{45}$$

Let $u(x)$, $v(x)$ be two independent solutions of Eq. (1). We obtain another set of independent solutions by considering $u(x + d)$ and $v(x + d)$, since conditions are identically the same at x and $x + d$. This means that we must have linear relations

$$\begin{aligned} u(x + d) &= a_1 u(x) + b_1 v(x), \\ v(x + d) &= a_2 u(x) + b_2 v(x), \end{aligned} \tag{46}$$

expressing the new set in terms of the old one. Here we remember Eq. (13), which specifies that

$$u(x + d)v'(x + d) - v(x + d)u'(x + d) = u(x)v'(x) - v(x)u'(x) = C;$$

hence the relation

$$a_1 b_2 - a_2 b_1 = 1. \tag{47}$$

The (a, b) matrix has a determinant 1. A suitable linear combination of the original u, v may simplify the relations (45) by making this matrix diagonal. We choose

$$U(x) = u + hv \tag{48.a}$$

and want to obtain

$$U(x + d) = \xi U(x). \tag{48.b}$$

This means that

$$a_1 + a_2 h = \xi,$$

$$b_1 + b_2 h = \xi h$$

and eliminating h, that

$$\begin{vmatrix} a_1 - \xi & a_2 \\ b_1 & b_2 - \xi \end{vmatrix} = \xi^2 - (a_1 + b_2)\xi + 1 = 0. \tag{49.a}$$

The product of the two roots of this equation is 1, and we have

$$\xi + \xi^{-1} = a_1 + b_2 \, . \tag{49.b}$$

The relation (48) can be expressed differently as

$$U_1(x + d) = \xi U_1(x), \qquad U_1 = e^{\mu x}\Phi_1(x) \tag{50}$$

$$\xi = e^{\mu d}, \ \Phi_1 \text{ periodical, period } d$$

and similarly

$$U_2(x + d) = \xi^{-1} U_2(x), \qquad U_2 = e^{-\mu x}\Phi_2(x)$$

with two different functions Φ_1, Φ_2 both of period d. This is the general statement of Floquet's theorem.

When $F(x)$ is an *even function*, it is easy to see by symmetry reasons that

$$\Phi_2(x) = \Phi_1(-x) \tag{51}$$

in which case (and only in this case) we obtain the conditions (2) or (4) of L.B.H. 1.[6]

As for Eq. (49.b), it now reads

$$2 \cosh (\mu d) = a_1 + b_2 , \tag{52}$$

a general result of great importance, as will be seen in the next sections. The condition (47) concerning the (a, b) matrix and the final relation (52) are exactly similar to those obtained in the theory of filters.

7. Stopping and passing bands, general discussion. The first essential step in the discussion of Hill's equation is to obtain the value of the exponent μ. A method yielding the μ value was discussed in L.B.H. 1 and resulted in a relation (L.B.H. 1, Eq. 18) of the following structure:

$$\cosh (\mu d) = \psi(d), \tag{53}$$

where ψ was a certain function of d. The method developed by Whittaker results in a formula of similar type (L.B.H. 1, Eq. 23) and our Eq. (52) is also similarly built. The solution of (53) will give

$$\mu d = \beta_0 + i\alpha_0 \tag{54}$$

where β_0 is the attenuation constant per period d, and α_0 the propagation constant, determined modulus 2π. From the form of Eq. (53) it is obvious that if α_0 be a solution, then

$$\alpha_0' = \alpha_0 + 2n\pi \qquad (n \text{ integral})$$

is another one. The function ψ may depend upon a variable coefficient k. In the case of Eq. (2), k may be the variable frequency ω, or it may represent the energy E in a problem of wave mechanics.

In such problems the discussion of Eq. (53) proceeds as follows. We first plot ψ as a function of k, and obtain a curve such as the one represented in Fig. 4 (this curve

