—NOTES—

ON SOME STATISTICAL PROPERTIES OF HYDRODYNAMICAL AND MAGNETO-HYDRODYNAMICAL FIELDS*

By T. D. LEE (*University of California, Berkeley*)

1. Introduction. It is now recognized through the works of Taylor, Kolmogoroff, Heisenberg, Batchelor and many others that turbulence, rather than laminar flow, is statistically the most natural phenomenon exhibited by a non-conducting fluid. From the knowledge of magneto-hydrodynamical waves,[1] it is also believed that in a fluid of high electrical conductivity, magneto-turbulence may be the most natural phenomenon. The purpose of this paper is to study some of the statistical properties of a fluid and their relations to the theory of turbulence. The discussion of magneto-turbulence leads to the conclusion that in the usual Kolmogoroff region, where the behavior of the fluid is independent of detailed mechanisms of energy supply, the magnetic energy will be in equipartition with the kinetic energy and that the energy spectrum of either field will be proportional to $k^{-5/3}$ where k is the wave number.

The ergotic motion of an idealized fluid with no dissipation of heat is studied in section 2 and the existence of a Lionville Theorem is proved. The application of a dimensional consideration and some of these statistical properties to the steady state of magneto-turbulence is discussed in section 3.

2. Ergodic motion of an idealized fluid. Let us first consider an incompressible fluid with zero viscosity and zero electrical conductivity. If a certain amount of energy is given to this fluid, we wish to know statistically how this energy will be subdivided into various modes of possible motion. Let $\mathbf{v}(\mathbf{r})$, $p(\mathbf{r})$, ρ be the velocity, pressure and density of the fluid. The Euler's equation of motion gives†

$$\mathbf{v}^{\cdot} = -(\mathbf{v}, \nabla)\mathbf{v} - \frac{\nabla p}{\rho} \tag{1}$$

and

$$\nabla \cdot \mathbf{v} = 0. \tag{2}$$

It is convenient to analyze the motion of the fluid in terms of the Fourier components of its velocity. We write

$$p(\mathbf{r}) = \sum_{\mathbf{k}} p(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{3}$$

$$\mathbf{v}(\mathbf{r}) = \sum_{\mathbf{k}} \mathbf{v}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\mathbf{k}} [\boldsymbol{\alpha}(\mathbf{k}) + i\boldsymbol{\beta}(\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{r}} \tag{4}$$

with $\qquad p(\mathbf{k}) = p(-\mathbf{k})^*, \qquad \mathbf{v}(\mathbf{k}) = \mathbf{v}(-\mathbf{k})^*, \qquad \boldsymbol{\alpha}(\mathbf{k}) = \boldsymbol{\alpha}(-\mathbf{k})$

and $\qquad \boldsymbol{\beta}(\mathbf{k}) = -\boldsymbol{\beta}(-\mathbf{k}).$

*Received April 2, 1951.

[1]H. Alfven, Arkiv. f. Mat. Astr. o. Fys. 29B, 2 (1943).

†A time derivative is indicated by a dot following the letter.

Equations (1) and (2) can then be written as

$$p(\mathbf{k}) = -\frac{\rho}{k^2} \sum_{\mathbf{k}'} [\mathbf{v}(\mathbf{k} + \mathbf{k}')\cdot\mathbf{k}][\mathbf{v}(-\mathbf{k}')\cdot\mathbf{k}], \tag{5}$$

$$\dot{\mathbf{v}}(\mathbf{k}) = -i \sum_{\mathbf{k}'} [\mathbf{v}(\mathbf{k} + \mathbf{k}')\cdot\mathbf{k}]\mathbf{v}(-\mathbf{k}') + \frac{i\mathbf{k}}{k^2} \sum_{\mathbf{k}'} [\mathbf{v}(\mathbf{k} + \mathbf{k}')\cdot\mathbf{k}][\mathbf{v}(-\mathbf{k}')\cdot\mathbf{k}], \tag{6}$$

and

$$\mathbf{v}(\mathbf{k})\cdot\mathbf{k} = 0. \tag{7}$$

For mathematical convenience we shall treat the three components of $\boldsymbol{\alpha}(\mathbf{k})$ and $\boldsymbol{\beta}(\mathbf{k})$ as independent but regard (7) as a constraint applied to the initial condition of the fluid. For from (6), if (7) is true at a particular moment it is always true at any other time.

