ON FREE VIBRATIONS WITH AMPLITUDINAL LIMITS*

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1. Consider a differential equation of the form

$$\ddot{y} + a_1(t)\dot{y} + a_2(t)y = 0 \tag{1}$$

or, more generally (placing

$$x_1 = y, \qquad x_2 = \dot{y} \tag{2}$$

and $n = 2$), a system of n differential equations

$$\dot{x}_i = \sum_{k=1}^{n} a_{ik}(t)x_k , \qquad (i = 1, \cdots , n), \tag{3}$$

where every coefficient function $a(t)$ is given as continuous for large positive t, say for $t^0 \leq t < \infty$. If A denotes the matrix (a_{ik}), and x the vector $(x_1 , \cdots , x_n)$, then (3) can simply be written as

$$\dot{x} = A(t)x. \tag{4}$$

Since n can be replaced by $2n$, there is no loss of generality in assuming that a_{ik} , x_i in A, x are real-valued. This will always be assumed in what follows.

The trivial solution, $x(t) \equiv 0$, of (4) will be excluded. Then, if $x(t)$ is any solution vector of (4), there cannot exist any t_0 for which $x(t_0) = 0$ (where 0 is the zero vector). For, on the one hand, $x(t) \equiv 0$ is a solution of (4) satisfying the initial condition $x(t_0) = 0$ and, on the other hand, any initial condition determines a solution $x(t)$ of (4) uniquely. Accordingly, if $r(t)$ denotes the length of the vector $x(t)$, then

$$x(t) = r(t)e(t), \text{ where } r(t) > 0 \text{ and } | e(t) | = 1 \qquad (r = | x |). \tag{5}$$

The positive scalar $r(t)$ and the unit vector $e(t)$ will respectively be referred to as the amplitude and phase factor of the solution $x(t)$.

Various conditions are known which, when satisfied by the matrix function $A(t)$, will assure for the system (4) the following type of asymptotic behavior†: Every non-trivial solution vector $x(t)$ of (4) tends, as $t \to \infty$, to a non-vanishing limit vector; in other words, every amplitude $r(t)$ tends to a finite, positive limit $r(\infty)$, and every phase factor $e(t)$ to a certain unit vector $e(\infty)$. From the point of view of the theory of vibrations, there are two objections to any criterion of this type.

The first objection is that, although such a criterion can be made applicable by using the method of the variation of constants**, it cannot apply directly to simplest vibration problems. In fact, even if the problem is that given by the case $a_1(t) = 0$, $a_2(t) = 1$ of (1), that is, by the linear oscillator $\ddot{y} + y = 0$, it is seen from (2), where $x_1 = c \cos (t - \gamma)$, $x_2 = -c \sin (t - \gamma)$ in the present case, that the above phase factor $e(t)$ fails to tend to a limit; although the limit, $r(\infty)$, of the amplitude exists and is positive, since $r(t) = | c | = \text{const.} > 0$ (unless $x(t) \equiv 0$).

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†A. Wintner, *On linear asymptotic equilibria*, Amer. J. Math. **71**, 853–858 (1949).

A. Wintner, *Small perturbations*, Amer. J. Math. **67, 417-430 (1945).

The second objection is that any criterion of the type in question becomes too severe by necessity. For, if only $r = |x|$, rather than $x = re$ (that is, r *and* e) is desired, then there becomes involved that issue which, on the one hand, is precisely the hard part of the problem and which, on the other hand, was not required at all. In fact, if the phase factor of a solution is known, then the amplitude of the solution (hence, the solution itself) can be obtained by a quadrature.

In order to see this, it is sufficient to observe that, since $r^2 = x.x$, hence $r\dot{r} = x.\dot{x}$, scalar multiplication of (4) by x gives $r\dot{r} = x.A(t)x$. It follows therefore from (5) that $r\dot{r} = r^2 e.A(t)e$, hence $(\log r)^{\cdot} = e.A(t)e$. Consequently, if $e = e(t)$ is known, then log $r(t)$ follows by a quadrature.

2. In what follows, a criterion will be developed which deals *only* with the problem of asymptotic amplitudes, without any reference to phase factors, and is therefore free of the above objections. In other words, a condition will be deduced which is not violated by vibration problems (such as $\ddot{y} + y = 0$) and which, when satisfied by the coefficient matrix $A(t)$, is sufficient to ensure for the corresponding system (4) the following property:

The amplitude function, $r(t) = |x(t)|$, of every non-trivial solution vector, $x = x(t)$, of (4) tends to a finite, non-vanishing limit, as $t \to \infty$. For the sake of brevity, such a system (4) will be said to be of *type* (*).

