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A NOTE ON RIEMANN'S METHOD APPLIED TO THE DIFFUSION EQUATION*

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Summary. The method of Riemann (or Green) is applied to the integration of the parabolic equation in m dimensions and two variables. The result is applied to the two phase diffusion problem. The reduction to tractable equations is not simple, however, and the results of the investigation appear to be essentially negative.

1. Introduction. The method of Riemann for integrating partial differential equations has received considerable attention in the case of the hyperbolic equation[1/6], where it has proved to be a most elegant approach. Its application to equations of the parabolic type has, on the other hand, received considerably less attention.**

Pascal[7] has considered the special case of fixed boundaries (in space), while Rademacher and Rothe[8] have considered the more general one-dimensional case where the dependent variable is specified over an arbitrary boundary but did not apply it to the solution of particular problems. The following note will be restricted to the case of two independent variables, but not necessarily to one dimension. The extension to an arbitrary number of independent variables may be made along the lines indicated by Webster.[9]

2. Riemann integration. The equation to be integrated is

$$L(u) = \frac{\partial}{\partial x}\left[x^{m-1}\frac{\partial}{\partial x}u(x, t)\right] - \kappa x^{m-1}\frac{\partial}{\partial t}u(x, t) = -Q(x, t). \tag{1}$$

If (1) is regarded as the equation of heat conduction, then $u(x, t)$ is the temperature, x is a space coordinate, t is the time coordinate, κ is the diffusion constant, and Q is the source strength. The space is m dimensional, and corresponds to one dimension,

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[1]H. Bateman, *Partial differential equations of mathematical physics*, Dover Press, New York, 1943, p. 126 ff.

[2]R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, J. Springer, Berlin, 1936, vol. 2, p. 311 ff.

[3]A. G. Webster, *Partial differential equations of mathematical physics*, G. E. Stechert and Co., New York, 1927, p. 244 ff.

[4]P. Frank and R. v. Mises, *Differentialgleichungen der Physik*, Friedr. Vieweg & Sohn, Braunschweig, 1925 (7th Ed.), vol. 1, p. 605 ff.

[5]E. Pascal, *Repertorium der Höheren Analysis*, B. G. Teubner, Leipzig, 1929, vol. 3, pp. 1175-1187.

[6]E. J. B. Goursat, *Cours d'analyse*, Gauthier-Villars, Paris, 1935, vol. 3, p. 308 ff.

**References (5)-(8) were brought to the author's attention by the reviewer, who also pointed out that the method under discussion is designated as Green's method therein. It is the author's impression that Green applied the method only to the elliptic equation, and that Riemann developed the generalization to other equations and to an arbitrary number of variables, cf. the discussion given by Webster, *loc. cit.* (3), p. 239. See also J. Hadamard, *Lectures on Cauchy's problem in linear partial differential equations*, Yale Univ. Press, New Haven, 1923, p. 57.

[7]Pascal, *loc. cit.*, p. 1178.

[8]Frank and v. Mises, *loc. cit.*, chapter by H. Rademacher and E. Rothe, p. 646 ff.

[9]Webster, *loc. cit.*, p. 257 ff.

two dimensions with axial symmetry, or three dimensions with spherical symmetry for $m = 1, 2,$ or 3, respectively.

The operator which is adjoint to L is defined by

$$M(v) = \frac{\partial}{\partial x}\left[x^{m-1}\frac{\partial}{\partial x}v(x,\,t)\right] + \kappa x^{m-1}\frac{\partial}{\partial t}v(x,\,t). \tag{2}$$

If the combination $vL(u) - uM(v)$ is integrated over a surface S in the (ξ, τ) plane, which is bounded by the curve C, it can be shown that[10]

$$\iint_S [vL(u) - uM(v)]\,d\xi\,d\tau = \oint_C \left\{\left[u\frac{\partial v}{\partial \xi} - v\frac{\partial u}{\partial \xi}\right]\cos(n,\,\xi) + \kappa uv\cos(n,\,\tau)\right\}\xi^{m-1}\,dl \tag{3}$$

where (ξ, τ) are running (x, t) coordinates, and $\cos(n, \xi)$ and $\cos(n, \tau)$ are the direction cosines of the inwardly directed normal to C with respect to the ξ and τ axes.

