GRAVITY WAVES OF FINITE AMPLITUDE

III. Steady, Symmetrical, Periodic Waves in a Channel of Finite Depth*

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Introduction. Part I of this series** is concerned with the theory of finite amplitude gravity waves in a channel of infinite depth, Part II deals with the problem of the solitary wave. In the present paper, the finite amplitude wave problem is considered for a channel of finite depth and it will be shown that Parts I and II are special cases of this problem. The basic approach will not be explained in detail here since it is given in Part I, but the problem will be solved independently of the earlier parts.

1. Statement of problem. We deal with an incompressible fluid in a channel with a horizontal base, the fluid having been set in motion in such a way that the flow is two-dimensional and irrotational. The fluid motion is periodic with wave length λ in

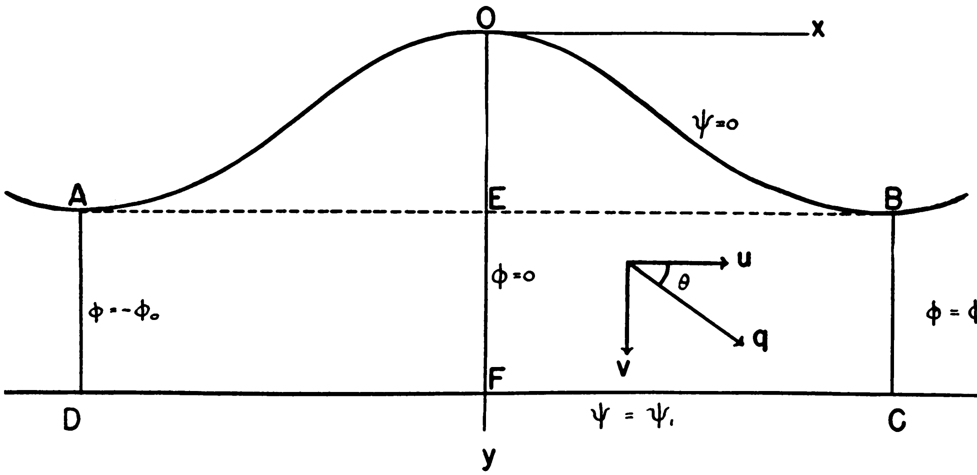

FIG. 1. $AB = \lambda$, $OE = a$, $AD = h$.

the horizontal x-direction, and is assumed to be symmetrical about a vertical line through any crest or trough of the wave pattern. It is assumed that the wave profile travels from right to left (see Figure 1) with velocity c. The axes Ox, Oy move with the crest O, Ox being horizontal and Oy vertically downwards. Relative to this set of axes the fluid motion is steady. A velocity potential ϕ and a stream function ψ exist for this motion and are defined by

$$\begin{aligned} d\phi &= u\,dx + v\,dy, \\ d\psi &= -v\,dx + u\,dy, \end{aligned} \tag{1.1}$$

where (u, v) are the velocity components relative to $O(x, y)$.

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From the symmetry assumption it follows that ϕ is constant along the verticals through $O, B \cdots$. We take $\phi = 0$ along OF, $\phi = -\phi_0$ along AD and $\phi = \phi_0$ along BC. The lower boundary is a streamline which we take to be $\psi = \psi_1$ and the upper free surface is the streamline $\psi = 0$. Over the upper free surface the pressure p is everywhere constant. If q denotes the velocity of a fluid particle relative to $O(x, y)$ and θ its inclination to the x axis, then $u + iv = qe^{i\theta} = ce^{\tau+i\theta}$. Suppose q_0 is the relative velocity at O of the fluid particles, then Bernoulli's equation applied to the free surface leads to

$$q^2 = q_0^2 + 2gy, \qquad \psi = 0, \tag{1.2}$$

where q and y refer to values on the free surface. By differentiation with respect to ϕ Levi-Civita transforms this equation to the form

$$\frac{\partial \theta}{\partial \psi} = -\frac{g}{c^3} e^{-3\tau} \sin \theta, \qquad \psi = 0. \tag{1.3}$$

The method used here consists in replacing this condition by

$$\frac{\partial \theta}{\partial \psi} = -\frac{gl}{c^3} e^{-3\tau} \sin 3\theta, \qquad \psi = 0 \tag{1.4}$$

where l is a constant at our disposal. An alternative form of (1.4) is

$$\mathscr{I}\left\{i \frac{d\xi}{dw} + \frac{gl}{c^3} e^{3\xi}\right\} = 0, \qquad \psi = 0 \tag{1.5}$$

where $\mathscr{I}$ denotes "imaginary part of", $w = \phi + i\psi$ and $\xi = -\tau + i\theta$.

