NOTE ON THE KINEMATICS OF PLANE VISCOUS MOTION*

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In 1911 G. Hamel (Göttinger Nachrichten, Math.-Phys. Kl. 1911, 261-270) obtained an interesting result, which may be stated as follows:

Let R be a finite plane region with boundary B. Then the equation $\Delta\psi = F$ possesses a solution ψ for which both ψ and $\partial\psi/\partial n$ vanish on B if and only if F satisfies

$$\int FU \, dx \, dy = 0, \tag{1}$$

U being an arbitrary harmonic function.

In other words, for the existence of a solution with this double boundary condition, it is necessary and sufficient that the function F be orthogonal to the linear space of harmonic functions.

The hydrodynamical interpretation of Hamel's theorem is as follows. For an incompressible fluid moving in the plane with vorticity ω, we have

$$u_x + v_y = 0, \qquad v_x - u_y = 2\omega, \tag{2}$$

and there is a stream-function ψ such that

$$u = -\psi_y, \qquad v = \psi_x, \qquad \Delta\psi = 2\omega. \tag{3}$$

Thus Hamel's theorem tells us that in order that a given distribution of vorticity may be consistent with vanishing velocity on the boundary B (the usual boundary condition for a viscous fluid in a fixed container), it is necessary and sufficient that

$$\int \omega U \, dx \, dy = 0, \tag{4}$$

U being an arbitrary harmonic function.

However, inspection of Hamel's proof (loc. cit. p. 266) shows that he made use of a Green's function of the second type, i.e. a harmonic function G_2 with a singularity $\log r$ and making $\partial G_2/\partial n = 0$ on B. There is, of course, no such function for Laplace's equation, since this singularity and this boundary condition are inconsistent.

Not knowing of Hamel's work, I obtained Hamel's result in 1935 in a rather special case (Proc. London Math. Soc. 40 (1935), 23-36) in a different way.** In the present note the theorem is extended to include compressibility.

Theorem: A compressible viscous fluid moves inside a fixed connected boundary B, on which the velocity vanishes. An expansion $\theta(x,y)$ and a vorticity $\omega(x,y)$ are consistent with this boundary condition if, and only if,

$$\int (2\omega U - \theta V) \, dx \, dy = 0, \tag{5}$$

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***Footnote added in proof* (*Feb. 20, 1950*): The result (4) has recently been proved by J. Kampé de Fériet (Math. Mag. **21**, 71-79(1947); Ann. Soc. Sci. Bruxelles (I) **62**, 11-18(1948).

where U is an arbitrary harmonic function and V the conjugate harmonic function, such that

$$U_x = V_y, \qquad U_y = -V_x. \tag{6}$$

In purely mathematical language, equation (5) is a necessary and sufficient condition for the consistency of the equations

$$u_x + v_y = \theta, \qquad v_x - u_y = 2\omega, \qquad (u)_B = 0, \qquad (v)_B = 0. \tag{7}$$

Proof: Let l, m be the direction cosines of the outward normal to B. Let θ and ω be arbitrarily assigned. Let u', v' satisfy

$$u'_x + v'_y = \theta, \qquad v'_x - u'_y = 2\omega, \qquad (lu' + mv')_B = 0. \tag{8}$$

It is well known that the solution (u', v') is unique, since the two partial differential equations define (u', v') to within the gradient of a harmonic function, and the normal derivative of the latter on B is then given by the last of (8).

Denoting the integral in (5) by I, we have

$$\begin{aligned} I &= \int (2\omega U - \theta V)\, dx\, dy \\ &= \int [U(v'_x - u'_y) - V(u'_x + v'_y)]\, dx\, dy \\ &= \int [u'(U_y + V_x) + v'(V_y - U_x)]\, dx\, dy \\ &\quad + \int_B [U(lv' - mu') - V(lu' + mv')]\, ds \end{aligned} \tag{9}$$

or, by (6) and (8),

$$I = \int_B U(lv' - mu')\, ds. \tag{10}$$

Now if $I = 0$ for arbitrary harmonic U, it follows that

$$(lv' - mu')_B = 0, \tag{11}$$

since the values of U on B may be arbitrarily assigned. Combining (8) with (11) we get $(u')_B = 0$, $(v')_B = 0$; thus *the condition* $I = 0$ *is sufficient.*

On the other hand, if (7) are consistent, then $(u')_B = 0$, $(v')_B = 0$, and so, by (10), $I = 0$; thus *the condition* $I = 0$ *is necessary.*

We get Hamel's theorem on putting $\theta = 0$.