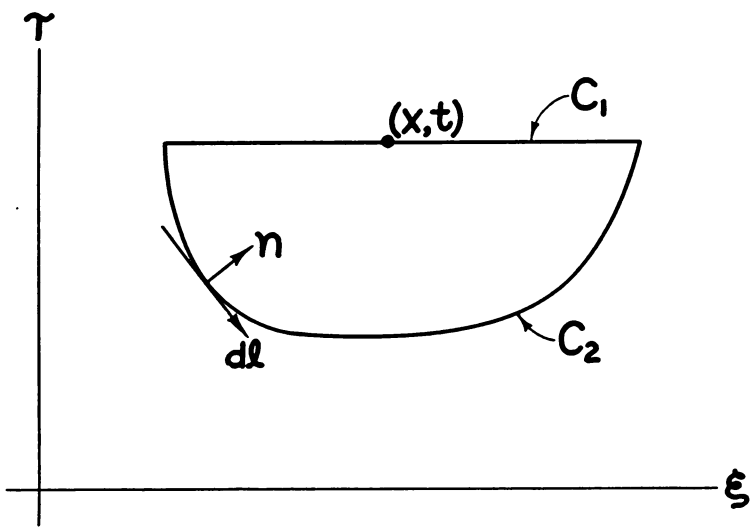

FIG. 1. Curve in (ξ, τ) plane for integration of Eq. (3).

The identity (3) will now be applied to a curve consisting of two segments C_1 and C_2, as shown in Fig. 1. C_1 is a portion of the line $\tau = t$, and C_2 is a curve for which $\tau \leq t$, but is otherwise arbitrary. Moreover, it will be assumed that

$$M[v(\xi,\,\tau)] = 0, \tag{4}$$

$$\kappa\xi^{m-1}v(\xi,\,t) = \delta(x - \xi), \tag{5}$$

where

$$\delta(x - \xi) = 0, \qquad x \neq \xi \tag{6a}$$

$$\int_{x-\epsilon}^{x+\epsilon} \delta(x - \xi)\,d\xi = 1, \qquad \epsilon > 0 \tag{6b}$$

[10]This follows directly from the more general result derived by Webster, *loc. cit.*, p. 245.

i.e., $v(\xi, \tau)$ is a solution to the adjoint equation (4) which reduces (essentially) to the Dirac delta function, as defined by (6), at $\tau = t$. Now along the curve C_1

$$dl \cos(n, \xi) = 0, \tag{7a}$$

$$dl \cos(n, \tau) = -d\xi, \tag{7b}$$

$$\begin{aligned}\oint_{C_1} \left\{\left[u \frac{\partial v}{\partial \xi} - v \frac{\partial u}{\partial \xi}\right] \cos(n, \xi) + \kappa uv \cos(n, \tau)\right\}\xi^{m-1}\, dl \\ = -\oint_{C_1} u(\xi, t)\,\delta(x - \xi)\, d\xi, \\ = -u(x, t).\end{aligned} \tag{7c}$$

Substituting $L(u) = -Q$ from (1), $M(v) = 0$ from (4), and the line integral along C_1 from (7c), (3) reduces to

$$\begin{aligned}u(x, t) = \iint_S Q(\xi, t)v(x, t; \xi, \tau)\, d\xi\, d\tau \\ + \oint_{C_2} \left\{\left[v \frac{\partial u}{\partial \xi} - u \frac{\partial v}{\partial \xi}\right] d\tau + \kappa uv\, d\xi\right\}\xi^{m-1}\end{aligned} \tag{8}$$

where the adjoint function v is now regarded as a function of both the parameters (x, t) and the running coordinates (ξ, τ).

3. Comparison with the method of characteristics. The foregoing solution is a degenerate case of the method of characteristics, as applied, e.g., by Volterra to the wave equation.[11] The two characteristic lines utilized in the solution of the hyperbolic equation in two coordinates merge into a single characteristic for the parabolic equation. Moreover, in applying the method of characteristics to the hyperbolic equation, the curve C_2 is restricted to intersecting a given characteristic only once, whereas in the present case the curve necessarily intersects the characteristic twice. It may also be remarked that the adjoint functions appropriate to the hyperbolic equation are generally singular along the entire characteristics through the point at which the solution is desired, whereas in the parabolic equation the adjoint function is singular only at that point.

The latter property is perhaps more closely related to the solution of the elliptic equation, where the adjoint function is singular at a point, although this point is then located within C, rather than on its boundary.

4. Boundary conditions. It will be assumed that the boundary condition is of the open Dirichlet-Neumann (mixed) type, viz.

$$u(\xi, \tau) + \beta(\xi, \tau) \frac{\partial}{\partial \xi} u(\xi, \tau) = \xi^{m-1} f(\xi, \tau), \qquad (\xi, \tau) \text{ on } C_2\,. \tag{9}$$

[11]Webster, *loc. cit.*, p. 266 ff.