The problem of the first approximation can now be stated as follows: ξ is a regular function of w in $0 \leq \psi \leq \psi_1$, is periodic and of period $2\phi_0$ in the ϕ direction, and is such that

$$\begin{aligned}
&\text{(a)} \qquad \mathscr{I}\{\xi\} = 0, \qquad \psi = \psi_1 ;\\
&\text{(b)} \qquad \mathscr{I}\{\xi\} = 0, \qquad \phi = 0, \qquad \phi = \pm\phi_0 ;\\
&\text{(c)} \qquad \mathscr{I}\left\{i \frac{d\xi}{dw} + \frac{gl}{c^3} e^{3\xi}\right\} = 0, \qquad \psi = 0.
\end{aligned} \tag{1.6}$$

In order to complete the statement of the problem it is necessary to postulate one further condition. In the infinitely deep channel problem and the solitary wave problem the actual fluid motion at infinity is specified ($\psi = \infty$ in the former, $\phi = \pm\infty$ in the latter) and in the present problem we must specify the actual flow at a convenient point of the fluid medium. This will be mentioned again later.

2. The method of solution. We first write $\chi = e^{-3\xi}$ and the problem in (1.6) may be restated as follows: χ is a regular function of w in $0 \leq \psi \leq \psi_1$, is periodic and of period $2\phi_0$ in the ϕ direction, and is such that

$$\begin{aligned}
&\text{(a)} \qquad \mathscr{I}\{\chi\} = 0, \qquad \psi = \psi_1 ;\\
&\text{(b)} \qquad \mathscr{I}\{\chi\} = 0, \qquad \phi = 0, \qquad \phi = \pm\phi_0 ;\\
&\text{(c)} \qquad \mathscr{I}\left\{\frac{i\, d\chi/dw - (3gl)/c^3}{\chi}\right\} = 0, \qquad \psi = 0.
\end{aligned} \tag{2.1}$$

Here, χ is defined within and on the boundaries of an infinite strip and we consider next how we may continue χ analytically throughout the whole of the w-plane. For this purpose we consider first of all the continuation of χ across $\psi = \psi_1$ into the strip $\psi_1 \leq \psi \leq 2\psi_1$. The line $\psi = 2\psi_1$ is the image of $\psi = 0$ in $\psi = \psi_1$; let P be any point of $\psi = 0$ and let Q be the image point of P in $\psi = \psi_1$. Suppose w_P is the value of w at P and w_Q that at Q. From Schwarz's principle of symmetry, if we wish to continue χ analytically across $\psi = \psi_1$, since $\mathscr{I}(\chi) = 0$ on $\psi = \psi_1$ we must define the value of χ at Q, namely $\chi(w_Q)$, as follows

$$\chi(w_Q) = \overline{\chi(w_P)}. \tag{2.2}$$

Here the bar denotes the conjugate, that is, if $\chi(w_P) = \alpha + i\beta$ then $\chi(w_Q) = \alpha - i\beta$. The functions α and β vary with ϕ only as P moves along the line $\psi = 0$. The function χ is defined in a similar way throughout the upper strip. In terms of α and β, condition (2.1c) becomes

$$\alpha \frac{d\alpha}{d\phi} + \beta\left(\frac{d\beta}{d\phi} + \frac{3gl}{c^3}\right) = 0, \qquad \psi = 0. \tag{2.3}$$

It is easily verified that (2.3) is the condition also that the function

$$g(\chi) = \frac{[i\, d\chi/dw + (3gl)/c^3]}{\chi} \tag{2.4}$$

has zero imaginary part on $\psi = 2\psi_1$. Thus, if we write

$$f(\chi) = \frac{[i\, d\chi/dw - (3gl)/c^3]}{\chi} \tag{2.5}$$

then we have $\mathscr{I}\{f(\chi)\} = 0$ on $\psi = 0$ and $\mathscr{I}\{g(\chi)\} = 0$ on $\psi = 2\psi_1$, the only change being the reversal of the gravitational term. Thus we do not have perfect symmetry about $\psi = \psi_1$, but the result suggests, and we proceed tentatively from this point, that we build up the following image or periodic picture in the $\pm\psi$ directions (see Figure 2).