Let us now consider a phase space with $\alpha_x(\mathbf{k})$, $\alpha_y(\mathbf{k})$, $\alpha_z(\mathbf{k})$, $\beta_x(\mathbf{k})$, $\beta_y(\mathbf{k})$, $\beta_z(\mathbf{k})$. $\cdots$, $\alpha_x(\mathbf{k}')$, $\alpha_y(\mathbf{k}')$, $\cdots$. as its coordinate axes.[2] In this space each point, compatible with the initial condition (7), represents a dynamical state of the fluid. The trajectory of this point governed by (6) describes the subsequent motion of the fluid. Differentiating (6), we have

$$\frac{\partial \dot{\alpha}_i(\mathbf{k})}{\partial \alpha_i(\mathbf{k})} + \frac{\partial \dot{\beta}_i(\mathbf{k})}{\partial \beta_i(\mathbf{k})} = 0; \qquad i = x, y, z. \tag{8}$$

This means that the density of a group of representative points in this phase space remain constant along their trajectories. Just as in an ordinary mechanical system, the existence of this Lionville Theorem suggests that the a priori probability of finding this system in any region of this phase space is proportional to the volume of this region.[3] From (6) the law of conservation of energy can be written as

$$\sum_{\mathbf{k}} [\boldsymbol{\alpha}^2(\mathbf{k}) + \boldsymbol{\beta}^2(\mathbf{k})] = \text{constant} \tag{9}$$

Thus, we find that statistically every mode of the Fourier components of velocity must be in energy equipartition. Let $F(k)$ be the spectrum of kinetic energy per unit volume. In the equilibrium case, we find

$$F(k) \propto k^2. \tag{10}$$

Next we discuss the corresponding case of an incompressible fluid with zero viscosity and infinite conductivity. Let $\mathbf{j}(\mathbf{r})$, $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ be the electric current, electric field and magnetic field respectively. The hydrodynamical equation becomes

$$\dot{\mathbf{v}} = -(\mathbf{v}, \nabla)\mathbf{v} - \frac{\nabla p}{\rho} + \frac{\mathbf{j} \times \mathbf{H}}{\rho c}. \tag{11}$$

Due to the infinite conductivity, we have

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{H}. \tag{12}$$

[2]To avoid the difficulties of an infinite number of degrees of freedom, we may introduce in (3) and (4) an upper limit on k corresponding to, say, the inverse of atomic dimensions.

[3]Although the ergodic hypothesis may be true for a three dimensional fluid it is not true for a two-dimensional case. The existence of a conservation law on vorticity forbids the fulfillment of ergodic hypothesis for a two-dimensional fluid. See T. D. Lee, J. App. Phy. (in press).

Furthermore, **j**, **E** and **H** are connected by the usual Maxwell Equations. On eliminating **j** and **E** from (11), (12) and the Maxwell Equations, the equation of motion of this fluid can be written as

$$\mathbf{v}^{\cdot} = -(\mathbf{v}, \nabla)\mathbf{v} - \frac{\nabla}{\rho}\left(p + \frac{\mathbf{H}^2}{8\pi}\right) + \frac{1}{4\pi\rho}(\mathbf{H}, \nabla)\mathbf{H}, \tag{13}$$

$$\mathbf{H}^{\cdot} = -(\mathbf{v}, \nabla)\mathbf{H} + (\mathbf{H}, \nabla)\mathbf{v} \tag{14}$$

and

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{H} = 0. \tag{15}$$

The term

$$-\nabla\left(\frac{\mathbf{H}^2}{8\pi}\right) + \frac{1}{4\pi}(\mathbf{H}, \nabla)\mathbf{H}$$

in (13) is the force acting on the fluid due to Maxwell stress. In an infinitely conducting fluid the magnetic lines are always frozen in the matter. Equation (14) means that in a system moving with the matter the increment of **H** is given by the extension of the magnetic lines of forces due to relative motions of matter in different places.