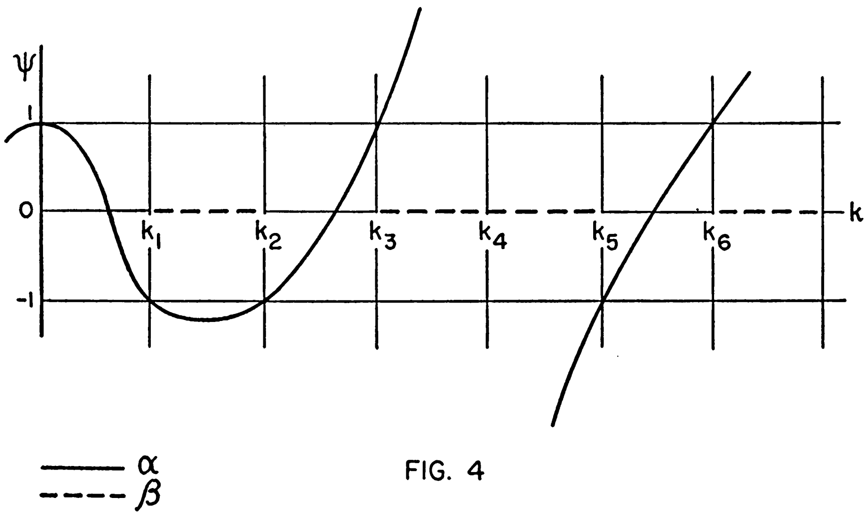

FIG. 4

[6]Since Floquet's theorem was not used through the rest of the paper L.B.H. 1, this omission was of no practical consequence.

was drawn so as to show some typical features of the problem). We obtain *passing bands* when

$$\beta_0 = 0, \qquad \text{or} \qquad -1 < \psi < +1. \tag{55}$$

When, on the contrary, we obtain

$$|\psi| > 1 \tag{56}$$

this means that $\beta_0 \neq 0$ and there is a *stopping band*:

$$\psi > 1, \qquad \alpha_0 = 0 \qquad \text{or} \pm 2n\pi,$$

$$\psi < -1, \qquad \alpha_0 = \pm\pi \qquad \text{or} \pm (2n+1)\pi.$$

The α, β curves corresponding to Fig. 4 are plotted in Fig. 5 as an illustration of the general procedure. It is interesting to notice that stopping bands always correspond to

$$\alpha_0 = \pm m\pi \qquad (m \text{ integral}) \tag{57}$$

or, if we introduce a new quantity a,

$$a = \frac{\mu}{2\pi i}, \qquad a_r = \pm\frac{m}{2\pi d},$$

a_r being the real part of a. This condition corresponds to the general results about the

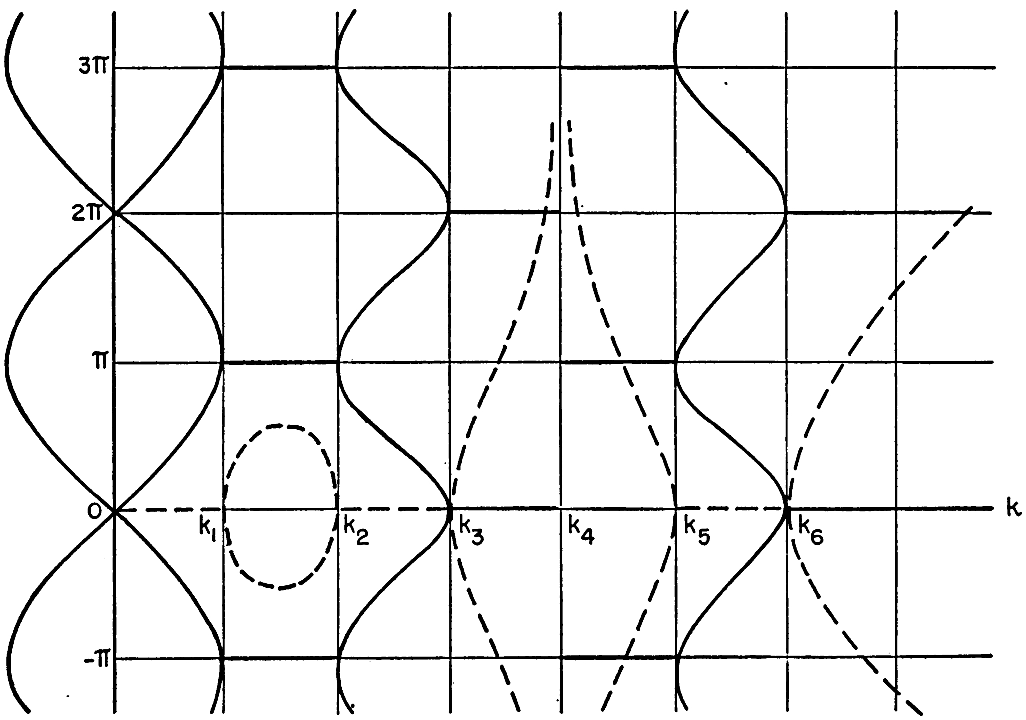

FIG. 5

limits of the Brillouin zones in one-dimensional problems.[7] There is, however, a fundamental difference between the method followed in the present paper and the discussion given in the book just quoted. The method previously used was designed to give information only about the passing bands. Propagation was assumed from the beginning and the μ value was chosen as the independent variable

$$\mu = 2\pi i a, \tag{58}$$

where $a = 1/\lambda$ is real, λ being the wave length. Equations (1) and (45) were then discussed in order to obtain the corresponding value of the parameter k (k being the frequency in case of Eq. (2)). This method gave us $k(a)$ in the passing bands and indicated the stopping band only indirectly by the fact that some k intervals could not be obtained in the discussion.