In the breaking case a zero of χ appears on $\psi = 0$ and we introduce, first of all, a zero of χ at $w = -i[(\pi/\nu) - 3\psi_1]$, where ν is a real parameter which is such that $\pi/\nu \geq 3\psi_1$. We introduce a similar zero at the reflection of this point in $\psi = \psi_1$, that is at $w = i[(\pi/\nu) - \psi_1]$. This we do because $\mathscr{I}(\chi) = 0$ on $\psi = \psi_1$, in fact any distribution of singularities on $\phi = 0$ must be symmetrical about $\psi = \psi_1$. We do not consider the image of the zero at $w = -i[(\pi/\nu) - 3\psi_1]$ in the free surface $\psi = 0$ from the analogy with the solution in Part I, but consider next the "image" of the zero $w = i[(\pi/\nu) - \psi_1]$ in the free surface $\psi = 0$ where we must satisfy $\mathscr{I}\{f(\chi)\} = 0$. The simple zero of χ at $w = i[(\pi/\nu) - \psi_1]$ will, in general, be a simple pole of $f(\chi)$ and we introduce, therefore, a simple pole at $w = -i[(\pi/\nu) - \psi_1]$. Similarly the "image" of the simple zero at $w = -i[(\pi/\nu) - 3\psi_1]$ in the line $\psi = 2\psi_1$ will be a simple pole at $w = i[(\pi/\nu) + \psi_1]$, and since this pole has to be reflected in $\psi = \psi_1$, it follows that the poles at $w = -i[(\pi/\nu) - \psi_1]$ and $w = i[(\pi/\nu) + \psi_1]$ are double poles. Proceeding further, the image of the pole at $w = -i[(\pi/\nu) - \psi_1]$ in $\psi = 2\psi_1$ is a zero at $w = i[(\pi/\nu) + 3\psi_1]$ while the pole at $w = i[(\pi/\nu) + \psi_1]$ will have as image a zero at $w = -i[(\pi/\nu) + \psi_1]$. We now introduce at $\psi = \psi_1 + 2\pi/\nu$ and $\psi = \psi_1 - 2\pi/\nu$ lines along which $\mathscr{I}(\chi) = 0$ and at $\psi = 2\pi/\nu$ and $\psi = -2\pi/\nu + 2\psi_1$ lines along which $\mathscr{I}\{f(\chi)\} = 0$ and $\mathscr{I}\{g(\chi)\} = 0$ respectively. This pattern is then repeated indefinitely in the $\pm\psi$ directions along $\phi = 0$. Similar patterns exist on all the lines $\phi = \pm 2m\phi_0$ where m is any integer. There are no zeros nor poles of χ at any other points in the w-plane.

Since we have made $\mathcal{I}\{f(\chi)\} = 0$ along $\psi = 2\pi/\nu$ and $\mathcal{I}\{g(\chi)\} = 0$ along $\psi = 2\psi_1$ it follows that $\mathcal{I}(\chi) = 0$ along $\psi = \psi_1 + \pi/\nu$, this being the converse of Schwarz's principle of symmetry. Hence $\mathcal{I}(\chi) = 0$ on the perimeter of the rectangle $FKHG$ and

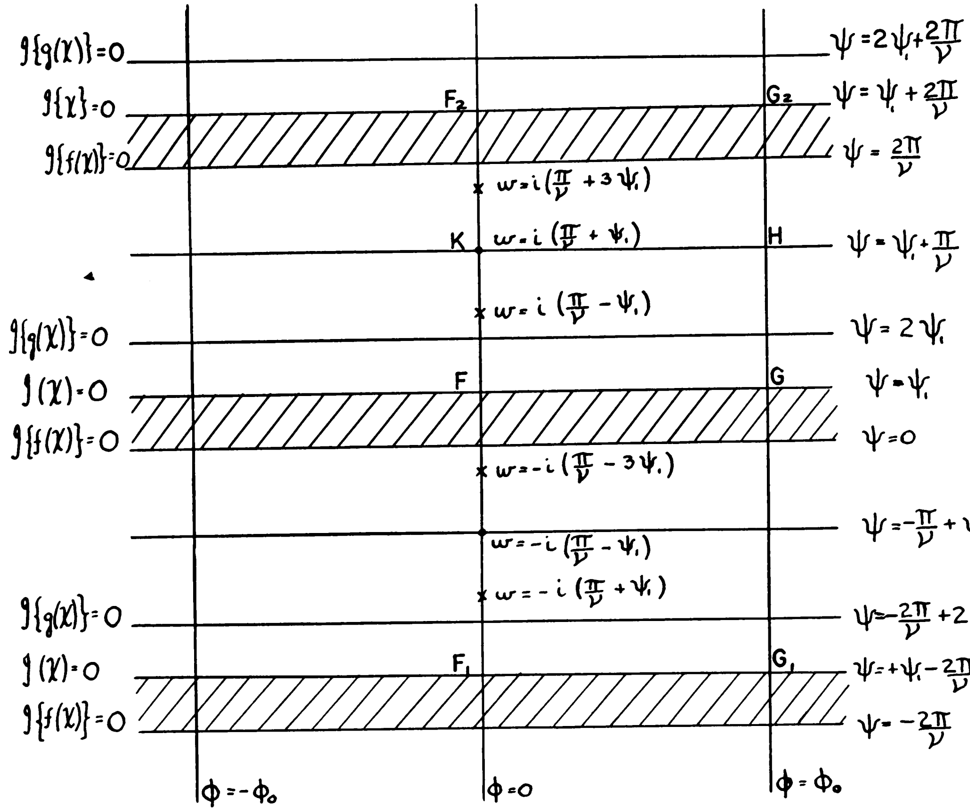

FIG. 2. x indicates a zero of χ, $\bullet$ indicates a pole of χ.

this rectangle can, therefore, be mapped on the upper half of the χ-plane and its perimeter into the real axis in the χ-plane. The mapping function is obtained from the Schwarz-Christoffel Theorem. Suppose the value of χ at F is A; at G let $\chi = A(1 - \mu)$ and at H let $\chi = A[1 - (\mu/k)^2]$, A, μ and k being real quantities. Then, since χ is infinite at K the transformation required is the solution of the differential equation

$$\frac{dw}{d\chi} = B\left(1 - \frac{\chi}{A}\right)^{-1/2}\left(\mu - 1 + \frac{\chi}{A}\right)^{-1/2}\left(\frac{\mu}{k^2} - 1 + \frac{\chi}{A}\right)^{-1/2}, \tag{2.6}$$

