From this follows the convergence of (13) for $s < s_0$, the interior of the singularity circle, and the convergence of (14) for $s > s_0$. The reader may wish to compare this with reference [1], pp. 46-49.

**References**

1. Chaplygin, *Gas jets*, Moscow 1902, see also NACA TM 1063.

**ON THE METHOD OF INVERSION IN THE TWO-DIMENSIONAL THEORY OF ELASTICITY**

**By E. STERNBERG and R. A. EUBANKS (Illinois Institute of Technology)**

**1. Introduction.** The method of inversion, originally introduced by J. H. Michell [1], has led to a variety of technically significant solutions to “plane” problems in the theory of elasticity [2], [3], [4], [5]. The usefulness of Michell’s stress-field transformation stems from its invariant properties which assure the preservation of an important class of boundary conditions. In the present note we show that any conformal stress-field transformation which preserves the principal-stress trajectories for every choice of the antecedent Airy function, is essentially a Michell transformation.

**2. The Michell transformation.** The inversion theorem of Michell may be stated as follows. Let $U(z, \bar{z})$ be real and biharmonic in a region $R$ of the $z$-plane, and let $R^*$ be the image of $R$ with respect to the mapping

\[ f = w(z) = \frac{az + b}{cz + d}, \]

where $h^2 = |w'|^2$ and $g$ is the inverse of $w$, is biharmonic in $R^*$. The stress fields generated by $U$ and $U^*$, considered as Airy functions in $R$ and $R^*$ respectively, are related according to

\[ \sigma^* + i\tau^* = \lambda(\sigma + i\tau) + p, \]

\[ \lambda = \frac{1}{h^2}, \quad p = 2(\lambda_{z\bar{z}}U - \lambda_{z}U_{\bar{z}} - \lambda_{\bar{z}}U_{z}). \]
Here $[\sigma, \tau]$ and $[\sigma^*, \tau^*]$ are the normal and shearing stresses on any arc $\Gamma$ and on its image $\Gamma^*$ with respect to the mapping $\zeta = w(z)$.

Moreover, the stress-field transformation characterized by the mapping (1) and the law of Airy function association (2) has the invariant properties:

(A) The images of the principal stress trajectories of $R$ are the principal stress trajectories of $R^*$.

(B) If any arc $\Gamma$ of $R$ is acted on by constant normal tractions only, so is its image $\Gamma^*$ of $R^*$.

(C) A concentrated load acting at a point $z_0$ of a boundary arc $\Gamma$ of $R$, and including a certain angle with $\Gamma$, is carried into a concentrated load of the same magnitude acting at $w(z_0)$ and including the same angle with the image arc $\Gamma^*$ of $R^*$.

Properties (A) and (B) were also established by Y. P. Jensen and D. L. Holl [6] by aid of derivatives of non-analytic (polygenic) functions [7], [8].

3. A converse of the inversion theorem. We now prove the following theorem. Let $R$ be a region of the $z$-plane and let $R^*$ be the image of $R$ with respect to the conformal mapping

$$\zeta = w(z), \quad w'(z) = \frac{dw}{dz} = he^{i\delta} \neq 0. \quad (4)$$

Moreover, for every Airy function $U(z, \bar{z})$, bi-harmonic in $R$, let there exist an Airy function $U^*(\zeta, \bar{\zeta})$, bi-harmonic in $R^*$, such that the corresponding stress-field transformation preserves principal stress trajectories, and the image field of stress is purely hydrostatic only if the antecedent field has the same property. Then $w(z)$ is given by (1) and $U^*$ is given by (2), i.e., the transformation is a Michell transformation.

To establish the theorem, we recall a result of Jensen and Holl [6] who showed that

$$\sigma + i\tau = \gamma_H(z, \bar{z}, \theta)$$

$$= 2U_{ss} + 2U_{s\bar{s}}e^{-2i\theta},$$

where

$$H_{(s, s)} = 2U_s \quad (6)$$

and $\gamma_H$ is the directional derivative of $H$ along $\Gamma$, $\theta$ being the inclination of $\Gamma$. In view of (5), property (A) is equivalent to the statement

$$\gamma_{H^*}[w(z), w(\bar{z}), \theta + \delta] = -\gamma_{H^*} \quad (7)$$

whenever

$$\gamma_H(z, \bar{z}, \theta) = -\gamma_{H^*}, \quad (8)$$

provided

$$H^*(\zeta, \bar{\zeta}) = 2U^*_s. \quad (9)$$

Equations (7), (8) by aid of (5) become

$$U_{ss} = U_{ss}e^{4i\theta}, \quad (10)$$

*This restriction is essential in order to rule out the trivial transformation which carries all antecedent stress distributions into hydrostatic fields of stress.