As before, we make a Fourier expansion of the field variables. In addition to (3) and (4) we have

$$\mathbf{H} = \sum_{\mathbf{k}} \mathbf{H}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\mathbf{k}} [\mathbf{g}(\mathbf{k}) + i\mathbf{h}(\mathbf{k})]e^{i\mathbf{k}\cdot\mathbf{r}} \tag{16}$$

with $\mathbf{g}(\mathbf{k}) = \mathbf{g}(-\mathbf{k})$ and $\mathbf{h}(\mathbf{k}) = -\mathbf{h}(-\mathbf{k})$.
In terms of these Fourier components (13), (14), (15) can be written as

$$\begin{aligned}\mathbf{v}^{\cdot}(\mathbf{k}) = i\sum_{\mathbf{k}'}&\left\{-[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}]\mathbf{v}(-\mathbf{k}') + \frac{1}{4\pi\rho}[\mathbf{H}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}]\mathbf{H}(-\mathbf{k}')\right\}\\ &+ \frac{i\mathbf{k}}{k^2}\sum_{\mathbf{k}'}\left\{[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][\mathbf{v}(-\mathbf{k}')\cdot\mathbf{k}] - \frac{1}{4\pi\rho}[\mathbf{H}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][\mathbf{H}(-\mathbf{k}')\cdot\mathbf{k}]\right\},\end{aligned} \tag{17}$$

$$\mathbf{H}^{\cdot}(\mathbf{k}) = i\sum_{\mathbf{k}'}\{-[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}]\mathbf{H}(-\mathbf{k}') + [\mathbf{H}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}]\mathbf{v}(-\mathbf{k}')\} \tag{18}$$

and

$$\mathbf{v}(\mathbf{k})\cdot\mathbf{k} = \mathbf{H}(\mathbf{k})\cdot\mathbf{k} = 0 \tag{19}$$

From (17)-(19) it can be verified readily that

$$\frac{\partial\alpha_i^{\cdot}(\mathbf{k})}{\partial\alpha_i(\mathbf{k})} + \frac{\partial\beta_i^{\cdot}(\mathbf{k})}{\partial\beta_i(\mathbf{k})} = 0,$$

and

$$\frac{\partial g_i^{\cdot}(\mathbf{k})}{\partial g_i(\mathbf{k})} + \frac{\partial h_i^{\cdot}(\mathbf{k})}{\partial h_i(\mathbf{k})} = 0; \qquad i = x, y, z. \tag{20}$$

The law of conservation of energy now becomes

$$\sum_{k}\left\{\frac{\rho}{2}[\boldsymbol{\alpha}^2(\mathbf{k}) + \boldsymbol{\beta}^2(\mathbf{k})] + \frac{1}{8\pi}[\mathbf{g}^2(\mathbf{k}) + \mathbf{h}^2(\mathbf{k})]\right\} = \text{constant}. \tag{21}$$

Let us now consider the phase space with $\alpha_x(\mathbf{k})$, $\beta_x(\mathbf{k})$, $g_x(\mathbf{k})$, $h_x(\mathbf{k})$, $\alpha_y(\mathbf{k})$, $\beta_y(\mathbf{k})$, $g_y(\mathbf{k})$, $h_y(\mathbf{k})$, $\cdots$, $\alpha_x(\mathbf{k}')$, $\cdots$. etc. as its coordinate axes. Every point in this space, compatible with (19), represents a dynamical state of this infinitely conducting fluid. Equation (20) suggests again that the a priori probability of finding this system in any region in this phase space is proportional to the volume of that region. Hence we find that in the equilibrium distribution every mode of the Fourier components of magnetic field and velocity field must be in energy equipartition. Let $M(k)$ be the corresponding energy spectrum of magnetic field per unit volume, then we have

