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INDUCED MASS WITH FREE BOUNDARIES*

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The concept of induced mass, and some of its properties, are extended to the case of an incompressible liquid having a free surface. The usual[1] treatment of the case of a non-viscous fluid extending to infinity must be considerably changed to do this.

1. Minimum principle. Let R be a region filled with an incompressible liquid, bounded in part by a *wetted wall* W, and in part by a *free surface* S at constant pressure p_0. The region R moves with the liquid, and may extend to infinity in some directions.

Suppose the fluid *accelerated from rest*, by an acceleration of W whose inward normal component toward the liquid is an arbitrary function $f(\mathbf{x})$ of position. Letting $\mathbf{u} = \mathbf{u}(\mathbf{x}; t)$ denote liquid velocity and $\mathbf{a} = \partial\mathbf{u}/\partial t$ denote acceleration, clearly

$$\text{Div } \mathbf{a} = \Sigma\partial(\partial u_k/\partial t)/\partial x_k = \partial(\Sigma\partial u_k/\partial x_k)/\partial t = 0, \tag{1}$$

by incompressibility. Similarly, the Navier-Stokes equations with *gravity neglected* are[2]

$$Du_i/Dt = -\partial p/\rho\partial x_i + \nu\nabla^2 u_i\,. \tag{2}$$

Since $\mathbf{u} = 0$ initially, the *initial* acceleration therefore satisfies $a_i = Du_i/Dt = -\partial p/\rho\partial x_i$, or, setting $A = (p_0 - p)/\rho$,

$$\mathbf{a} = \nabla A, \qquad \text{where} \qquad p = p_0 - \rho A, \qquad \textit{initially}. \tag{3}$$

Combining (3) with (1), we get

$$\nabla^2 A = 0 \qquad \text{in } R. \tag{4}$$

The free surface condition is simply

$$A = 0 \qquad \text{on} \qquad S. \tag{5}$$

Finally, by continuity, we have

$$a_{\text{normal}} = \partial A/\partial n = f(\mathbf{x}) \qquad \text{on } W. \tag{6}$$

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[1]Given in [1], Chap. V; in [3]; and in Chap. VI of Lamb's *Hydrodynamics*.

[2]Here and below D/Dt is the substantial derivative $\partial/\partial t + u_k\,\partial/\partial x_k$, while $\partial/\partial\, n$ denotes the inward normal derivative on the surface of the liquid.

By potential theory,[3] conditions (4) to (6) uniquely determine A; we shall call A the *acceleration potential* associated with the given acceleration of W; by (3), A determines p.

THEOREM 1. The *acceleration kinetic energy*

$$T = \frac{1}{2} \rho \iiint_R \nabla A \cdot \nabla A \, dR \tag{7}$$

is minimized by the free surface condition (5), relative to all other volume-conserving flows in R satisfying (6).

Remark. The acceleration kinetic energy is half the second time derivative of the ordinary kinetic energy.

Proof. Let $\nabla A + \mathbf{b}$ be any other volume-conserving flow satisfying (6). Then

$$\text{Div } \mathbf{b} = \text{Div } \mathbf{b} + \nabla^2 A = \text{Div } (\nabla A + \mathbf{b}) = 0$$

Consider now the expanded acceleration kinetic energy

$$T = \frac{1}{2} \rho \int_R (\nabla A + \mathbf{b}) \cdot (\nabla A + \mathbf{b}) \, dR$$

$$= T_0 + \rho \int_R (\nabla A \cdot \mathbf{b}) \, dR + \frac{1}{2} \rho \int_R (\mathbf{b} \cdot \mathbf{b}) \, dR.$$

Since the last term is positive unless $\mathbf{b}$ vanishes identically, the theorem will be proved if we can show that the middle integral is zero. But since Div $\mathbf{b} = 0$, clearly

$$\text{Div } (A \cdot \mathbf{b}) = A \text{ Div } \mathbf{b} + (\nabla A) \cdot \mathbf{b} = \nabla A \cdot \mathbf{b}.$$

Hence, by the Divergence Theorem, letting b_n denote the outward normal component of $\mathbf{b}$,

$$\int_R (\nabla A \cdot \mathbf{b}) \, dR = -\int_W A b_n \, dS - \int_S A b_n \, dS. \tag{8}$$

Since $\nabla A + \mathbf{b}$ satisfies (6) on W, $b_n = 0$ on W and the first surface integral in (8) vanishes. By (5), the second surface integral vanishes; hence the proof is complete.