Here, on the contrary, we start with any given k value and compute the corresponding complex μ (54) that shows either pure propagation or attenuation. Instead of computing $k(\mu)$ we investigate the behavior of $\mu(k)$, and we obtain information about the properties of the stopping bands in addition to the characteristics of the passing bands. Thus the graph of Fig. 5 shows the variation of the attenuation β in addition to the propagation coefficient α. Former graphs contained only α.

8. Stopping and passing bands with the B.W.K. method. The important question to be discussed next is: how much information about passing and stopping bands can we obtain through the B.W.K. method? The answer is rather surprising: no practical answer is obtained through the direct B.W.K. procedure of Sec. 2, and only the second method of Sec. 5 can yield a reasonable solution of the problem. We gave a hint at these essential limitations of the original B.W.K. procedure at the end of Sec. 4. Let us explain it in so many words.

The B.W.K. procedure of Secs. 1 and 2 makes use of waves (Eq. 3) that may be more or less distorted but always propagate freely without exhibiting any reflection. If, on the other hand, we investigate the cause for the stopping bands of the preceding section, we discover that they are due to a phenomenon of cumulative reflection: each cell of length d reflects back a certain amount of the incident wave, and all these elementary wavelets, propagating backward, happen to be sufficiently "in phase" to produce a large reflected wave that takes up most of the energy of the incident wave. Such a result can be obtained only through the procedure of Sec. 5, which takes into account the possible local reflections; the method of Sec. 2 cannot give any information on such problems.

The discussion will be clearer if we start with a problem involving no process of approximation. Let us take the case of Eq. (4), the rigorous solution of which is given by (3). The function $G(x)$ is any arbitrary real periodic function of x (period d). Here we are dealing with an equation of Hill's type, where $F(x)$ is represented by Eq. (5). The two independent solutions are

$$\left.\begin{matrix} u(x) \\ u^*(x) \end{matrix}\right\} = G^{-1/2} e^{\pm iS}, \qquad S = \int_a^x G\, dx; \tag{59}$$

[7] L. Brillouin, *Wave propagation in periodic structures*, McGraw Hill, New York, 1946 (p. 6 Fig. 2.4, p. 8 Fig. 2.7, p. 15 Fig. 3.9, p. 57 Fig. 15.4, p. 68 Fig. 17.2, p. 103 Fig. 27.1, p. 113, pp. 118-120, pp. 143-145).

hence

$$u(x + d) = G(x)^{-1/2}e^{-iS(x+d)}$$

and

$$S(x + d) = S(x) + \int_x^{x+d} G\,dx = S(x) + \int_0^d G\,dx. \tag{60}$$

Comparing (60) and (50) we obtain

$$e^{\mu d} = \exp\left(-i\int_0^d G\,dx\right)$$

or

$$\mu d = -i\int_0^d G\,dx + 2n\pi i, \qquad (n \text{ integral}) \tag{61}$$

$$\mu = -i\overline{G} + ni\frac{2\pi}{d},$$

where

$$\overline{G} = \frac{1}{d}\int_0^d G\,dx \tag{62}$$

is simply the average value of G. Hence the B.W.K. method yields directly the very important quantity μ, together with the fact that μ is defined only modulus $i\ 2\pi/d$.

The function $G(x)$ may depend in any arbitrary way upon a parameter k, but so long as G remains real the formula (62) always retains its validity. Instead of the types of curves shown in Fig. 5 we obtain a graph of the type of Fig. 6, with curves intersecting each other when $\alpha = m\pi$.

This proves that our Eq. (4) exhibits very exceptional features and represents a poor approximation to the general Hill equation. It has a continuous passing band and no stopping bands whatsoever.

9. The Mathieu-Hill equation discussed with the method of Sec. 5. The discussion of Sec. 8 shows that the B.W.K. method does not lead to a practical solution of equations of the Mathieu-Hill type. The method developed in Sec. 5 will give us the necessary correction and enable us to obtain a much better approximate solution.