$$M(k) = F(k) \propto k^2. \tag{22}$$

3. Turbulence and magneto-turbulence. In the case of a real fluid, due to the energy dissipation by either viscosity or electric resistance, a steady state is reached only if there is a constant source of energy supply. This situation can best be represented by a constant energy flow from small wave number regions to large wave number regions and then in turn a transformation into heat. The existence of this energy transfer makes it necessary that the energy spectrum must be quite different from the equilibrium case of an idealized fluid. However, since this energy transfer is only between different $|\mathbf{k}|$ values, it is expected that all modes of Fourier components of the same $|\mathbf{k}|$ values may be still in energy equipartition. We shall discuss only the statistical properties of magneto-turbulence in some detail as the corresponding properties of turbulence are well known and also can be obtained readily from the following discussion by putting $\mathbf{H} = 0$.

A. Steady case of an inviscid infinitely conducting fluid with a constant energy flow. Let us consider again an incompressible fluid with zero viscosity and infinite conductivity. If an amount of energy is supplied at a certain wave number, then due to the existence of non-linear terms in the equations (17) and (18), this energy is scattered between the magnetic field and the velocity field, and also is transferred within the different wave numbers of both fields. Due to the availability of more volume in phase space for high k values, statistically this energy is always transported toward higher and higher wave numbers. Let us now impose a boundary condition that a constant rate of energy ϵ per unit volume is supplied at wave numbers smaller than a certain k_1 and the same rate of energy is taken out at a k_2 much larger than k_1 . Since the transfer of energy from low k to high k is of a statistical nature, it is expected that there exists a significant region, defined by limits ko and kc ($ko < kc$) between k_1 and k_2 such that in this region the behavior of the field depends only on ϵ but not on the boundary and the detailed mechanism of energy supply. This region will be called as "Kolmogoroff region" since Kolmogoroff[4] was the first one who pointed out the existence of such a region in ordinary turbulence. Let $E_v(k)$ and $E_H(k)$ be the rate of energy transfer per unit volume at k, within the velocity field and within the magnetic field respectively. In the Kolmogoroff region all functions can depend only on ϵ, ρ and k. From dimensional considerations the only forms that $F(k)$, $M(k)$, $E_v(k)$ and $E_H(k)$ can take are

$$F(k) = C_v \rho^{1/3} \epsilon^{2/3} k^{-5/3}, \tag{23}$$

$$M(k) = C_H \rho^{1/3} \epsilon^{2/3} k^{-5/3}, \tag{24}$$

$$E_v(k) = \kappa_v \epsilon, \tag{25}$$

[4]A. N. Kolmogoroff, A.N.C.R. Acad. Sci. U.R.S.S. **30**, 301 (1941); **32**, 16 (1941).

$$E_H(k) = \kappa_H \epsilon, \tag{26}$$

with

$$\kappa_v + \kappa_H = 1. \tag{27}$$

C_v , C_H , κ_v , κ_H are numerical constants. Equations (25) and (26) mean that the rate of energy transfer $E_v(k)$ and $E_H(k)$ must be constant at every k. Consequently, there can not be any energy transfer between the magnetic energy and kinetic energy in this Kolmogoroff region. From the result of previous section we have that for the equilibrium distribution every mode of the Fourier components of this idealized fluid must be in energy equipartition. A deviation from this distribution necessarily requires an energy flow. Hence, the presence of energy flow from low $|\mathbf{k}|$ to high $|\mathbf{k}|$ makes the spectra $F(k)$ and $M(k)$ different from k^2 law. Similarly, since there is no energy transfer between the magnetic field and the velocity field we can conclude that *for any fixed value of* $|\mathbf{k}|$ *the magnetic energy must be in equipartition with the kinetic energy.* We may write

$$C_v = C_H . \tag{28}$$

Similar arguments also lead to the conclusion that both magnetic field and velocity field are isotropic in the Kolmogoroff region. If the lower limit of Kolmogoroff region ko, is comparable to k_1 and the upper limit kc, is much greater than ko, one may normalize $F(k)$ in terms of the average velocity, v_0 , of the fluid. We have

$$\frac{\rho}{2} v_0^2 \cong \int_{k_0}^{\infty} F(k)\, dk.$$

Hence,

$$M(k) = F(k) \cong \frac{1}{3} \rho k_0^{2/3} v_0^2 k^{-5/3} \tag{29}$$

in the Kolmogoroff region.