When S is void, our result reduces to the classical case ([2], p. 84). There is however no relation between the kinetic energy for *steady* flow with a free boundary, which is always infinite in an infinite non-viscous stream, and virtual mass. In this respect, the case of flows with free boundaries is unlike the classical case.

2. General interpretation. The acceleration potential A described in Sec. 1 has a very general interpretation. Let a fluid motion, defined by an "undisturbed" velocity field $\mathbf{U} = \mathbf{U}(x; t)$, be altered at time $t = 0$ by an additional instantaneous normal acceleration $f(\mathbf{x})$ of W. Because of incompressibility, we have for the perturbed velocity field $\mathbf{U} = \mathbf{u}$,

$$\text{Div } \mathbf{u} = 0, \qquad \text{whence} \qquad \text{Div } \mathbf{a} = 0, \tag{9}$$

where $\mathbf{a} = \partial \mathbf{u} / \partial t$ denotes the additional acceleration.

[3]For existence and uniqueness theorems, see O. D. Kellogg, *Potential theory*, pp. 218, 315. Although such theorems have been proved only for a restricted class of infinite regions, there is no reason to doubt their general validity.

Subtraction of the Navier-Stokes equations for $\mathbf{U}$ from those for $\mathbf{U} + \mathbf{u}$ gives

$$\frac{\partial u_i}{\partial t} = -u_k \frac{\partial U_i}{\partial x_k} - U_k \frac{\partial u_i}{\partial x_k} - u_k \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \; . \tag{10}$$

Here p denotes the increment in the pressure field due to the additional acceleration of W; moreover, this is true even if gravity is considered. Since the perturbation begins at $t = 0$, clearly $\mathbf{u}(x; 0) = 0$, whence (10) reduces to

$$a_i = -\partial p / \rho \partial x_i \, , \qquad \text{at} \qquad t = 0. \tag{11}$$

Since the rest of the reasoning leading to (4) to (7) applies, we get the following result.

THEOREM 2. The instantaneous pressure distribution required to accelerate a moving incompressible fluid is the same as if the fluid were at rest.

The preceding result can be extended to the case of an "impulsive" velocity change, by integrating (10) over a short interval of time, which is then allowed to tend to zero. It is readily seen that if $\mathbf{u}$ is uniformly bounded and $\int\int \nabla^2 \mathbf{u} \, dR \, dt = o(1)$ under these circumstances, and if the normal impulse per unit area is defined as $p^* = \mathrm{Lim}_{\Delta t \to 0} \int_0^{\Delta t} p dt$, we get the limiting analog of equation (3),

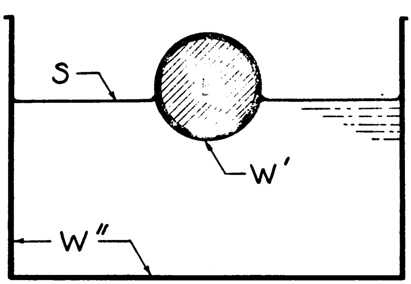
$$u_i = -\partial p^* / \rho \partial x_i \qquad \text{on} \qquad W. \tag{3*}$$

From (9), the fact that $p^* = 0$ on the free surface S, and continuity we get similarly equations analogous to (4) to (6). From these we deduce the following result.

THEOREM 3. The impulse $p^*(\mathbf{x})$ per unit area required to produce an additional normal "impulsive velocity" $u_n = f(\mathbf{x})$ of W, is $-\rho A$, where A is the acceleration potential of Sec. 1.

In the case of the irrotational motion of a non-viscous fluid, since $\nabla^2 \mathbf{u} = 0$, the assumption about $\nabla^2 \mathbf{u}$ (which is hard to prove rigorously) is superfluous.

3. Induced mass tensor. To define an *induced mass tensor*, let $W = W' + W''$ consist in part of the wetted area W' of an accelerated *missile*, and otherwise of the fixed walls W'' of a *container*. The case of a ball, floating on the surface of the water in a pail (Fig. 1) is typical.