We start with the solutions u and $v = u^*$ of Eq. (59), that satisfy a B.W.K. differential equation (39)

$$u'' + H(x)u = 0,$$

where

$$H = G^2 - \frac{3}{4}\left(\frac{G'}{G}\right)^2 + \frac{1}{2}\frac{G''}{G} \tag{39}$$

according to Eq. (4). The $G(x)$ function is periodic and exhibits the same period d as the original Hill equation

$$f'' + F(x)f = 0. \tag{1}$$

We assume that the function G has been obtained through the procedure of Sec. 2 (Eqs. 9, 10) and that $H(x)$ differs very little from $F(x)$. We now use the second approximation represented by Eq. (44b), and we obtain two independent (approximate) solutions of (1):

$$U(x) = u(x) + g_1(x) = A(x)u + B(x)u^*,$$
$$U^*(x) = u^*(x) + g_1^*(x) = B^*(x)u + A^*(x)u^*, \tag{63}$$

with

$$A(x) = 1 - \frac{1}{c}\int_{y=x_0}^{y=x} (H - F)uu^*\,dy,$$
$$B(x) = \frac{1}{c}\int_{y=x_0}^{y=x} (H - F)u^2\,dy, \qquad c = 2i, \tag{64}$$

Since Eq. (43) yields

$$h_1'(y) = (H - F)u \qquad H - F \text{ real}, \tag{65}$$

for this second approximation. Our solutions (63) will satisfy relations (46)

$$U(x + d) = a_1U(x) + b_1U^*(x),$$
$$U^*(x + d) = a_2U(x) + b_2U^*(x), \tag{66}$$

where obviously

$$a_2 = b_1^*, \qquad b_2 = a_1^*,$$

and Floquet's exponent results from Eq. (52)

$$\cosh(\mu d) = 1/2(a_1 + b_2) = \mathcal{R}e\,(a_1), \tag{67}$$

$\mathcal{R}e$ meaning "real part of." Hence the only thing we need is the coefficient a_1 in Eq. (66). We can easily obtain this information from Eq. (63), taking

$$x = x_0 + d,$$

$$U(x_0 + d) = A(x_0 + d)u(x_0 + d) + B(x_0 + d)u^*(x_0 + d). \tag{68}$$

The discussion of Sec. (8) shows that

$$u(x_0 + d) = \exp(\mu_0 d)\,u(x_0),$$

$$\mu_0 = -i\bar{G}$$

but, at x_0 we have $A = 1$, $B = 0$, hence

$$u(x_0) = U(x_0)$$

and Eq. (68) reads

$$U(x_0 + d) = A(x_0 + d)\exp(\mu_0 d)U(x_0) + B(x_0 + d)\exp(-\mu_0 d)U^*(x_0). \tag{69}$$

Comparing Eqs. (66) and (69) we obtain

$$a_1 = A(x_0 + d)\exp(\mu_0 d) = \exp(-i\bar{G}d)\left[1 - \frac{1}{2i}\int_{x_0}^{x_0+d} (H - F)uu^*\,dy\right] \tag{70}$$

according to (64). This gives us the coefficient a_1 we need in Eq. (67):

$$\cosh(\mu d) = \Re e \exp(-i\bar{G}d)\left[1 + \frac{i}{2} J\right] \tag{71}$$

with real

$$J = \int_{x_0}^{x_0+d} (H - F)uu^* \, dy;$$

hence

$$\cosh(\mu d) = \cos(\bar{G}d) + 1/2 \, J \sin(\bar{G}d), \tag{72}$$

and we finally obtain the Floquet exponent we were looking for. *Passing* bands correspond to conditions when

$$-1 \leq \cosh(\mu d) \leq +1,$$

and stopping bands appear when

$$|\cosh(\mu d)| > 1 \tag{73}$$

as explained in Sec. 7. The B.W.K. method of Sec. 8 never gave any stopping band. Our second approximation (72) may give stopping bands when

$$\bar{G}d = m\pi + \epsilon,$$

$$\cosh(\mu d) = (-1)^m(1 + 1/2 \, J\epsilon) + \cdots . \tag{74}$$

These stopping bands appear in the neighborhood of the points $\bar{G}d = m\pi$ where the curves of Fig. 6 intersect each other.

This proves that the B.W.K. method must be completed with the approximation process of Sec. 5 in order to yield a solution of Hill's equation. The procedure sketched in this final section seems to represent one of the most practical approaches to the problem of Hill's equation, in addition to those cases that were discussed in a previous paper (L.B.H. 1).