If $\beta \equiv 0$ the boundary condition is simply of the open Dirichlet type, whereas if $\beta = \infty$ (with f replaced by $\beta^{-1}f$), it is of the open Neumann type. (The adjective "open" implies that boundary conditions are not applied over a closed curve in the ξ, τ plane, i.e., the curve is open in the direction of positive time.) Substituting the condition (9) in Eq. (8) to eliminate $(\partial u/\partial \xi)$, it is found that the explicit presence of u in the integrand may be eliminated by requiring the adjoint function to satisfy the boundary condition.

$$\kappa v(x, t; \xi, \tau) \cos (n, \tau) + \left[\frac{\partial}{\partial \xi} v(x, t; \xi, \tau) + \beta^{-1}(\xi, \tau) v(\xi, \tau)\right] \cos (n, \xi) = 0, \qquad (\xi, \tau) \text{ on } C_2 \tag{10}$$

whence Eq. (8) reduces to the alternative forms

$$u(x, t) = \iint_S Q(\xi, \tau) v(x, t; \xi, \tau)\, d\xi\, d\tau + \oint_{C_2} \beta^{-1}(\xi, \tau) f(\xi, \tau) v(x, t; \xi, \tau)\, d\tau, \tag{11a}$$

$$= \iint_S Q(\xi, \tau) v(x, t; \xi, \tau)\, d\xi\, d\tau + \oint_{C_2} f(\xi, \tau)\left[\kappa v(x, t; \xi, \tau)\, d\xi - \frac{\partial}{\partial \xi} v(x, t; \xi, \tau)\, d\tau\right]. \tag{11b}$$

The result (11) replaces the problem of solving Eq. (1), subject to the inhomogeneous boundary condition (9), with the problem of solving the adjoint equation (4), subject to the homogeneous boundary condition (10) and the singular condition (5). It remains to show that the latter solution exists and is unique. The proof is evidently the same as that for Eq. (1), if the sign of t is changed, and it follows that a necessary condition is that the boundary condition on v be open in the negative τ direction (note that (5) closes the boundary condition in the positive τ direction). Now, since $\cos (n, \tau)$ and $\cos (n, \xi)$ cannot vanish simultaneously, the only conditions for which Eq. (10) is indeterminate are

$$\left.\begin{aligned} \cos (n, \xi) &= 0 \\ \beta(\xi, \tau) &= 0 \end{aligned}\right\} \quad (\xi, \tau) \text{ on } C_2 . \tag{12}$$

Hence, for the smallest value of τ (the initial time) on C_2, assuming the curve to be continuous so that $\cos (n, \xi) = 0$ at the minimum, β must vanish, and it is therefore necessary, cf. Eq. (9), that u be explicitly stated there. The physical implication is that the phenomenon described must have an initially prescribed state.

5. Reduction for fixed boundaries. Perhaps the simplest contour C_2 consists of two lines of constant ξ plus a line of constant $\tau(<t)$. For convenience, the last line may be taken as $\tau = 0$ with no loss of generality. The first two lines will be chosen as $\xi = x_1$ and $\xi = x_2$, as shown in Fig. 2. (Choosing the line $\xi = 0$, would be a rather special case for $m > 1$.)

The boundary conditions (9) and (12) reduce to

$$\xi = x_i : u(x_i\,, \tau) + \beta_i \frac{\partial}{\partial \xi} u(x_i\,, \tau) = x_i^{1-m} f_i(\tau), \qquad i = 1, 2 \tag{13}$$

$$\tau = 0 : u(\xi, 0) = u_0(\xi) \tag{14}$$

while Eq. (10) reduces to

$$v(x, t; x_i\,, \tau) + \beta_i \frac{\partial}{\partial \xi} v(x, t; x_i\,, \tau) = 0, \qquad i = 1, 2 \tag{15}$$

and Eq. (11b) becomes

$$\begin{aligned} u(x, t) = \int_0^t d\tau \int_{x_1}^{x_2} Q(\xi, \tau) v(x, t; \xi, \tau)\, d\xi \\ + \kappa \int_{x_1}^{x_2} u_0(\xi) v(x, t; \xi, 0) \xi^{m-1}\, d\xi \\ + \int_0^t f_1(\tau) \frac{\partial}{\partial \xi} v(x, t; x_1\,, \tau)\, d\tau \\ - \int_0^t f_2(\tau) \frac{\partial}{\partial \xi} v(x, t; x_2\,, \tau)\, d\tau \end{aligned} \tag{16}$$