B. Effects of viscosity and conductivity. The existence of viscosity and finite conductivity gives a particular mechanism of energy dissipation in the above discussion. Let μ and σ be the viscosity and electric conductivity respectively. We have the expression for energy dissipation per unit volume, $[\epsilon]_{\text{dissipation}}$.

$$[\epsilon]_{\text{dissipation}} = \mu \,|\, \nabla \times \mathbf{v} \,|^2 + \frac{\mathbf{j}^2}{\sigma} \cong \mu \,|\, \nabla \times \mathbf{v} \,|^2 + \frac{c^2 \,|\, \nabla \times \mathbf{H} \,|^2}{(4\pi)^2 \sigma} . \tag{30}$$

Equation (30) can be written in terms of the energy spectrum as

$$\left[\frac{dF(k)}{dt}\right]_{\text{viscosity}} = -\frac{2\mu k^2}{\rho} F(k) \tag{31}$$

and

$$\left[\frac{dM(k)}{dt}\right]_{\text{finite conductivity}} = -\frac{c^2 k^2}{2\pi\sigma} M(k). \tag{32}$$

The dependence on k^2 in (31) and (32) means that the effects of viscosity and finite conductivity are important only at high k values. The upper limit kc of the Kolmogoroff region may be estimated as the wave number at which the energy transfer between wave numbers is comparable to the heat dissipation. From (17) and (18) the efficiency of energy transfer between different wave numbers can be estimated as

$$\frac{1}{\kappa_v}\left[\frac{1}{F(k)}\frac{dF(k)}{dt}\right]_{\text{transfer}} \simeq \frac{1}{\kappa_H}\left[\frac{1}{M(k)}\frac{dM(k)}{dt}\right]_{\text{transfer}} \simeq \sqrt{F(k)k^3/\rho} \tag{33}$$

Hence kc is given approximately by

$$\left(\frac{\mu}{\rho} + \frac{c^2}{4\pi\sigma}\right) \simeq \sqrt{F(k_c)k_c^3/\rho}. \tag{34}$$

Using (29), (34) can be written as

$$\frac{k_c}{k_0} \simeq R^{3/4} \tag{35}$$

where R is the modified Reynolds number defined by

$$R \equiv \rho v_0 \Big/ \left(\mu + \frac{c^2 \rho}{4\pi\sigma}\right) k_0 \; . \tag{36}$$

Equation (35) means that the range of the Kolmogoroff region increases with increasing Reynolds number. The behavior of the energy spectrum between ko, the lower limit of the Kolmogoroff region, and the boundary k_1 must depend on some extra-dimensional quantities related to the boundary or the detailed mechanism of energy supply. Hence no universal energy spectrum can be given in that region. The importance of this transition region will depend on the efficiency of energy coupling between the magnetic field and the velocity field and also on the detailed manner as how the energy is supplied. In ordinary turbulence it is assumed that such a transition is unimportant and this assumption is experimentally verified.[5] It may be natural to make the same assumption here. Due to the lack of experimental evidences, such assumption is always subject to criticisms and future verifications. But if this is the true case, then at high Reynolds number the total mechanical energy must be comparable to the total magnetic energy in a conducting fluid.

The author is indebted to Professors E. Fermi and S. Chandresekhar for discussions.

[5]See e.g. Heisenberg, Zeit. f. Phy. 124, 628 (1947).

ON THE INVERSION OF THE VOLTERRA INTEGRAL EQUATION*

By B. GROSS (*The National Institute of Technology, Rio de Janeiro*)

When the given kernel of the Volterra integral equation can be represented as a Laplace transform, the same representation is obtained for the resolving kernel of the equation. For this case the solution is given in explicit form.

The Volterra integral equation

$$f(x) = g(x) + \int_0^x k(x - y)g(y)\, dy \qquad (x > 0) \tag{1}$$

*Received May 8, 1951.