We let A^1, A^2, A^3 denote the acceleration potentials in R for unit translations of W' parallel to the axes; let A^4, A^5, A^6 denote those for unit rigid rotations of W' about the coordinate axes; in all cases we assume $\partial A / \partial n = 0$ on W'', and $A = 0$ on S. We then define the symmetric 6×6 induced mass tensor (matrix) $\| T_{hk} \|$, as in the ordinary case ([1], p. 154), by

$$T_{hk} = \rho \iiint_R \nabla A^h \cdot \nabla A^k \, dR = T_{kh} \tag{12}$$

Our aim is to show that the T_{hk} have most of their familiar properties. Although the preceding definition of the T_{hk} was given in principle by L. I. Sedov [4], he did not deduce the properties proved below.

First, note that the diagonal components T_{hh} of induced mass, satisfy the conditions of Theorem 1. From this fact the following result follows immediately.

Corollary 1. The diagonal components T_{hh} of induced mass are increased if either (i) the region R occupied by liquid is increased (i.e., the free boundaries are pushed out), or (ii) free surface area is replaced by container walls or wetted missile area.

Corollary 2. Let a volume ΔV of liquid be replaced by missile. The new translation induced mass T'_{hh} satisfies the inequality.

$$T'_{hh} \geq T_{hh} - \rho \Delta V \tag{13}$$

Proof. Let $R' = R - \Delta V$ denote the reduced volume occupied by liquid. Consider the acceleration field of R, under which ΔV is accelerated as a rigid body under h-translation, while R' is given the acceleration corresponding to T'_{hh}. By Theorem 1, $T'_{hh} + \rho \Delta V$ will exceed T_{hh}, proving formula (13). This result can be extended to rotational components, if $\rho \Delta V$ is replaced by the appropriate rigid moment of inertia.

Again, we have by Green's second identity

$$-T_{hk} = \rho \int_S A^h \frac{\partial A^k}{\partial n} dS + \rho \int_{W'} A^h \frac{\partial A^k}{\partial n} dS + \rho \int_{W''} A^h \frac{\partial A^k}{\partial n} dS$$

By (5), $A^h = 0$ on S, and so the first summand is zero; by (6), $\partial A^k / \partial n = 0$ on W'', hence the last summand is zero.

Using (3) and reversing signs, we therefore have

$$T_{hk} = \int_{W'} (p_h - p_0) \frac{\partial A^k}{\partial n} dS, \tag{14}$$

where p_0 is the free surface pressure, and p_h is the pressure on W' under unit h-acceleration from rest. Also on W', $\partial A^k / \partial n = \partial x^k / \partial n$ $[k = 1, 2, 3]$ and $\partial A^4 / \partial n = x_2 \partial x_3 / \partial x_n - x_3 \partial x_2 / \partial x_n$ etc., by definition. Hence, cancelling out the effect of the constant pressure p_0 over the missile, we have

Theorem 4. The tensor component T_{hk} represents the total h-component of initial *pressure force* required to produce a unit k-acceleration of the missile.

In view of the linearity of (3) to (6) in A and p, we deduce immediately the following corollary.

Corollary 1. If, at any instant, the configuration and flow field are given, the instantaneous effect of an additional acceleration of the missile, with components $a_1, \cdots a_6$, is to produce components $\sum_{k=1}^{6} T_{hk} a_k$ of reaction in the liquid.

This result will be compared with experimental data in Sec. 5.

4. Momentum interpretation. If there is no container, so that $W = W'$ and $W'' = 0$, momentum interpretations of the T_{hk} are also possible. I shall show this directly from momentum considerations. (To make this rigorous, one must consider, as in [1] or [3], the convergence of the momentum integrals involved.)

Let the missile, supposed of finite diameter, be given initial acceleration from rest, and let C by any cylinder parallel to the x_h-axis, whose finite cross-section contains the wetted area $W = W'$ of the missile. The total h-momentum[4] in C is finite; moreover,

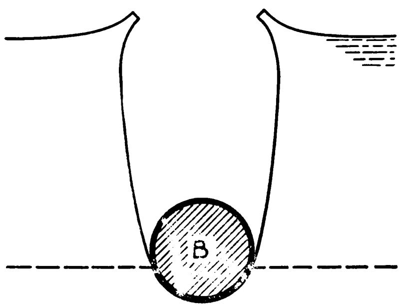
[4]By h-momentum, we mean linear momentum parallel to the x_h-axis.

since the liquid is at rest initially, the rate of convection of fluid h-momentum out of C is zero. Since the fluid is non-viscous, the shear stresses across this boundary are also zero; hence the rate of total transfer of h-momentum across the boundary of C is zero. We conclude that the time rate of increase of h-momentum of the liquid in C, is equal to the h-component of pressure force.