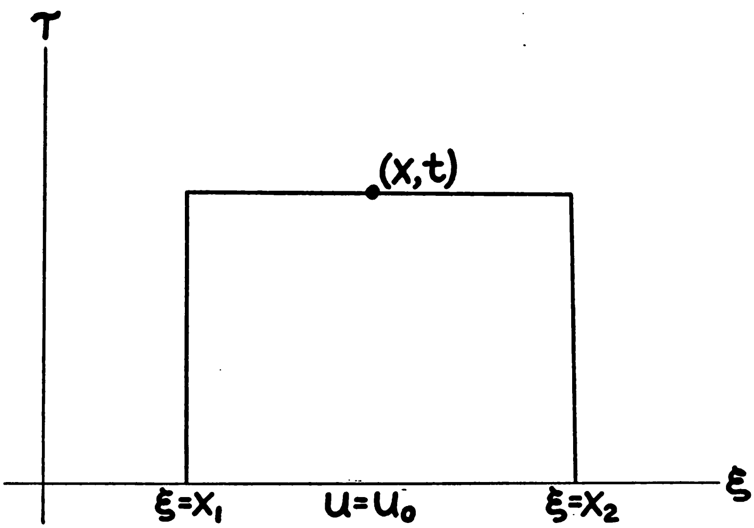

Fig. 2. Curve in (ξ, τ) plane for fixed boundaries.

The result (16) is generally deduced directly from Green's theorem, rather than the more general identity of Eq. (3).

6. Application to two phase problems. As an example of a more general type of problem to which the result (11) could be applied, the freezing or melting of a solid will be considered. It will be assumed that the substance undergoing such a change of state is located between two fixed boundaries $\xi = x_1$, x_2. Between these two boundaries lies a moving boundary, designated $x_{12}(\tau)$, as shown in Fig. 3. The temperature on the two sides of this boundary will be designated as u_1 and u_2.

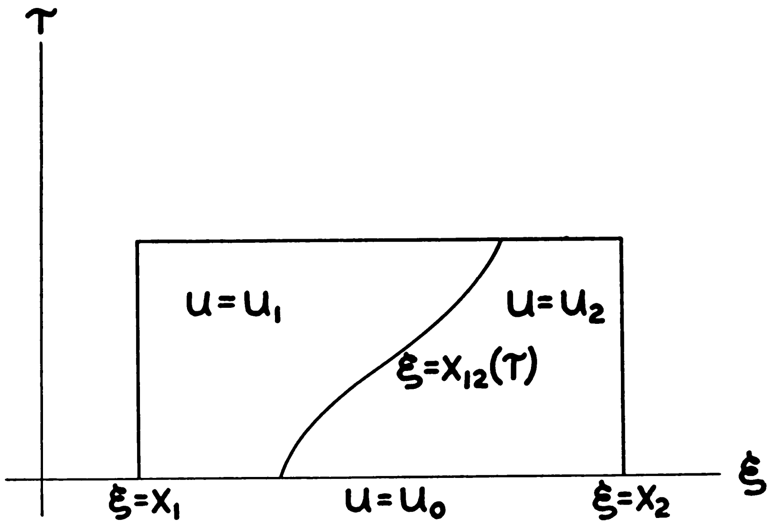

Fig. 3. Curve in (ξ, τ) plane for two phase problems.

At the boundaries $\xi = x_1$, $\xi = x_2$, and $\tau = 0$ the boundary conditions (13) and (14) are appropriate, while the boundary conditions at x_{12} may be written

$$u_1(x_{12}, t) = u_2(x_{12}, t) = 0, \tag{17}$$

$$\left[k_1 \frac{\partial}{\partial x} u_1(x_{12}, t) - k_2 \frac{\partial}{\partial x} u_2(x_{12}, t)\right] = \lambda\rho \frac{dx_{12}(t)}{dt} \tag{18}$$

where (17) states that the boundary is one at which freezing occurs, and (18) follows from a heat balance of a small element including a section of the boundary, k being the conductivity, ρ the density, and λ the latent heat. The result (8) may be applied separately to the regions to the right and left of the boundary x_{12}, where

$$dl_{1_2} = \mp d\tau \sec(n, \xi_{1_2}). \tag{19}$$

The adjoint functions in the two regions satisfy Eq. (5) at $\tau = t$, Eq. (14) at $\xi = x_1$ and $\xi = x_2$, and