B being a complex constant in the general case. The appropriate solution of (2.6) is

$$\chi = A\left\{1 - \mu\,\mathrm{sn}^2\left[\frac{K}{\phi_0}(w - i\psi_1),\, k\right]\right\} \tag{2.7}$$

where sn (α, k) is* the usual Jacobian Elliptic Function with parameter k, whose real period is $4K$ and whose imaginary period is $2iK'$. In obtaining (2.7) from (2.6) the constant B has been so chosen as to make χ have period $2\phi_0$ in the ϕ direction; in this connection we use the relation $\operatorname{sn}(\alpha + 2K) = -\operatorname{sn}\alpha$. Furthermore the arbitrary constant of integration arising from the integration of (2.6) has been chosen so that χ is real when $\psi = \psi_1$; this enables us to satisfy (2.1a). We can determine μ as follows. Since the poles of sn (α, k) occur at points congruent to iK' and $2K + iK'$, on comparison with Figure 2, we obtain

$$\frac{\pi}{\nu} = \frac{\phi_0 K'}{K} \tag{2.8}$$

We know also from Figure 2 that a zero of χ occurs at $w = i[(\pi/\nu) - \psi_1]$ and hence

$$1 - \mu \operatorname{sn}^2\left[\frac{iK}{\phi_0}\left(\frac{\pi}{\nu} - 2\psi_1\right), k\right] = 0 \tag{2.9}$$

If we use (2.8) and the relation $\operatorname{sn}(\alpha + iK') = 1/k \operatorname{sn}\alpha$ it follows from (2.9) that μ is given by

$$\mu = k^2 \operatorname{sn}^2\left(\frac{2iK\psi_1}{\phi_0}, k\right) \tag{2.10}$$

and thus the required solution is $\chi = e^{-3\xi} = e^{3\tau - 3i\theta}$ where

$$\chi = A\left\{1 - k^2 \operatorname{sn}^2\left(\frac{2iK\psi_1}{\phi_0}, k\right) \operatorname{sn}^2\left[\frac{K}{\phi_0}(w - i\psi_1), k\right]\right\}. \tag{2.11}$$

Before proceeding further we shall verify that χ in (2.11) satisfies all the conditions enunciated in (2.1). The conditions of regularity in $0 \leq \psi \leq \psi_1$, of periodicity in the ϕ direction and condition (2.1a) are certainly satisfied. To demonstrate that (2.1b) is satisfied we use Jacobi's Imaginary Transformation, namely

$$\begin{aligned} \operatorname{sn}(iu, k) &= \frac{i \operatorname{sn}(u, k')}{\operatorname{cn}(u, k')}, \\ \operatorname{cn}(iu, k) &= \frac{1}{\operatorname{cn}(u, k')}, \\ \operatorname{dn}(iu, k) &= \frac{\operatorname{dn}(u, k')}{\operatorname{cn}(u, k')}, \end{aligned} \tag{2.12}$$

where $k' = (1 - k^2)^{1/2}$ is the complementary modulus. It then follows, when $\phi = 0$, that χ is real; similarly when $\phi = \phi_0$ we have $\operatorname{sn}[(K/\phi_0)(w - i\psi_1)] = \operatorname{sn}[K + (iK/\phi_0)(\psi - \psi_1)]$ and since $\operatorname{sn}(\alpha + K) = \operatorname{cn}\alpha/\operatorname{dn}\alpha$ the results (2.12) apply and make χ real. Thus condition (2.1b) is satisfied on $\phi = 0$ and $\phi = \phi_0$ and elsewhere by periodicity. We consider finally condition (2.1c) or what is equivalent (1.4).

The addition formula for the sn function leads to the result

*When the parameter is not specified it will be understood that it is k.

$$\operatorname{sn}\frac{K}{\phi_0}(w - i\psi_1) = \frac{\operatorname{sn}(K\phi/\phi_0)\operatorname{cn}[iK(\psi - \psi_1)/\phi_0]\operatorname{dn}[iK(\psi - \psi_1)/\phi_0]}{1 - k^2\operatorname{sn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi - \psi_1)/\phi_0]} + \frac{\operatorname{cn}(K\phi/\phi_0)\operatorname{dn}(K\phi/\phi_0)\operatorname{sn}[iK(\psi - \psi_1)/\phi_0]}{1 - k^2\operatorname{sn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi - \psi_1)/\phi_0]} \tag{2.13}$$

