theorems, apply to the problems discussed above. We are justified then in using formal expansion theorems, such as (12), (22) in the solution of problems of the above type.

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A NOTE ON A VECTOR FORMULA*

By H. LOTTRUP KNUDSEN (*The Royal Technical University of Denmark, Copenhagen*)

Of some vector formulas compiled in a recent paper[1] the one discussed in the present note seems to be of general interest in field theory.

1. Derivation of the vector formula. Let $\mathbf{B}(\mathbf{r})$ denote a vector function of the position vector $\mathbf{r}$, satisfying sufficient continuity and differentiability conditions, and let A denote a closed surface and V the region of space bounded by this surface. Using conventional vector notation we may then state Gauss' theorem in the following way

$$\int_A d\mathbf{a}\cdot\mathbf{B} = \int_V dv\ \nabla\cdot\mathbf{B}. \tag{1}$$

Letting $\varphi(\mathbf{r})$ denote a scalar function and $\mathbf{\Phi}(\mathbf{r})$ a dyade function, both possessing sufficient continuity and differentiability properties, we may derive the following equations from Gauss' theorem

$$\int_A d\mathbf{a}\ \varphi = \int_V dv\ \nabla\varphi, \tag{2}$$

$$\int_A d\mathbf{a}\cdot\mathbf{\Phi} = \int_V dv\ \nabla\cdot\mathbf{\Phi}.^{2} \tag{3}$$

Substituting in equations (2) and (3)

$$\varphi = \mathbf{r}\cdot\mathbf{B}, \tag{4}$$

$$\mathbf{\Phi} = \mathbf{r}\mathbf{B}, \tag{5}$$

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[1] H. L. Knudsen, *Nogle vektorformler og deres anvendelse,* (Some vector formulas and their application), Fysisk Tidsskrift, Copenhagen, to be published.

[2] M. Lagally, *Vorlesungen über Vektor-Rechnung,* dritte Auflage, Leipzig, 1944.

and using

$$\nabla \cdot \mathbf{r} = 3, \tag{6}$$

$$\nabla \mathbf{r} = \boldsymbol{\varepsilon}, \tag{7}$$

where $\boldsymbol{\varepsilon}$ is the unit dyade, we obtain

$$\begin{aligned} \int_A d\mathbf{a}\ \mathbf{r}\cdot\mathbf{B} &= \int_V dv\ \nabla(\mathbf{r}\cdot\mathbf{B}) = \int_V dv\ [\nabla\mathbf{r}\cdot\mathbf{B} + \nabla\mathbf{B}\cdot\mathbf{r}] \\ &= \int_V dv\ [\boldsymbol{\varepsilon}\cdot\mathbf{B} + \nabla\mathbf{B}\cdot\mathbf{r}] = \int_V dv\ [\mathbf{B} + \nabla\mathbf{B}\cdot\mathbf{r}], \end{aligned} \tag{8}$$

$$\int_A d\mathbf{a}\cdot\mathbf{r}\mathbf{B} = \int_V dv\ \nabla\cdot(\mathbf{r}\mathbf{B}) = \int_V dv\ [\nabla\cdot\mathbf{r}\mathbf{B} + \mathbf{r}\cdot\nabla\mathbf{B}] = \int_V dv\ [3\mathbf{B} + \mathbf{r}\cdot\nabla\mathbf{B}]. \tag{9}$$

Subtracting equation (8) from (9) we find

$$\int_A d\mathbf{a}\cdot\mathbf{r}\mathbf{B} - \int_A d\mathbf{a}\ \mathbf{B}\cdot\mathbf{r} = \int_V dv\ [2\mathbf{B} + \mathbf{r}\cdot\nabla\mathbf{B} - \nabla\mathbf{B}\cdot\mathbf{r}] \tag{10}$$

or introducing the unit dyade $\boldsymbol{\varepsilon}$

$$\int_A d\mathbf{a}\cdot[\mathbf{r}\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\mathbf{r}]\cdot\mathbf{B} = \int_V dv\ [2\mathbf{B} + \mathbf{r}\cdot\nabla\mathbf{B} - \nabla\mathbf{B}\cdot\mathbf{r}]. \tag{11}$$

We consider now the special case where $\mathbf{B}(\mathbf{r})$ is an irrotational field, i.e. we assume that

$$\nabla \times \mathbf{B} = 0. \tag{12}$$

When this condition is satisfied, the dyade $\nabla\mathbf{B}$ is symmetrical. We have then

$$\mathbf{r}\cdot\nabla\ \mathbf{B} = \nabla\mathbf{B}\cdot\mathbf{r}. \tag{13}$$

Introducing this equation in (11) we finally obtain the following equation for an irrotational field $\mathbf{B}(\mathbf{r})$

$$\frac{1}{2}\int_A d\mathbf{a}\cdot[\mathbf{r}\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\mathbf{r}]\cdot\mathbf{B} = \int_V dv\ \mathbf{B}. \tag{14}$$