A corresponding result holds for moments of momentum; in this case C must be a solid of revolution containing W, with the axis about which the moment is taken, for axis of symmetry. Combining with Theorem 2, we get the following result. (We define an "h-curve" to be a straight line parallel to the x_h-axis, if $h = 1, 2, 3$; and to be a circle perpendicular to the x_{h-3}-axis, with center on the x_{h-3}-axis, if $h = 4, 5, 6$.)

THEOREM 5. Let a missile be given a unit k-acceleration from rest, in a liquid bounded by the missile wetted area and a free surface. Then the rate of increase of the h-component of liquid momentum in any region C bounded by h-curves, which contains W, is exactly T_{hk} initially.

5. Application. we can apply the corollary of Theorem 4 to the case of a missile B travelling vertically in a cavity, as in Fig. 2. The reaction to acceleration of B will be at least as great as if the liquid were confined to the underside of a plane bounding the wetted area of B (indicated as a dashed line in Fig. 2).



In this case, as first noted by von Kármán [6], the acceleration potential can be obtained by symmetry, using the reflection principle. It follows that the instantaneous inertia opposed by the liquid to the acceleration of B, should be *between* 50 per cent *and* 100 per cent *of that offered if there were no cavity, and the same area were wetted.*

This is not necessarily the same as the change in the cavity drag coefficient due to *steady* acceleration or deceleration (cf. Sec. 1, end). In fact, data obtained at the Naval Ordnance Laboratory[5], indicate that this change probably corresponds to less than 25 per cent of that occurring if there were no cavity.

The preceding results also apply to the case of impact on water, for which the momentum interpretation of §4 may be of interest. In this case, a rigorous lower bound to the loss of energy at impact is obtained by Theorem 1. Unfortunately, about 50 per cent of the energy lost at impact is presumably absorbed by the energy of compression (and radiated as acoustic energy); hence this bound is excessively low.

However, as the impact phase has been extensively discussed by other authors ([4], [5], [6]), especially in connection with the landing of seaplane floats, we shall not discuss it further here.

[5]A. May and J. Woodhull, "The virtual mass of a sphere entering water vertically", *Journal of Applied Physics* 21 1285-9 (1950).

6. Correction for hydrostatic force. Since gravity has been neglected above, it is interesting to have a rough estimate of the effect of gravity on the pressure exerted by a liquid on a missile moving through it with wetted area W' trailed by a cavity (Fig. 2). We suppose the liquid incompressible, and bounded by W', container walls W'', and a free surface S. The additional instantaneous acceleration $\mathbf{b}$ due to a vertical gravity field with intensity g satisfies $\mathbf{b} = g\nabla B$, where $\nabla^2 B = 0$, $B = y$ (depth coordinate) on S, and $\partial B/\partial n = 0$ on $W' + W'' = W$; the associated hydrostatic pressure is $\rho g(y - B)$.

For given boundary configurations S and W, the resulting "hydrostatic acceleration potential" gB can be most easily estimated using an electrolytic tank, and the results interpreted in terms of the dimensionless *cavity buoyancy coefficient*

$$C_H = \frac{\text{hydrostatic bouyancy force}}{\rho g \times \text{mean depth} \times \text{horizontal projection of } W'}$$

In this way, C_H was estimated[7] for three two-dimensional cavity flows, having profiles similar to that of Fig. 2. The *cavity buoyancy coefficients averaged about* 25 per cent.

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ON THE NON-UNIQUENESS OF PERIODIC SOLUTIONS FOR AN ASYMMETRIC LIÉNARD EQUATION*

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The following result has been stated by H. Serbin [5, Theorem II]. Let $f(x)$, $g(x)$ be continuous for $-\infty < x < \infty$, and let

$$f(x) < 0, \qquad -x_1' < x < x_1, \tag{1.0}$$

$$f(x) > 0, \qquad x < -x_1', \qquad x_1 < x, \tag{1.1}$$

where $x_1' > 0$ and $x_1 > 0$. Let

$$\int_0^\infty f(x)\,dx > 0 \tag{1.2}$$

$$xg(x) > 0, \qquad x \neq 0 \tag{1.3}$$

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