From (2.12) we see that cn $(i\alpha)$ and dn $(i\alpha)$ are real while sn $(i\alpha)$ is imaginary, hence we obtain from (2.11)

$$\alpha = e^{3\tau}\cos 3\theta$$

$$= A\left\{1 - k^2\operatorname{sn}^2\left(\frac{2i\psi_1K}{\phi_0}\right)\frac{\operatorname{sn}^2(K\phi/\phi_0)\operatorname{cn}^2[iK(\psi - \psi_1)/\phi_0]\operatorname{dn}^2[iK(\psi - \psi_1)/\phi_0]}{[1 - k^2\operatorname{sn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi - \psi_1)/\phi_0]]^2} + \frac{\operatorname{cn}^2(K\phi/\phi_0)\operatorname{dn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi - \psi_1)/\phi_0]}{[1 - k^2\operatorname{sn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi - \psi_1)/\phi_0]]^2}\right\}, \tag{2.14}$$

$$\beta = ie^{3\tau}\sin 3\theta$$

$$= 2Ak^2\operatorname{sn}^2\left(\frac{2i\psi_1K}{\phi_0}\right)\operatorname{sn}(K\phi/\phi_0)\operatorname{cn}(K\phi/\phi_0)\operatorname{dn}(K\phi/\phi_0)\operatorname{sn}[iK(\psi - \psi_1)/\phi_0] \times \frac{\operatorname{cn}[iK(\psi - \psi_1)/\phi_0]\operatorname{dn}[iK(\psi - \psi_1)/\phi_0]}{[1 - k^2\operatorname{sn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi - \psi_1)/\phi_0]]^2}. \tag{2.15}$$

Using (2.14) and (2.15) we can evaluate $\partial\theta/\partial\psi$. Since $i\tan 3\theta = \beta/\alpha$ we have

$$3ie^{6\tau}\frac{\partial\theta}{\partial\psi} = \alpha\frac{\partial\beta}{\partial\psi} - \beta\frac{\partial\alpha}{\partial\psi} \tag{2.16}$$

and we substitute now for α and β from (2.14) and (2.15). To simplify the analysis we write

$$S = \operatorname{sn}\frac{K\phi}{\phi_0}, \qquad C = \operatorname{cn}\frac{K\phi}{\phi_0}, \qquad D = \operatorname{dn}\frac{K\phi}{\phi_0},$$
$$s_1 = \operatorname{sn}\frac{iK\psi_1}{\phi_0}, \qquad c_1 = \operatorname{cn}\frac{iK\psi_1}{\phi_0}, \qquad d_1 = \operatorname{dn}\frac{iK\psi_1}{\phi_0}. \tag{2.17}$$

We then obtain at $\psi = 0$,

$$\frac{3\phi_0e^{6\tau}(\partial\theta/\partial\psi)(1 - k^2s_1^2S^2)^5}{2k^2KA^2\operatorname{sn}^2(2i\psi_1K/\phi_0)SCD}$$

$$= \left\{(1 - k^2S^2s_1^2)^2 - k^2\operatorname{sn}^2\left(\frac{2i\psi_1K}{\phi_0}\right)(S^2c_1^2d_1^2 + C^2D^2s_1^2)\right\}\{(c_1^2d_1^2 - s_1^2d_1^2 - k^2s_1^2c_1^2) \times (1 - k^2s_1^2S^2) + 4k^2s_1^2c_1^2d_1^2S^2\} - 2k^2\operatorname{sn}^2\left(\frac{2i\psi_1K}{\phi_0}\right)s_1^2c_1^2d_1^2 \times \{(1 - k^2s_1^2S^2)[S^2(d_1^2 + k^2c_1^2) - C^2D^2] - 2k^2S^2(S^2c_1^2d_1^2 + C^2D^2s_1^2)\}. \tag{2.18}$$

The detailed reduction of the right hand side of (2.18) can be omitted here and we quote merely the final simplified version of (2.18), namely

$$\frac{3\phi_0 e^{6\tau}\,(\partial\theta/\partial\psi)(1 - k^2S^2s_1^2)^2}{2k^2KA^2\,\mathrm{sn}^2\,[(2i\psi_1K)/\phi_0]SCD} = \frac{(1 - 2s_1^2 + k^2s_1^4)(1 - 2k^2s_1^2 + k^2s_1^4)}{1 - k^2s_1^4}\,. \tag{2.19}$$

If we now use the formulae for sn $2u$, cn $2u$ and dn $2u$ in terms of sn u, cn u and dn u given in Whittaker and Watson's Modern Analysis p. 498 Ex. 5 it follows that the right hand side of (2.19) is simply

$$(1 - k^2s_1^4)\,\mathrm{cn}\left(\frac{2i\psi_1K}{\phi_0}\right)\mathrm{dn}\left(\frac{2i\psi_1K}{\phi_0}\right)$$

so that

$$e^{6\tau}\frac{\partial\theta}{\partial\psi}\bigg|_{\psi=0} = \frac{2}{3}\frac{k^2KA^2}{\phi_0}\,\mathrm{sn}^2\left(\frac{2i\psi_1K}{\phi_0}\right)(1 - k^2s_1^4)\,\mathrm{cn}\left(\frac{2i\psi_1K}{\phi_0}\right) \cdot \mathrm{dn}\left(\frac{2i\psi_1K}{\phi_0}\right)\frac{SCD}{(1 - k^2s_1^2S^2)^2}\,. \tag{2.20}$$

