AXIALLY SYMMETRICAL JET MIXING OF A COMPRESSIBLE FLUID*

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1. Introduction. The problem of turbulent jet mixing of an incompressible fluid was first analyzed successfully by Tollmien[1] in 1926 by the application of Prandtl's mixing length theory. He solved the following three problems:—

(1) Mixing of a parallel stream with the adjacent fluid at rest.
(2) Two-dimensional jet from a very narrow opening issuing into medium at rest.
(3) Axially symmetrical jet escaping from very small opening into medium at rest.

The corresponding laminar problem has been theoretically analyzed by Schlichting[2] in 1933. Neither of these solutions, however, holds for finite opening, i.e. at a short distance from the opening.

Kuethe,[3] in 1935, extended Tollmien's results to the case of a two-dimensional jet issuing into a medium not at rest and also worked out an approximate method for the computation of the velocity profile in the initial part of a round jet issuing into medium at rest. Squire and Trouncer,[4] in 1944, extended Kuethe's results to the case of a round jet issuing into a uniform stream by assuming certain velocity profiles across the jet. Abramovich,[5] in 1939, extended Tollmien's problem, i.e. the mixing of a parallel stream with the adjacent medium at rest, to the case of a compressible fluid. He considered the effects of compressibility due to high temperature and those due to high subsonic speed separately and obtained some approximate solutions. Reichardt,[6] in 1941, from the experimental data, suggested the constant exchange coefficient over each cross-section of the mixing zone for the free turbulence problem and Görtler,[7] in 1942, reexamined Tollmien's problems by the application of Reichardt's theory and obtained some improvement in the velocity profiles.

In 1949, the present author,[8] investigated the problems of the flow of a two-dimensional jet from a finite opening of a compressible fluid exhausting into uniform stream and of the mixing of two uniform streams of compressible fluid. Both the laminar and the turbulent cases were considered. The effects of compressibility due to large temperature difference and those due to high velocity were considered simultaneously.

In the present paper, this analysis is extended to the case of an axially symmetrical jet of a compressible fluid exhausting into a uniform stream. The flow of the jet is assumed to be under full expansion from a nozzle, i.e. the pressure of the flow at the exit of the nozzle is exactly equal to that of the surrounding stream. The pressure gradient in the jet is assumed to be negligible. Both the laminar and the turbulent cases will be considered.

The first part of this paper is concerned with the laminar flow. The usual assumptions of boundary layer theory adopted to simplify the Navier-Stokes equations. A solution by the method of small perturbations is first obtained. Then the exact solution is examined. A numerical integration method is used to compute the velocity and the temperature distributions in the jet for the exact solution.

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The second part of the paper is concerned with the turbulent flow. The fundamental equation of motion of an axially symmetrical jet is derived by using Taylor's hypothesis concerning the transport of vorticity and Reichardt's assumption of free turbulence, i.e. the assumption that the exchange coefficient over each cross-section of the mixing zone is constant. By suitable transformation of variables, the equations of turbulent flow become identical to those of laminar flow. Hence the solution of Part I can be used to the turbulent flow provided that the empirical quantity of eddy viscosity has been evaluated experimentally.

The Laminar Jet Mixing

2. Fundamental equations. The equations of motion for the mixing of an axially symmetrical jet in a viscous compressible fluid are

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left(\mu r \frac{\partial u}{\partial r} \right) \tag{1}$$

$$\frac{\partial p}{\partial r} = 0 \tag{2}$$

These equations are obtained under the usual assumptions of boundary layer theory, and the pressure p is assumed to be constant. The x axis is taken along the axis of the jet, and r is the radial distance. The quantities u and v are the x and r components of the velocity, respectively, ρ is the density, and μ is the coefficient of viscosity of the fluid.

The equation of continuity is

$$\frac{\partial(\rho u)}{\partial x} + \frac{1}{r} \frac{\partial(\rho v r)}{\partial r} = 0 \tag{3}$$

The equation of energy is

$$\rho u \frac{\partial}{\partial x} (C_p T) + \rho v \frac{\partial}{\partial r} (C_p T) = \frac{1}{r} \frac{\partial}{\partial r} \left(\lambda r \frac{\partial T}{\partial r} \right) + \mu \left(\frac{\partial u}{\partial r} \right)^2 \tag{4}$$

where C_p is the specific heat at constant pressure, λ is the coefficient of heat conduction, and T is the temperature.