If a prime denotes the operation of transposing a matrix $A = (a_{ik})$, that is, if $A' = (a_{ki})$, then the criterion in question can be formulated as follows:

Let λ *denote the least, and* μ *the greatest, characteristic number (eigenvalue) of the symmetric matrix* $(1/2)(A + A')$, *where* A *is any real matrix. For large positive* t, *say for* $t^0 \leq t < \infty$, *let* $A = A(t)$ *be a continuous function, and suppose that the corresponding continuous functions* $\lambda = \lambda(t)$, $\mu(t)$ *are integrable over the half-line* $t^0 \leq t < \infty$, *that is, that the integrals*

$$\lim_{T\to\infty} \int^{T} \lambda(t)\, dt, \qquad \lim_{T\to\infty} \int^{T} \mu(t)\, dt \tag{6}$$

are convergent. Then the system (4) *is of type* (*).

It is worth emphasizing that $\int^{\infty} |\lambda(t)|\, dt = \infty$ or $\int^{\infty} |\mu(t)\, dt = \infty$ is allowed, that is, that the *absolute* convergence of the integrals (6) is *not* required.

3. First, if $\xi = (\xi_1, \cdots, \xi_n)$ is any vector, then

$$\xi.A\xi = \tfrac{1}{2}\xi.(A + A')\xi \tag{7}$$

is an identity, since

$$\xi.A\xi = \sum_{i=1}^{n}\sum_{i=1}^{n} a_{ik}\xi_i\xi_k \text{ and } \tfrac{1}{2}\xi.(A + A')\xi = \sum_{i=1}^{n}\sum_{k=1}^{n} \tfrac{1}{2}(a_{ik} + a_{ki})\xi_i\xi_k\,,$$

where $A = (a_{ik})$, $A' = (a_{ki})$. On the other hand, if λ denotes the least, and μ the greatest, eigenvalue of a real, symmetric matrix B, then λ is the minimum, and μ the maximum, attained by the (real) quadratic form $\xi.B\xi$ on the unit sphere, $|\xi| = 1$. If this fact is applied to the matrix $B = (1/2)(A + A')$, it follows from (7) that

$$\lambda \leq \xi.A\xi \leq \mu \text{ if } |\xi| = 1. \tag{8}$$

Next, if $r = r(t)$ denotes the amplitude, and $e = e(t)$ the phase factor, of an arbitrary non-trivial solution vector $x = x(t)$ of (4), then, as verified at the end of Section 1, the logarithmic derivative of $r(t)$ is identical with $e(t).Ae(t)$, where $A = A(t)$. Since the hypothesis, $|\xi| = 1$, of the inequalities (8) is satisfied by the vector $\xi = e(t)$, it now follows from (8) that

$$\lambda(t) \leq d \log r(t)/dt \leq \mu(t). \tag{9}$$

Finally, if $t^0 < u < v$, integration of (9) between $t = u$ and $t = v$ gives

$$\int_u^v \lambda(t)\, dt \leq \log r(v) - \log r(u) \leq \int_u^v \mu(t)\, dt.$$

On the other hand, the convergence of the improper integrals (6) means that

$$\int_u^v \lambda(t)\, dt \rightarrow 0 \text{ and } \int_u^v \mu(t)\, dt \rightarrow 0 \text{ if } u \rightarrow \infty,\ v \rightarrow \infty.$$

But the last two formula lines imply that $\log r(v) - \log r(u) \rightarrow 0$ as $u \rightarrow \infty$, $v \rightarrow \infty$. This means that the logarithm of $r(t)$ tends to a finite limit as $t \rightarrow \infty$. Since this is equvalent to the statement that $r(t)$ itself tends to a finite non-vanishing limit, the proof is complete.

A GENERALIZATION OF ALFREY'S THEOREM FOR VISCO-ELASTIC MEDIA*

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1. Introduction. For the non-homogeneous stresses in isotropic incompressible visco-elastic media characterized by linear relations between the components of stress, strain and their derivatives with respect to time, T. Alfrey has shown (Ref. 1) that in the case of the first boundary value problem, the stress distribution is identical with that in an incompressible elastic material under the same instantaneous surface forces. A similar result was obtained for the second boundary value problem where the displacements at the boundary are specified. It is the purpose of the present note to generalize this theorem to isotropic compressible media for problems involving body forces. Only the first boundary value problem will be discussed, as the corresponding theorem on the second boundary value problem is self-evident.

2. First boundary value problem. Let the displacements along the x, y, z directions be u, v, w. Then the typical expressions** of the six strain components can be written as:

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{1}$$

If the six stress components are denoted by σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}, the components

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**Throughout this note, only typical expressions are explicitly given; other expressions can be readily obtained by cyclic permutations.