From (2.15) we have

$$ie^{3\tau}\sin 3\theta\bigg|_{\psi=0} = -2Ak^2\,\mathrm{sn}^2\left(\frac{2i\psi_1K}{\phi_0}\right)\frac{SCDs_1c_1d_1}{(1 - k^2s_1^2S^2)^2}\,, \tag{2.21}$$

hence combining (2.20) and (2.21) we obtain

$$\frac{\partial\theta}{\partial\psi}\bigg|_{\psi=0} = -\frac{1}{3}\left\{\frac{2iKA}{\phi_0}\,\frac{\mathrm{cn}\,[(2i\psi_1K)/\phi_0]\,\mathrm{dn}\,[(2i\psi_1K)/\phi_0]}{\mathrm{sn}\,[(2i\psi_1K)/\phi_0]}\right\}e^{-3\tau}\sin 3\theta. \tag{2.22}$$

If we now return to the fundamental boundary condition (1.11) we observe that it may be satisfied by (2.11) provided that

$$\frac{3gl}{c^3} = \frac{2iKA}{\phi_0}\,\frac{\mathrm{cn}\,[(2i\psi_1K)/\phi_0]\,\mathrm{dn}\,[(2i\psi_1K)/\phi_0]}{\mathrm{sn}\,[(2i\psi_1K)/\phi_0]}\,. \tag{2.23}$$

This is the formula for the velocity of propagation of the waves. Although the right hand side contains an i in the numerator it is real since cn (iu) and dn (iu) are real while sn (iu) is imaginary.

This solution of the problem is completed by defining A. If we postulate that the actual fluid velocity at $w = i\psi_1$ is U, the relative velocity at this point is $U + c$ hence from (2.11), since $\chi = e^{-3\xi} = (u - iv)^3/c^3$, we obtain

$$A = \left(\frac{U}{c} + 1\right)^3. \tag{2.24}$$

The solution of the problem can now be written in the form

$$\left(\frac{dw}{dz}\right)^3 = (U + c)^3\left\{1 - k^2\,\mathrm{sn}^2\left(\frac{2i\psi_1K}{\phi_0}\,,\,k\right)\mathrm{sn}^2\left[\frac{K}{\phi_0}(w - i\psi_1),\,k\right]\right\} \tag{2.25}$$

and the wave velocity formula in the form

$$\frac{3gl}{(U + c)^3} = \frac{2iK}{\phi_0}\,\frac{\mathrm{cn}\,[(2i\psi_1K)/\phi_0]\,\mathrm{dn}\,[(2i\psi_1K)/\phi_0]}{\mathrm{sn}\,[(2i\psi_1K)/\phi_0]}\,. \tag{2.26}$$

This wave velocity formula can be expressed in fairly simple terms by using the Jacobi and Landen Transformations. In the first place, applying the Jacobi transformation to the right hand side of (2.26) we obtain

$$\frac{3gl}{(U+c)^3} = \frac{2K}{\phi_0} \frac{\mathrm{dn}\,[(2\psi_1 K)/\phi_0,\, k']}{\mathrm{sn}\,[(2\psi_1 K)/\phi_0,\, k']\;\mathrm{cn}\,[(2\psi_1 K)/\phi_0,\, k']} . \tag{2.26'}$$

Landen's transformation states that

$$\mathrm{sn}\left\{(1+k')u,\, \frac{1-k'}{1+k'}\right\} = (1+k')\,\frac{\mathrm{sn}\,(u,k)\;\mathrm{cn}\,(u,k)}{\mathrm{dn}\,(u,k)} ;$$

hence

$$\frac{3gl}{(U+c)^3} = \frac{2(1+k)K/\phi_0}{\mathrm{sn}\,\{2(1+k)K\psi_1/\phi_0\,,\,(1-k)/(1+k)\}} . \tag{2.27}$$

If we multiply both sides of this by ψ_1 the right hand side is then in non-dimensional form.

The breaking case is at once evident from Figure 2, for this corresponds to the case when the zero at $w = -i[(\pi/\nu) - 3\psi_1]$ lies on $\psi = 0$, that is the breaking case occurs when

$$\frac{\pi}{\nu} = 3\psi_1 . \tag{2.28}$$

Using (2.8) this may be written in the form

$$\frac{K\psi_1}{\phi_0} = \frac{1}{3} K'. \tag{2.29}$$

This result is easily verified also by writing $\chi = 0$ in (2.11) and solving for $K\psi_1/\phi_0$ when $w = 0$. This quantity π/ν is one of the important parameters of the problem for we may note also that the small amplitude problem corresponds to large values of π/ν, when the system of singularities moves away from the region $0 \leq \psi \leq \psi_1$.

