numbers is comparable to the heat dissipation. From (17) and (18) the efficiency of energy transfer between different wave numbers can be estimated as

$$\frac{1}{\kappa_v}\left[\frac{1}{F(k)}\frac{dF(k)}{dt}\right]_{\text{transfer}} \simeq \frac{1}{\kappa_H}\left[\frac{1}{M(k)}\frac{dM(k)}{dt}\right]_{\text{transfer}} \simeq \sqrt{F(k)k^3/\rho} \tag{33}$$

Hence kc is given approximately by

$$\left(\frac{\mu}{\rho} + \frac{c^2}{4\pi\sigma}\right) \simeq \sqrt{F(k_c)k_c^3/\rho}. \tag{34}$$

Using (29), (34) can be written as

$$\frac{k_c}{k_0} \simeq R^{3/4} \tag{35}$$

where R is the modified Reynolds number defined by

$$R \equiv \rho v_0 \Big/ \left(\mu + \frac{c^2\rho}{4\pi\sigma}\right)k_0 . \tag{36}$$

Equation (35) means that the range of the Kolmogoroff region increases with increasing Reynolds number. The behavior of the energy spectrum between ko, the lower limit of the Kolmogoroff region, and the boundary k_1 must depend on some extra-dimensional quantities related to the boundary or the detailed mechanism of energy supply. Hence no universal energy spectrum can be given in that region. The importance of this transition region will depend on the efficiency of energy coupling between the magnetic field and the velocity field and also on the detailed manner as how the energy is supplied. In ordinary turbulence it is assumed that such a transition is unimportant and this assumption is experimentally verified.[5] It may be natural to make the same assumption here. Due to the lack of experimental evidences, such assumption is always subject to criticisms and future verifications. But if this is the true case, then at high Reynolds number the total mechanical energy must be comparable to the total magnetic energy in a conducting fluid.

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[5]See e.g. Heisenberg, Zeit. f. Phy. 124, 628 (1947).

ON THE INVERSION OF THE VOLTERRA INTEGRAL EQUATION*

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When the given kernel of the Volterra integral equation can be represented as a Laplace transform, the same representation is obtained for the resolving kernel of the equation. For this case the solution is given in explicit form.

The Volterra integral equation

$$f(x) = g(x) + \int_0^x k(x-y)g(y)\,dy \qquad (x > 0) \tag{1}$$

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has proved to be of interest for the solution of many problems in physics. Its solution can be reduced to that of the simpler equation[1]

$$k(x) = m(x) + \int_0^x k(x - t)m(t)\,dt. \tag{2}$$

Here, $k(x)$ is the given kernel and $m(x)$ the resolving kernel of Eq. (1). Volterra[1] has given a solution of this equation in form of a series development. Doetsch[2] has solved it with the aid of the Laplace transform and Titchmarsh[3] with the aid of Fourier transforms. All these solutions however allow for very general conditions to be imposed upon the given kernel.

In physics integral equations of this type mainly occur in the theory of dielectric[4] and elastic[5] relaxation processes. In these applications $k(x)$ belongs to a very particular class of functions; it can be shown to satisfy conditions which will allow it to be represented as a Laplace transform

$$k(x) = \int_0^\infty r(s)e^{-xs}\,ds. \tag{3}$$

The purpose of the present paper is the discussion of the solution of Eq. (2) under the condition that $k(x)$ is given by the expression (3).

Equation (2) is of the convolution type. Application of the Laplace integral method of inversion[6] gives therefore

$$M(p) = \frac{K(p)}{1 + K(p)}, \tag{4}$$

where

$$K(p) = \int_0^\infty e^{-px}k(x)\,dx, \tag{5a}$$

$$M(p) = \int_0^\infty e^{-px}m(x)\,dx. \tag{5b}$$

It follows from (3) and (5) that

$$K(p) = \int_0^\infty e^{-px}\,dx \int_0^\infty r(s)e^{-xs}\,ds. \tag{6}$$

Suppose that $r(s)$ is a real function of integrable square over $(0, \infty)$. Then the order of integration in Eq. (6) can be inverted and

$$K(p) = \int_0^\infty \frac{r(s)}{s + p}\,ds. \tag{7}$$

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[1]V. Volterra and S. Pérès. *Théorie des fonctionelles*, Paris, Gauthier Villars, 1938.

[2]G. Doetsch, Math. Annalen **89**, 192 (1923).

[3]E. C. Titchmarsh, *Introduction to the theory of Fourier Integrals*, Oxford, Clarendon Press, 1937, p. 311.

[4]B. Gross, Zeits. f. Physik **107**, 217 (1937). Phys. Rev. **57**, 57 (1940).

[5]B. Gross, J. Applied Phys. **18**, 212 (1947); **19**, 257 (1948).

[6]R. V. Churchill, *Modern operational mathematics in engineering*. McGraw-Hill, New York, 1944, pg. 42.

$K(p)$ is therefore a Stieltjes transform. As such it is regular in the p plane cut along the negative real axis and the functions $K(pe^{\pm i\pi})$ are also of integrable square.

Applying the complex inversion theorem to Eq. (4) one obtains

$$m(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K(p)}{1 + K(p)} e^{pz}\, dp, \qquad (c > 0). \tag{8}$$

Suppose also that

$$| 1 + K(p) | > \delta > 0.$$

Then the line of integration can be deformed into a loop round the negative real axis, so that ultimately

$$m(z) = \frac{1}{2\pi i} \int_0^\infty \left[\frac{K(pe^{-i\pi})}{1 + K(pe^{-i\pi})} - \frac{K(pe^{+i\pi})}{1 + K(pe^{+i\pi})} \right] e^{-pz}\, dp. \tag{9}$$

Equation (9) can be written in the form

$$m(z) = \int_0^\infty \bar{r}(p) e^{-pz}\, dp. \tag{10}$$

This shows that under the given conditions the resolving kernel too is represented by a Laplace transform, and

$$\bar{r}(p) = \frac{1}{2\pi i} \left[\frac{K(pe^{-i\pi})}{1 + K(pe^{-i\pi})} - \frac{K(pe^{+i\pi})}{1 + K(pe^{+i\pi})} \right]. \tag{11}$$

Here, $K(pe^{\pm i\pi})$ is a shorthand way for writing $\lim_{\delta\to 0} K(-p \pm i\delta)$. Then, according to Eq. (7),

$$K(pe^{\pm i\pi}) = \lim_{\delta \to 0} \int_0^\infty \frac{r(s)}{s - p \pm i\delta}\, ds, \tag{12}$$

and

$$\lim_{\delta\to 0} \int_0^\infty \frac{r(s)}{s - p \pm i\delta}\, ds = \int_0^\infty \frac{r(s)}{s - p}\, ds \mp i\pi r(p), \tag{13}$$

where the integral denotes the principal value.[7] Substitution of (13) into (11) and decomposition into real and imaginary parts gives

$$\bar{r}(p) = \frac{r(p)}{\left[1 + \int_0^\infty \frac{r(s)}{s - p}\, ds \right]^2 + \pi^2 r^2(p)} \tag{14}$$

Equations (3), (10), and (14) represent the solution of the given equation.

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[7]The evaluation of this limit represents a particular case of the calculus of the boundary values of a function along a given path. The expression (13) is immediately obtained from the general formulae given e.g. by A. Hurwitz and R. Courant, *Funktionentheorie*, Julius Springer, Berlin 1929, p. 333.