—NOTES—

A VARIATIONAL PROBLEM FOR THE ROOTS OF A CUBIC EQUATION*

(A contribution to the theory of servomechanisms)

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1. The problem. Let t denote the time, $u(t)$ a function of time and $v(t)$ the displacement of a servomechanism, which is to simulate $u(t)$. As regards the error

$$\eta(t) = v(t) - u(t)$$

many servomechanisms obey the equation

$$\eta''' + a_1\eta'' + a_2\eta' + a_3\eta = -u''' - a_4u'' \tag{1}$$

with constant coefficients $a_1, \cdots a_4$. The quality of the servomechanism depends on the roots of the algebraic equation

$$x^3 + a_1x^2 + a_2x + a_3 = 0. \tag{2}$$

These may be denoted by x_1, x_2, x_3. We consider the quantity

$$\xi = \text{Min}\,(-\Re x_1, -\Re x_2, -\Re x_3),$$

where $\Re x_i$ means the real part of x_i. Then $\xi > 0$ means stability, and the greater ξ, the faster the servomechanism follows sudden changes of $u(t)$. A positive ξ appears if and only if the well known Hurwitz-conditions are satisfied. These are

$$a_1 > 0, \qquad a_2 > 0, \qquad a_1a_2 > a_3 > 0.$$

If a_1 and a_2 are regarded as fixed, and a_3 as variable, ξ is a function of a_3. An interesting problem both from a mathematical and a technical point of view, is to find Max ξ. There is a simple formula for it, which does not seem to be as well known as its merits.

2. The solution. Introducing the classic notations

$$p = a_2 - \frac{a_1^2}{3}, \qquad q = a_3 - \frac{a_1a_2}{3} + \frac{2}{27}a_1^3,$$

$$D = -4p^3 - 27q^2,$$

we may represent the roots by

$$\begin{aligned}
x_1 &= -\frac{a_1}{3} + \epsilon\left[-\frac{q}{2} + \frac{1}{18}(-3D)^{1/2}\right]^{1/3} + \epsilon^2\left[-\frac{q}{2} - \frac{1}{18}(-3D)^{1/2}\right]^{1/3},\\
x_2 &= -\frac{a_1}{3} + \epsilon^2\left[-\frac{q}{2} + \frac{1}{18}(-3D)^{1/2}\right]^{1/3} + \epsilon\left[-\frac{q}{2} - \frac{1}{18}(-3D)^{1/2}\right]^{1/3},\\
x_3 &= -\frac{a_1}{3} + \left[-\frac{q}{2} + \frac{1}{18}(-3D)^{1/2}\right]^{1/3} + \left[-\frac{q}{2} - \frac{1}{18}(-3D)^{1/2}\right]^{1/3},
\end{aligned} \tag{3}$$

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where

$$\epsilon = -\frac{1}{2} + \frac{i}{2}(3)^{1/2}.$$

The quantity p depends on a_1 and a_2 only.

The roots x_1, x_2, x_3 are continuous functions of a_3. Hence the same holds for ξ. The quantity ξ vanishes for $a_3 = 0$ and for $a_3 = a_1a_2$. In the first case one root vanishes, in the latter $x = \pm i(a_2)^{1/2}$ represents two roots with vanishing real parts.

For $0 < a_3 < a_1a_2$ we have $\xi > 0$. Therefore ξ as a function of a_3 has a positive maximum in the interval $(0, a_1a_2)$.

The criterion for multiple roots is $D = 0$. For $D > 0$ three different real roots exist; for $D < 0$ one pair of conjugate complex roots exists. The roots x_1 and x_2 of the system (3) give these complex roots, while x_3 is real.

The only case where D may vanish is for $p \leq 0$. Vanishing takes place for two values of a_3 which we denote by $(\alpha - \beta)$ and $(\alpha + \beta)$. We can write

$$\alpha = \frac{a_1a_2}{3} - \frac{2a_1^3}{27}, \qquad \beta = 2\left(-\frac{p^3}{27}\right)^{1/2}.$$

All roots are continuously differentiable functions of a_3 in all intervals which do not contain $(\alpha - \beta)$ or $(\alpha + \beta)$. Starting from the well-known formulae

$$x_1 + x_2 + x_3 = -a_1,$$

$$x_1^2 + x_2^2 + x_3^2 = a_1^2 - 2a_2,$$

$$x_1^3 + x_2^3 + x_3^3 = -a_1^3 + 3a_1a_2 - 3a_3,$$

we find

$$\frac{dx_1}{da_3} = \frac{1}{(x_1 - x_2)(x_1 - x_3)},$$

$$\frac{dx_2}{da_3} = \frac{1}{(x_2 - x_1)(x_2 - x_3)}, \tag{4}$$

$$\frac{dx_3}{da_3} = \frac{1}{(x_3 - x_1)(x_3 - x_2)}.$$

Now, let a_3 be a value for which $D > 0$. Then all roots are real and different from one another. Hence $-\xi$ coincides with one root, say x_i, for a sufficient small neighbourhood of a_3. From (4) it follows that

$$\frac{d\xi}{da_3} = -\frac{dx_i}{da_3} \neq 0,$$

in this neighbourhood. Thus, a maximum of ξ can exist within the interval $(0, a_1a_2)$ only if $D \leq 0$.