By the theorem expressed through this equation the volume integral of an irrotational vector field over a region of space may be converted into a surface integral extended over the surface bounding this region. That this conversion can be carried out, is evident from the fact that an irrotational vector field $\mathbf{B}(\mathbf{r})$ may be expressed as the gradient of a certain scalar field, the potential $\varphi(\mathbf{r})$,

$$\mathbf{B} = \nabla\varphi. \tag{15}$$

The conversion of the volume integral into a surface integral follows then directly from (2). However, the theorem expressed by equation (14) has the advantage that by using this equation we may express the surface integral without knowing the potential $\varphi(\mathbf{r})$ of $\mathbf{B}(\mathbf{r})$.

The use of the theorem developed in this section will be illustrated by an example.

2. The force on a body in a central field of force. Let a body V, bounded by a closed surface A, be acted on by a central field of force. The center of the field is denoted by O and the force per unit volume of the body by $\mathbf{f}(\mathbf{r})$, where $\mathbf{r}$ is the position vector with O as origo. The force density $\mathbf{f}(\mathbf{r})$ may be expressed as

$$\mathbf{f}(\mathbf{r}) = g(|\mathbf{r}|)\boldsymbol{\varrho}, \tag{16}$$

where $g(|\mathbf{r}|)$ is a scalar function of the distance from the center of the field, and where $\boldsymbol{\varrho}$ denotes a unit vector coparallel with $\mathbf{r}$.

The force $\mathbf{F}$, with which the field of force acts on the body V, is expressed by

$$\mathbf{F} = \int_V \mathbf{f}\, dv. \tag{17}$$

A central field of force being irrotational, we may convert the volume integral in this expression into a surface integral, extended over the boundary A of V by using the equation (14), derived in the last section. We find

$$\mathbf{F} = \frac{1}{2}\int_A d\mathbf{a}\cdot[\mathbf{r}\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\mathbf{r}]\cdot\mathbf{f}. \tag{18}$$

Letting $\mathbf{n}$ denote the outward unit normal to A and denoting the scalar surface element by da, so that $d\mathbf{a} = \mathbf{n}\, da$, we may rewrite (18) as

$$\mathbf{F} = \frac{1}{2}\int_A [\boldsymbol{\varrho}\cos(\mathbf{n}, \boldsymbol{\varrho}) - \mathbf{n}]\,|\mathbf{r}|\, g(|\mathbf{r}|)\, da. \tag{19}$$

The expression in the square bracket in this equation has the simple geometrical meaning demonstrated in Fig. 1.

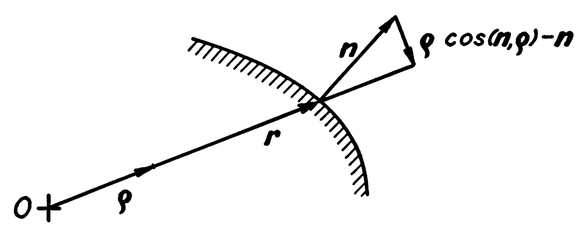

FIG. 1.

The application of the above developed expression (19) for the force on a body in a central field of force will be illustrated by two examples.

The force inversely proportional to the distance. Let us first consider a central field of force, in which the force is directed towards the center and is inversely proportional to the distance from the center. For such a field of force the function $g(|\mathbf{r}|)$ is expressed by

$$g(|\mathbf{r}|) = -K\,|\mathbf{r}|^{-1} \tag{20}$$

where K is a constant. By substituting this function in (19) we obtain the following expression for the force on the body in question

$$\mathbf{F} = -\frac{K}{2}\int_A [\boldsymbol{\varrho}\cos(\mathbf{n}, \boldsymbol{\varrho}) - \mathbf{n}]\, da = -\frac{K}{2}\int_A \boldsymbol{\varrho}\cos(\mathbf{n}, \boldsymbol{\varrho})\, da \tag{21}$$

as we have

$$\int_A \mathbf{n}\, da = 0, \tag{22}$$

this integral expressing the vector areal of the closed surface A.

As an example of the application of the formula (21) we shall calculate the force on a sphere with center P and radius R in a central field of force of the type discussed here, supposing that the center O of the field is situated on the surface of the sphere as shown in Fig. 2. From the symmetry it follows that the resulting force will be parallel

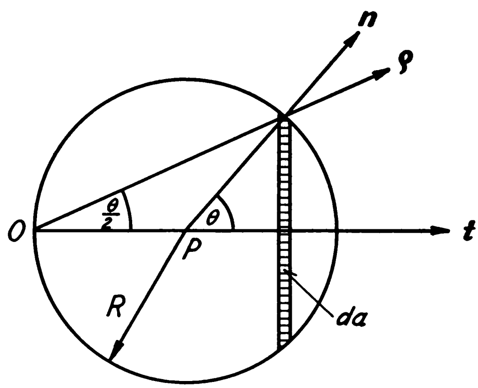

FIG. 2.