Thus far, we have discussed only the first step in the series of successive approximations according to the procedure of Sec. 5. It may be found necessary to go some steps further and to use the expansion (42), in which case our formulas (63) could still be maintained. But Eqs. (64) would read

$$A(x) = 1 - \frac{1}{2i} \int_{y=x_0}^{y=x} (H - F)(u + g_1 + \cdots + g_n)u^* \, dy \tag{75}$$

if the n first approximations of (41) were used. This would simply mean a similar correction in Eqs. (70) and (71). Then

$$J = \int_{x_0}^{x_0+d} (H - F)(u + g_1 + \cdots + g_n)u^* \, dy = J_r + iJ_i, \tag{76}$$

J would no longer be real and Eq. (72) should be modified to

$$\cosh(\mu d) = \left(1 - \frac{J_i}{2}\right) \cos(\bar{G}d) + \frac{1}{2} J_r \sin(\bar{G}d). \tag{77}$$

This should give a more accurate definition of the passing bands and stopping bands.

10. Conclusions. The principle of the B.W.K. method, as stated in Sec. 2, consists essentially in a procedure of successive approximations starting from a solution of Eq. (6), that simply represents the Hamilton-Jacobi equation of the problem:

$$G_0 = \frac{\delta S_0}{\delta x}, \qquad \left(\frac{\delta S_0}{\delta x}\right)^2 = F. \tag{78}$$

In a three-dimensional problem, the wave equation (1) reads

$$\nabla^2 f + Ff = 0, \tag{79}$$

and Eq. (6) yields the Hamilton-Jacobi equation

$$\left(\frac{\delta S_0}{\delta x}\right)^2 + \left(\frac{\delta S_0}{\delta y}\right)^2 + \left(\frac{\delta S_0}{\delta z}\right)^2 = F \tag{80}$$

as was shown in the paper in J. de Phys. 7, 337 (1926), Eq. 13.

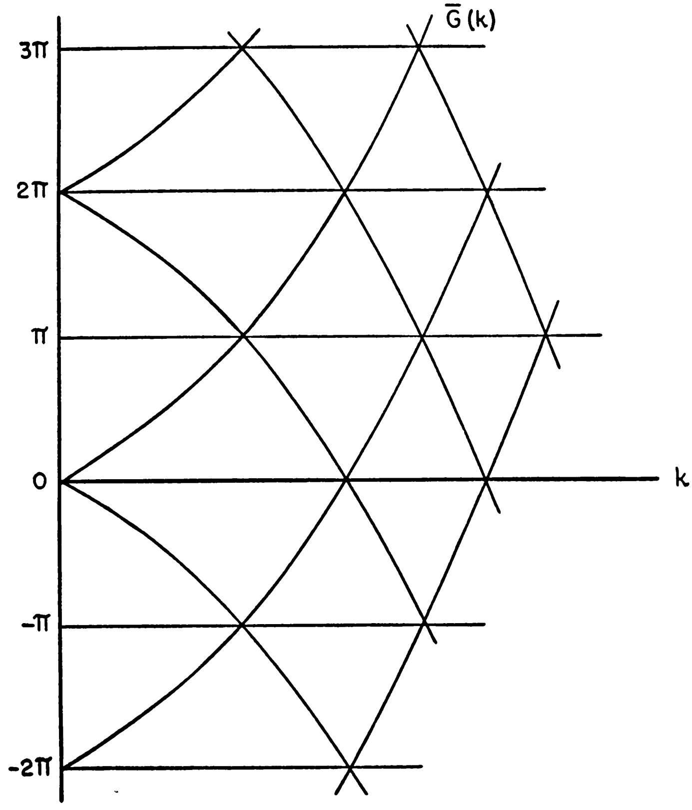

FIG. 6

The successive approximations of the B.W.K. procedure represent a semi-convergent series, similar to the well-known Debye series for Bessel functions (Sec. 4). This proves that there is a limitation to the accuracy of the method, and that nothing can be gained by increasing too much the number of terms in the series.

The reason for this is easy to understand and was explained in Sec. 3. The Hamilton-Jacobi equation always leads to pure waves, propagating without reflection, as exemplified in Eq. (3). Taking the example of wave-mechanics, where

$$F = 2m(E - V) \qquad (h = 2\pi) \tag{81}$$

we obtain in (80) the Hamilton-Jacobi equation of classical mechanics. Let us consider a problem of particles hitting a potential hill: they will go over the hill if their energy is large enough, or be completely reflected when their energy is too low, but we shall never obtain partial reflection with some particles reflected and some others climbing above the hill. This sort of thing, however, happens with the wave equation (79), and we discussed in Sec. 3 some typical instances where partial reflection must take place.

This proves that after obtaining the best possible approximation with the B.W.K. method of Sec. 2, we must turn back to the original wave equation and use the method of Sec. 5. The discussion of Hill's equation (Secs. 6-9) exemplifies the need for this correction and shows how to apply it to a practical problem.