**The partial derivatives of $U^*$ with respect to $\zeta$ and $\bar{\zeta}$ are to be evaluated at $\zeta = w(z)$.
Thus
\[ U_{ss}U_{\tau\tau}^*(\bar{w}')^2 = U_{ss}U_{\tau\tau}^*(w')^2, \]  
which implies that the function
\[ \phi(z, \bar{z}) = U_{ss}U_{\tau\tau}^*(w')^2 \]
is real-valued. Furthermore, assuming that the original stress distribution is not hydrostatic, so that \( U_{ss} \neq 0 \), it follows by hypothesis that \( U_{\tau\tau}^* \neq 0 \) and hence \( \phi \) does not vanish identically. For convenience, let
\[ \varphi(z, \bar{z}) = U_{ss}U_{ss}/\phi. \]
Equation (12) then appears as
\[ U_{ss} = \varphi(w')^2U_{\tau\tau}^*, \]
where \( \varphi \) is again real-valued. Equation (14) constitutes a necessary and sufficient condition for the preservation of the principal-stress trajectories.

We next apply to (14) the condition that \( U(z, \bar{z}) \) and \( U^*(\xi, \bar{\xi}) \) are both biharmonic, i.e., \( U_{ssss} = 0 \) and \( U_{\tau\tau\tau\tau}^* = 0 \). This leads to
\[ \left[ \frac{\varphi}{w} \frac{w''}{w'} + 2\varphi \right] U_{ssss} + \left[ \varphi_{ss} - \varphi_{,w} \frac{w''}{w'} - \frac{(2\varphi_s)^2}{\varphi} \right] U_{ss} = 0, \]
or, since (15) must hold for every bi-harmonic \( U \),
\[ \varphi \frac{w''}{w'} + 2\varphi_s = 0, \]
\[ \varphi_{ss} - \varphi_{,w} \frac{w''}{w'} - 2(\varphi_s)^2 = 0. \]
The complete solution of (16), subject to the requirement \( \varphi = \bar{\varphi} \), is
\[ \varphi(z, \bar{z}) = \kappa(w\bar{w})^{-1/2} = \kappa/h \]
with \( \kappa \) an arbitrary real constant. Noting that (16), (17) require \( \varphi_{ss} = 0 \), we conclude from (18) that
\[ \left( \frac{w''}{w'} \right)' - \frac{1}{2} \left( \frac{w''}{w'} \right)^2 = 0, \]
i.e., the Schwarzian derivative of \( w(z) \) vanishes. The complete solution of (19) is given by
\[ w(z) = \frac{az + b}{cz + d} \]
and since the mapping is to be \( (1, 1) \) we may put \( ad - bc = 1 \). In order to arrive at the law of Airy-function association we now integrate (14) by use of (18). This integration yields,
\[ U^* = \frac{h}{\kappa} [U + A\tilde{z} + \alpha\bar{z} + \tilde{\alpha}z + B], \]
where $A$, $B$ are arbitrary real constants, and $\alpha$ is an arbitrary complex number. It is readily confirmed by direct computation that the constants $A$, $B$, and $\alpha$ give rise to an arbitrary, uniform hydrostatic stress-field in the image domain $R^*$. The constant $\kappa$, on the other hand, affects merely the scale of the image stress-distribution. We may therefore put $A = B = \alpha = 0$, $\kappa = 1$. This completes the proof.

References


A MINIMUM PRINCIPLE FOR STRUCTURAL STABILITY*

By H. J. WEISS and G. H. HANDELMAN (Carnegie Institute of Technology)

1. Statement of the problem. In a recent paper, W. Prager has discussed the problem of structural stability from the following point of view. Consider a given configuration of a deformable body, referred to a fixed system of rectangular axes, $x_i (i = 1, 2, 3)$, under a set of stresses $\lambda \sigma_{ij}$ which are in equilibrium with given surface tractions. These stresses are prescribed only to within the arbitrary constant factor $\lambda$. The configuration is assumed to be stable if $\lambda$ is sufficiently small. A system of infinitesimal perturbation displacements $u_i$ is then applied, and the question is asked for what values of the factor $\lambda$ will the equilibrium become indifferent.

The solution to the problem leads to the following system of linear, homogeneous, second order, partial differential equations

$$J_{ii,i} = 0,$$

subject to the homogeneous boundary conditions on the surface

$$J_{ii}n_i = 0,$$

where

$$J_{ij} = [\tau_{ij} + \frac{1}{2}\lambda(\sigma_{ip}\varepsilon_{pi} - \sigma_{ip}\varepsilon_{qi}) - \lambda\sigma_{ip}\omega_{pi}].$$

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