There are two points which must be mentioned before concluding this section, the first is that the problem of the uniqueness of the solution (2.11) has not been solved. Secondly, the problem of higher approximations has not been attempted, but it seems fairly certain that the higher approximations may be derived as in Part II by using the more exact boundary condition

$$\frac{\partial\theta}{\partial\psi} = -\frac{g}{c^3}\left\{\frac{1}{3} e^{-3\tau}\sin 3\theta + \frac{4}{81} e^{-3\tau}\sin^3 3\theta + \frac{16}{729} e^{-3\tau}\sin^5 3\theta + \cdots\right\} \tag{2.30}$$

and to assume a solution of the form

$$e^{-3\xi} = A\left\{1 + \mu_1\,\mathrm{sn}^2\left[\frac{K}{\phi_0}(w - i\psi_1)\right] + \mu_3\,\mathrm{sn}^6\left[\frac{K}{\phi_0}(w - i\psi_1)\right] + \cdots\right\}, \tag{2.31}$$

where μ_1 and μ_3 are constants. The results of Parts I and II suggest that $-\mu_1$ is the same as the μ of (2.10) and

$$\mu_3 = \frac{1}{54}\,\mathrm{sn}^6\left(\frac{2iK\psi_1}{\phi_0}\,,\,k\right);$$

also if μ_3 and higher terms are neglected, the results are likely to be in error by about 13% at the breaking point but will be much less for a non-breaking wave (here we assume that $l = 1/3$).

An alternative method of deriving the solution (2.11) is to make use of the isolated wave singularity distribution, given in Part II, and to use theta functions for the more extended distribution of the present periodic wave problem. This has been done, but is no shorter than the method expounded here.

3. Special cases of the solution (2.11). We demonstrate first that the problems of the infinitely deep channel and the solitary wave are merely particular cases of the solution (2.11) to the order of approximation in this paper. For this purpose we use the connections between the Jacobian Elliptic Functions and trigonometric functions, namely

$$\operatorname{sn}\left(\frac{2Kx}{\pi}, k\right) = \frac{2\pi}{Kk}\left\{\frac{q^{1/2}\sin x}{1-q} + \frac{q^{3/2}\sin 3x}{1-q^3} + \cdots\right\},$$

$$\operatorname{cn}\left(\frac{2Kx}{\pi}, k\right) = \frac{2\pi}{Kk}\left\{\frac{q^{1/2}\cos x}{1+q} + \frac{q^{3/2}\cos 3x}{1+q^3} + \cdots\right\},$$

$$\operatorname{dn}\left(\frac{2Kx}{\pi}, k\right) = \frac{\pi}{2K} + \frac{2\pi}{K}\left\{\frac{q\cos 2x}{1+q^2} + \frac{q^2\cos 4x}{1+q^4} + \cdots\right\},$$

where

$$\frac{1-k'^{1/2}}{1+k'^{1/2}} = \frac{2q + 2q^9 + 2q^{25} + \cdots}{1 + 2q^4 + 2q^{16} + \cdots}.$$

As k tends to zero, q tends to zero and we have in the limit

$$\lim_{k\to 0}\left(\frac{k}{q^{1/2}}\right) = 4. \tag{3.1}$$

Furthermore as $k \to 0$, $K \to \pi/2$ and $K' \to \infty$, hence

$$\lim_{k\to 0}\operatorname{sn}\left(\frac{2Kx}{\pi}, k\right) = \sin x, \quad \lim_{k\to 0}\operatorname{cn}\left(\frac{2Kx}{\pi}, k\right) = \cos x, \quad \lim_{k\to 0}\operatorname{dn}\left(\frac{2Kx}{\pi}, k\right) = 1. \tag{3.2}$$

Let us consider now what happens to the solution of the previous section as the parameter k tends to zero. When ϕ_0 is constant we see from (2.8) that π/ν tends to infinity, that is, the system of singularities moves to an infinite distance from the strip $0 \le \psi \le \psi_1$. In the problem of Part I, namely $\psi_1 = \infty$, there is a line of zeros parallel to $\psi = 0$ lying in the half plane $\psi \le 0$ and to reproduce the results of Part I we, therefore, make $k \to 0$ and $\psi_1 \to \infty$ in such a way that one line of zeros remains in the finite part of the w-plane. As $k \to 0$ we thus obtain

$$\operatorname{sn}\left\{\frac{K}{\phi_0}(w - i\psi_1)\right\} \to \frac{1}{2i}\{e^{i\pi w/2\phi_0}e^{\pi\psi_1/2\phi_0} - e^{-i\pi w/2\phi_0}e^{-\pi\psi_1/2\phi_0}\},$$

$$\operatorname{sn}\left(\frac{2i\psi_1 K}{\phi_0}\right) \to \frac{1}{2i}\{e^{-\pi\psi_1/\phi_0} - e^{\pi\psi_1/\phi_0}\};$$

hence as $k \to 0$ and $\psi_1 \to \infty$

$$\operatorname{sn}\left\{\frac{K}{\phi_0}(w - i\psi_1)\right\} \sim \frac{1}{2i} e^{i\pi w/2\phi_0} e^{\pi\psi_1/2\phi_0}.$$

$$\operatorname{sn}\left(\frac{2i\psi_1 K}{\phi_0}\right) \sim -\frac{1}{2i} e^{\pi\psi_1/\phi_0}.$$