Next, we consider a value a_3 and a neighbourhood of it in which $D < 0$. In this case we may write

$$\Re x_1 = \Re x_2 = -\frac{a_1}{3} - \frac{1}{2}\left[-\frac{q}{2} + \frac{1}{18}(-3D)^{1/2}\right]^{1/3} - \frac{1}{2}\left[-\frac{q}{2} - \frac{1}{18}(-3D)^{1/2}\right]^{1/3}$$

and, furthermore

$$\xi = \text{Min}\,(-\Re x_1\,,\ -x_3).$$

The criterion for

$$x_3 = \Re x_1$$

is $q = 0$. This means that $a_3 = \alpha$.

If it is assumed that a_3 does not coincide with α, ξ is represented either by $-x_3$ or by $-\Re x_1$ for a sufficient small neighbourhood of a_3. From (4) it follows that

$$\frac{d\xi}{da_3} = -\Re\,\frac{dx_i}{da_3} \neq 0$$

with a suitable subscript i. This holds for the neighbourhood. Therefore a maximum does not exist if $D < 0$ and $q \neq 0$.

Summing up, we may state that the maximum of ξ is given either by $D = 0$ or by $q = 0$, $D < 0$.

The following three cases may occur.

1) $p > 0$: We find $D < 0$. Max ξ is determined by $q = 0$. We have

$$\text{Max}\,\xi = \frac{a_1}{3}\,; \qquad a_3 = \frac{a_1a_2}{3} - \frac{2}{27}\,a_1^3\,.$$

2) $p = 0$: We find $D = 0$, $q = 0$ for $a_3 = \alpha$ and

$$\text{Max}\,\xi = \frac{a_1}{3} \qquad \text{for} \qquad a_3 = \frac{a_1a_2}{3} - \frac{2}{27}\,a_1^3\,.$$

3) $p < 0$: It is obvious that $D > 0$ for $q = 0$. The maximum of ξ is obtained for $D = 0$. The roots then are given by

$$-x_1 = -x_2 = \frac{a_1}{3} - \left(\frac{q}{2}\right)^{1/3}$$

$$-x_3 = \frac{a_1}{3} + 2\left(\frac{q}{2}\right)^{1/3}.$$

For $a_3 = \alpha \pm \beta$, we have $q = \pm\beta$ and

$$\xi = \frac{a_1}{3} - \left(\frac{\beta}{2}\right)^{1/3} \qquad \text{for} \qquad a_3 = \alpha + \beta,$$

$$\xi = \frac{a_1}{3} - 2\left(\frac{\beta}{2}\right)^{1/3} \qquad \text{for} \qquad a_3 = \alpha - \beta.$$

We see that $a_3 = \alpha + \beta$ gives the greater value for ξ. As the maximum of ξ exists, at least one of the abscissae $a_3 = \alpha \pm \beta$ must be in the interval $(0, a_1a_2)$. Since D is negative for $a_3 = a_1a_2$ (where complex roots exist), the abscissa $a_3 = \alpha + \beta$ surely is in the interval.

Therefore, we may state that

$$\text{Max}\,\xi = \frac{a_1}{3} - \frac{1}{(3)^{1/2}}\,(-p)^{1/2} \qquad \text{for} \qquad a_3 = \frac{a_1a_2}{3} - \frac{2}{27}\,a_1^3 + \frac{2}{9}\,(3)^{1/2}(-p)^{3/2}.$$

Obviously we can combine all cases and write

$$\operatorname{Max} \xi = \Re\left\{\frac{a_1}{3} - \frac{1}{(3)^{1/2}}(-p)^{1/2}\right\}.$$

This is obtained for

$$a_3 = \Re\left\{\frac{a_1 a_2}{3} - \frac{2}{27}a_1^3 + \frac{2}{9}(3)^{1/2}(-p)^{3/2}\right\}.$$

PERIODIC MOTIONS OF A NON-LINEAR DYNAMIC SYSTEM*

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1. Introduction. It is a matter of common knowledge that many problems of dynamics are non-linear in character and that the linearization of such problems is an expediency adopted in the face of mathematical difficulties attending the non-linearity. Fortunately, the linearized treatment gives useful results in a number of problems. On the other hand, there are cases where the essential characteristics of the phenomenon are altered by the assumption of linearity.

In the absence of general methods for handling non-linear problems, it is desirable to investigate typical non-linear phenomena to as great detail as possible with the intention of arriving at general methods of analysis.

One such problem is van der Pol's equation

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0, \qquad \left(\dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}\right). \tag{1.1}$$

This defines the one-dimensional motion of a particle of unit mass suspended on a spring of unit stiffness and subjected to negative damping at low amplitudes and positive damping at high amplitudes.

The existence of periodic solutions of an equation of more general form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1.2}$$

was considered by Liénard, Ref. 1, who proved, for $g(x) = x$ and under fairly general conditions on $f(x)$ (see Ref. 2), that there exists a periodic solution. The proof used in Ref. 1 was reproduced in Ref. 3 for $g(x) \neq x$.

In the present paper, the problem is considered from a different point of view. The periodicity of the solution is shown to exist for a wider class of differential equations (1.2) which can best be described and compared as in the following table:

	$F(x) = \int_0^x f(x)\,dx$	$G(x) = \int_0^x g(x)\,dx$	$f(x),\ g(x)$
Lefschetz	$F(\infty) = \infty$	$G(\infty) > 0$	Symmetric
Present paper	$F(\infty) > 0$	$G(\infty) = \infty$	No symmetry required
	$F(\infty) = \infty$	$G(\infty) > 0$	No symmetry required

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