with the radius PO to the center of the field; in computing the force we therefore only need to retain the component in this direction of each of the differential contributions to the surface integral. Using the symbols shown in Fig. 2 we find from (21) the following expression for the force $\mathbf{F}$

$$\mathbf{F} = -\frac{K}{2}\int_0^{\pi} \mathbf{t}\cos\frac{\theta}{2}\cos\frac{\theta}{2}\, 2\pi R\sin\theta\, R\, d\theta = -K\pi R^2\mathbf{t}, \tag{23}$$

where the first factor $\cos\theta/2$ stands for $\cos(\boldsymbol{\varrho}, \mathbf{t})$, the second factor $\cos\theta/2$ for $\cos(\mathbf{n}, \boldsymbol{\varrho})$, and $2\pi R\sin\theta\, Rd\theta$ for da. In this equation $\mathbf{t}$ denotes a unit vector coparallel with OP. Through direct calaculation of the volume integral (17) we find, by dividing the sphere in conical shells with their apex at the center O of the field and introducing the angle $\alpha = \theta/2$ as an integration variable,

$$\mathbf{F} = -\int_0^{\pi/2}\int_0^{2R\cos\alpha} \mathbf{t}K\xi^{-1}\cos\alpha\,\xi\, d\alpha\, 2\pi\xi\sin\alpha\, d\xi = -K\pi R^2\mathbf{t}, \tag{24}$$

where $\cos\alpha$ stands for $\cos(\boldsymbol{\varrho}, \mathbf{t})$, $\xi d\alpha\, 2\pi\xi\sin\alpha\, d\xi$ for dv.
In the example discussed here a double integral has to be calculated for finding the force as a volume integral, whereas by using the vector formula (14) we get the force expressed as a single integral.

The Force Inversely Proportional to the Square of the Distance. For a central field of force in which the force is inversely proportional to the square of the distance from the center of the field and directed towards this center, the scalar function $g(|\mathbf{r}|)$ defined in (15) is expressed by

$$g(|\mathbf{r}|) = -K\,|\mathbf{r}|^{-2}, \tag{25}$$

where K is a constant. The force $\mathbf{F}$, with which this field acts on a body V, bounded by a closed surface A, may be found by substituting (25) in (19). We hereby find

$$\mathbf{F} = -\frac{K}{2}\int_A [\boldsymbol{\varrho}\cos(\mathbf{n}, \boldsymbol{\varrho}) - \mathbf{n}]\,|\mathbf{r}|^{-1}\,da. \tag{26}$$

For demonstrating the application of this formula we consider a sphere with center P and radius R in a central field of force of the type discussed here. The center O of the field is assumed to be situated on the surface of the sphere. The previously used Fig. 2 may also be applied as an illustration of the present problem. For reasons of symmetry the force will be parallel with the radius PO to the center of the field; in computing the surface integral we therefore retain only the component in this direction of the differential contributions. We find from (26)

$$\mathbf{F} = -\frac{K}{2}\int_0^{\pi} \mathbf{t}\left[\cos\frac{\theta}{2}\cos\frac{\theta}{2} - \cos\theta\right]\left[2R\cos\frac{\theta}{2}\right]^{-1} 2\pi R\sin\theta\,R\,d\theta = -\frac{4\pi}{3}KR\mathbf{t}, \tag{27}$$

where the first factor $\cos\theta/2$ stands for $\cos(\boldsymbol{\varrho}, \mathbf{t})$, the second factor $\cos\theta/2$ for $\cos(\mathbf{n}, \boldsymbol{\varrho})$, $\cos\theta$ for $\cos(\mathbf{n}, \mathbf{t})$, $[2R\cos\theta/2]^{-1}$ for $|\mathbf{r}|^{-1}$ and $2\pi R\sin\theta\,Rd\theta$ for da.

From the theory for Newtonian potentials[3] it is known that the force $\mathbf{F}$ from the field of force discussed here may be computed by assuming that the mass of the sphere is concentrated to its center. As the volume of the sphere is $4\pi/3\,R^3$ and the distance between the center of the sphere and the center of the field R, we therefore find the force $\mathbf{F}$ expressed as

$$\mathbf{F} = -(KR^{-2})\left(\frac{4\pi}{3}R^3\right)\mathbf{t}. \tag{28}$$

This expression is seen to be identical with the expression (27), found by using the surface integral method. The result from the potential theory used above was derived by carrying out a double integration. The calculation has consequently been simplified also in the present case by the use of the vector formula (14).

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THE CIRCULAR PLATE WITH ECCENTRIC HOLE*

By SAMUEL D. CONTE (*Wayne University*)

1. Introduction. It is well known that the equation governing the small deflections of a thin uniform plate, assumed homogeneous and isotropic, is

$$D\Delta\Delta w + P(x, y) = 0, \tag{1}$$

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