If the Prandtl's number $C_p \mu / \lambda$ is assumed to be unity and C_p to be constant, the temperature becomes a function of velocity only, i.e.

$$T = A + Bu - \frac{u^2}{2C_p} \tag{5}$$

where A and B are constants determined by the boundary conditions. The relation (5) is well-known for the two-dimensional case,[8] but it is interesting to find that it holds true also for axially symmetrical flow.

The pressure is assumed to be constant; hence

$$\rho^* = \frac{\rho}{\rho_0} = \frac{T_0}{T} = \frac{1}{T^*}, \tag{6}$$

where the subscript 0 represents some reference conditions.

The coefficient of viscosity may be expressed as

$$\mu^* = \frac{\mu}{\mu_0} = \left(\frac{T}{T_0} \right)^m = T^{*m}, \tag{7}$$

where m may be taken as a constant less than or equal to one depending on the temperature range investigated.

3. Solution by the method of small perturbations. In some practical cases, both the velocity and the temperature in the jet differ only slightly from those of the surrounding stream. It is advisable to find some approximate solution for this special case first.

Write

$$u = u_0 + u_1 , \qquad v = v_1 ,$$
$$\rho = \rho_0 + \rho_1 , \qquad \mu = \mu_0 + \mu_1 \tag{8}$$

where $u_1 \ll u_0$, $v_1 \sim u_1$, $\rho_1 \ll \rho_0$ and $\mu_1 \ll \mu_0$.

Substituting Eq. (8) into Eq. (1) neglecting the higher order terms, one has

$$\frac{\partial u_1}{\partial x} = \alpha^2 \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u_1}{\partial r}\right), \tag{9}$$

where

$$\alpha^2 = \frac{\mu_0}{u_0 \rho_0}$$

The initial conditions at the exit of the nozzle may be assumed as follows:

$$\begin{aligned} x &= 0 \\ u_1 &= u_{10} = \text{constant}, \qquad 0 \le r \le r_0 \\ u_1 &= 0, \qquad\qquad\qquad\quad r > r_0 . \end{aligned} \tag{10}$$

The solution of Eq. (9) with the boundary conditions of Eq. (10) is[9]

$$\frac{u_1}{u_{10}} = r_0 \int_0^\infty e^{-\lambda^2 \alpha^2 x} J_0(\lambda r) J_1(\lambda r_0)\, d\lambda, \tag{11}$$

where J_0 and J_1 are the Bessel functions of zero and first order respectively. The results are plotted in Fig. 1.

At $x = 0$

$$\frac{u_1}{u_{10}} = r_0 \int_0^\infty J_0(\lambda r) J_1(\lambda r_0)\, d\lambda = \begin{cases} 1, & r_0 > r, \\ 0, & r_0 < r. \end{cases} \tag{12}$$

This is the well-known discontinuous integral of Weber and Schafheitlin[10]. Thus the boundary condition (10) is satisfied.

4. Exact solution. To solve the problem more rigorously, one has to resort to Eqs. (1) and (3). By introducing the stream function ψ, which is defined by

$$\frac{\partial \psi}{\partial r^*} = \rho^* u^* r^*, \qquad \frac{\partial \psi}{\partial x^*} = -\rho^* v^* r^*, \tag{13}$$

where $\rho^* = \rho/\rho_0$, $u^* = u/u_0$, $r^* = r/r_0$ and $x^* = x/r_0$; the equation of continuity, Eq. (3), is automatically satisfied.

If the independent variables x and r are changed to x_1^* and ψ. Equation (1) becomes

$$\frac{\partial u^*}{\partial x_1^*} = \frac{\partial}{\partial \psi}\left(\mu^* \rho^* u^* r^{*2} \frac{\partial u^*}{\partial \psi}\right), \tag{14}$$