Thus from (2.11), as $k \to 0$, $\psi_1 \to \infty$

$$e^{-3\xi} \sim A\left\{1 - \left(\frac{1}{16} k^2 e^{3\pi\psi_1/\phi_0}\right) e^{i\pi w/\phi_0}\right\}. \tag{3.3}$$

The two limiting processes are independent of one another; if we now postulate that

$$\lim_{\psi_1 \to \infty} \lim_{k \to 0} \frac{1}{16} k^2 e^{3\pi\psi_1/\phi_0} = \text{constant} \tag{3.4}$$

and make $U = 0$ in (2.24), we obtain the solution of Part I, namely

$$e^{-3\xi} = 1 - Be^{i\pi w/\phi_0}. \tag{3.5}$$

The velocity of propagation formula (2.26) may be discussed in a similar way. With $U = 0$ the left hand side is $3gl/c^3$ and the right hand becomes

$$\frac{i\pi}{\phi_0} \frac{\cos\,[(\pi i\psi_1)/\phi_0]}{\sin\,[(\pi i\psi_1)/\phi_0]}$$

as $k \to 0$, so that as $\psi_1 \to \infty$ we obtain

$$\frac{3gl}{c^3} = \frac{\pi}{\phi_0} \tag{3.6}$$

which is similar to Part I.

Consider next what happens when we make k tend to its upper limit, namely unity. As $k \to 1$, $k' \to 0$, and thus we can deduce the relevant limits from Jacobi's Imaginary Transformation. We obtain from (2.8) and (2.12):

$$\operatorname{sn}\left\{\frac{K}{\phi_0}(w - i\psi_1),\, k\right\} = \operatorname{sn}\left\{\frac{\nu K'}{\pi}(w - i\psi_1),\, k\right\}$$

$$= i\,\frac{\operatorname{sn}\,\{(\nu K'/i\pi)(w - i\psi_1),\, k'\}}{\operatorname{cn}\,\{(\nu K'/i\pi)(w - i\psi_1),\, k'\}},$$

and as $k' \to 0$, since $K' \to (1/2)\pi$, we obtain

$$\lim_{k \to 1} \operatorname{sn}\left\{\frac{K}{\phi_0}(w - i\psi_1),\, k\right\} = \tanh\frac{1}{2}\nu(w - i\psi_1). \tag{3.7}$$

Similarly,

$$\lim_{k \to 1} \operatorname{sn}\left(\frac{2i\psi_1 K}{\phi_0},\, k\right) = i\tan\nu\psi_1\,. \tag{3.8}$$

It will be noted that by first substituting for the ratio K/ϕ_0 in the above expressions we avoid the difficulty of indeterminacy of this ratio. The solution (2.11) now becomes

$$\begin{aligned}\chi &= A\{1 + \tan^2 \nu\psi_1 \tanh^2 \tfrac{1}{2}\nu(w - i\psi_1)\}\\ &= A \sec^2 \nu\psi_1\{1 - \sin^2 \nu\psi_1 \operatorname{sech}^2 \tfrac{1}{2}\nu(w - i\psi_1)\}\end{aligned} \tag{3.9}$$

which is the isolated wave solution of Part II, A being equal to $\cos^2 \nu\psi_1$. Similarly the wave velocity formula (2.26′) becomes

$$\frac{3gl}{c^3} = \frac{(2 \cos^2 \nu\psi_1)[(K'\nu)/\pi]}{\sin [(2\psi_1\nu K')/\pi] \cos [(2\psi_1\nu K')/\pi]};$$

hence

$$\frac{3gl}{c^3} = \nu \cot \nu\psi_1$$

as in Part II.

We can summarize the results of varying parameters as follows:

(a) $k \to 0$, $\psi_1 \to \infty$; finite amplitude waves in a channel of infinite depth;

(b) $k \to 1$; isolated wave;

(c) $k \to 0$; ψ_1 , ϕ_0 finite; small amplitude waves in a channel of finite depth;

(d) $K\psi_1/\phi_0 = \frac{1}{3}K'$; breaking condition at the crest.

(e) $0 \leq K\psi_1/K'\phi_0 \leq 1/3$.

The result (e) follows from expressing the condition that π/ν must lie between $3\psi_1$ and infinity. The result (c) follows from the early part of this section for in this case formula (2.11) becomes

$$e^{-3\xi} = A\left\{1 + k^2 \sinh^2 \frac{\pi\psi_1}{\phi_0} \sin^2 \frac{\pi}{2\phi_0} (w - i\psi_1)\right\}$$

and, k being small, this approximates to the usual solution for small amplitude waves. Furthermore the wave velocity formula in this case becomes

$$\frac{g}{c^3} = \frac{\pi}{\phi_0} \coth \frac{\pi\psi_1}{\phi_0}$$

and with the usual assumptions $\psi_1 = ch$, $\phi_0 = \frac{1}{2}c\lambda$ this becomes

$$c^2 = \frac{g\lambda}{2\pi} \tanh \left(\frac{2\pi h}{\lambda}\right).$$