$$v_1(x, t; x_{12}, \tau) = v_2(x, t; x_{12}, \tau) = 0 \tag{20}$$

Substituting the conditions (13)-(15), (17), and (20) in Eq. (11b) and assuming $Q = 0$, the temperatures are found to be

$$u_1(x, t) = \kappa_1 \int_{x_1}^{x_{12}(0)} u_0(\xi) v_1(x, t; \xi, 0) \xi^{m-1}\, d\xi + \int_0^t f_1(\tau) \frac{\partial}{\partial \xi} v_1(x, t; x_1, \tau)\, d\tau, \tag{21a}$$

$$u_2(x, t) = \kappa_2 \int_{x_{12}(0)}^{x_2} u_0(\xi) v_2(x, t; \xi, 0) \xi^{m-1}\, d\xi - \int_0^t f_2(\tau) \frac{\partial}{\partial \xi} v_2(x, t; x_2, \tau)\, d\tau. \tag{21b}$$

The determining equation for x_{12} is now obtained by substituting Eqs. (21) in Eq. (18) with the result

$$k_1\kappa_1 \int_{x_1}^{x_{12}(0)} u_0(\xi)\,\frac{\partial}{\partial x} v_1(x_{12}\,,\,t;\,\xi,\,0)\xi^{m-1}\,d\xi - k_2\kappa_2 \int_{x_{12}(0)}^{x_2} u_0(\xi)\,\frac{\partial}{\partial x} v_2(x_{12}\,,\,t;\,\xi,\,0)\xi^{m-1}\,d\xi$$

$$+ k_1 \int_0^t f_1(\tau)\,\frac{\partial^2}{\partial x\,\partial\xi} v_1(x_{12}\,,\,t;\,x_1\,,\,\tau)\,d\tau + k_2 \int_0^t f_2(\tau)\,\frac{\partial^2}{\partial x\,\partial\xi} v_2(x_{12}\,,\,t;\,x_2\,,\,\tau)\,d\tau \quad (22)$$

$$= \lambda\rho\,\frac{d}{dt}\,x_{12}(t).$$

The solution of Eq. (22) for $x_{12}(t)$ will evidently be quite complex in practical applications; moreover, the function v_1 and v_2 must be determined first.* For $m = 1$, $x_2 = \infty$, and $u(x_1) = $ const., a solution can be given and was first obtained by Riemann,[12] using similarity solutions to the original partial differential equation (1). The physical problem investigated by Riemann was the freezing of a deep lake for constant surface temperature, and his approach yields the desired results with considerably less trouble than the solution of Eqs. (21), (22).

The problem of the growth of ice on cylindrical cables or spherical shells and the freezing of flat, cylindrical, or spherical ingots are all problems of great practical importance which fall in the category under consideration. Only the first of these problems has (to the author's knowledge) been treated analytically, and then only quite approximately.[13] The author has attempted to solve such problems through the formulation of Eqs. (21) and (22) but was unable to obtain anything approaching an analytical solution in closed form, except for the simple problem of freezing a slab with a (required) constant temperature gradient at the interface of freezing.

Perhaps the most expedient approximate approach is to assume an equation, say $x_{12}^{(0)}(t)$, for the propagation of the interface, introduce it in Eqs. (21) and solve for $u_1^{(1)}$ and $u_2^{(1)}$, substitute these in Eq. (18) to obtain the next approximation $x_{12}^{(1)}(t)$, solve for $u_1^{(2)}$ and $u_2^{(2)}$, etc. While the convergence of this process would appear to be quite rapid, the evaluation of the integrals in Eqs. (21) remains difficult (even when v_1 and v_2 are readily determinable), and it will generally be necessary to have recourse to numerical or graphical techniques.

7. Conclusion. While the method of Riemann brings about a rather general solution to the parabolic equation, it does not appear to present an immediately practical method of solving problems which are more complex than those which are readily treated by more standard methods. The general solution presented herein may nevertheless prove useful in leading to various approximate solutions, or it may be that the equations such as those presented herein may actually be solved analytically or on automatic computing machines.

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*In general, they will be analogous to the Green's functions for fixed boundaries.

[12]Frank and v. Mises, *loc. cit*, vol. 2, chapter by Riemann, 220-222.

[13]C. L. Perkins and L. B. Slichter, *Problem of ice formation*, J. Appl. Phys. **10**, 135 (1939).