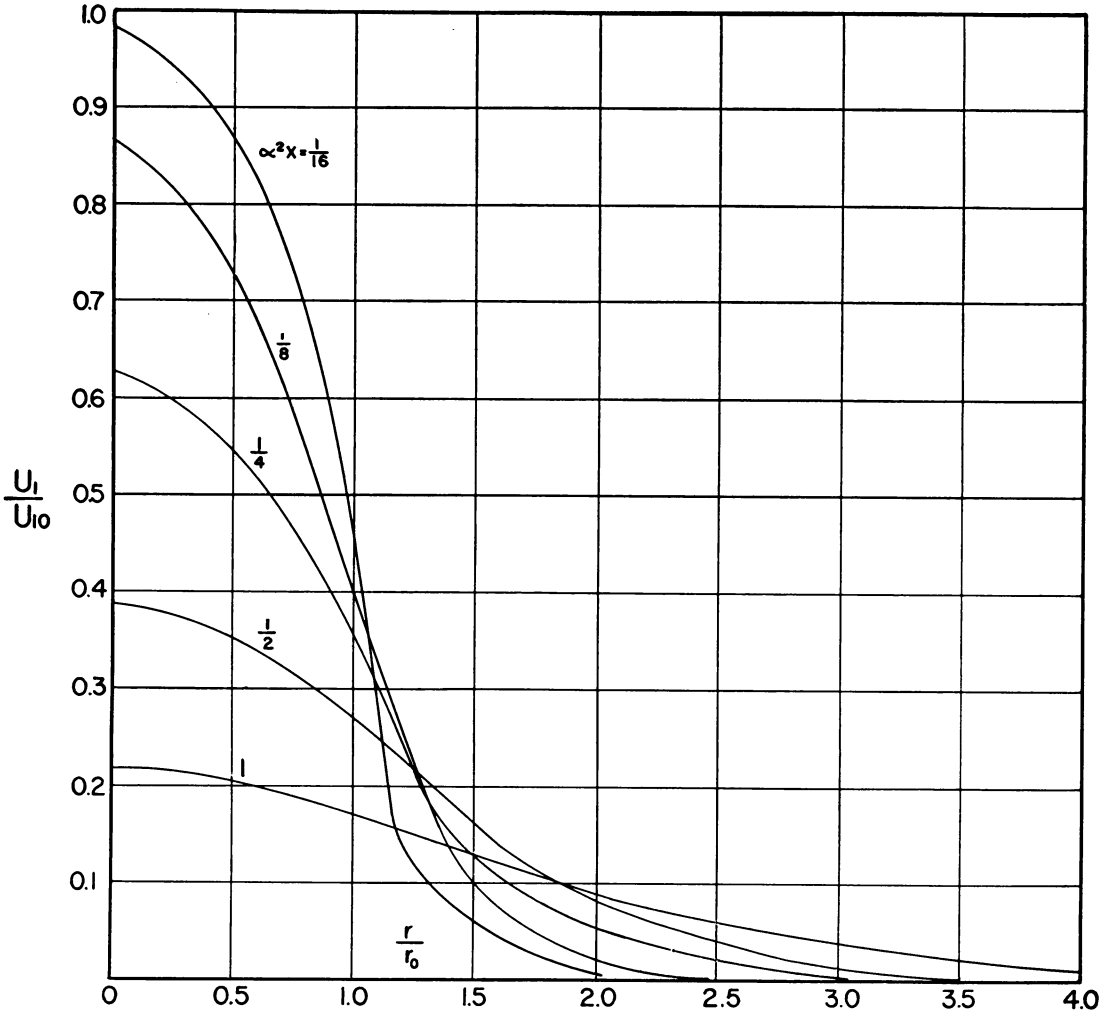

FIG. 1. Velocity distributions in axially symmetrical jet by small perturbation solution.

where $x_1^* = \alpha^2 x^*$. The initial conditions are now

$$x_1^* = 0, \qquad T^* = T_0^*(\psi), \qquad u^* = u_0^*(\psi) \tag{15}$$

Equation (14) can be integrated numerically by a finite difference method starting from the initial conditions (15) with the help of (5), (6) and (7). In the numerical integration, besides (15), the condition on the axis of the jet must be known which is found to be:—

$$\frac{\partial u^*}{\partial x_1^*} = \frac{2\mu^*}{\rho^* u^*} \frac{\partial^2 u^*}{\partial r^{*2}} \tag{16}$$

As soon as u^* and ρ^* are found in terms of ψ at given x^*, the radial distance r^* can be computed by the following formula

$$r^{*2} = 2 \int_0^{\psi} \frac{d\psi}{\rho^* u^*} \tag{17}$$

A typical example has been set up for IBM Machine calculation by Dr. H. Polachek and Mr. T. S. Walton of the Naval Ordnance Laboratory. The boundary condition at the exit of nozzle ($x^* = 0$) are

$$\left.\begin{aligned} u^* &= 1.1 \\ T^* &= 2.0 \\ \rho^* &= 0.5 \end{aligned}\right\} r^* < 1; \qquad \left.\begin{aligned} u^* &= 1.0 \\ T^* &= 1.0 \\ \rho^* &= 1.0 \end{aligned}\right\} r^* > 1$$

The following relations were also used in the numerical example:

$$T^* = -9.66 + 11.26u^* - 0.60u^{*2}, \qquad \mu^* = T^{*0.76}.$$

This represents a hot jet of flow in a cold stream. The result is shown in Fig. 2. To compare with the small perturbation solution, it shows that the velocity on the axis of

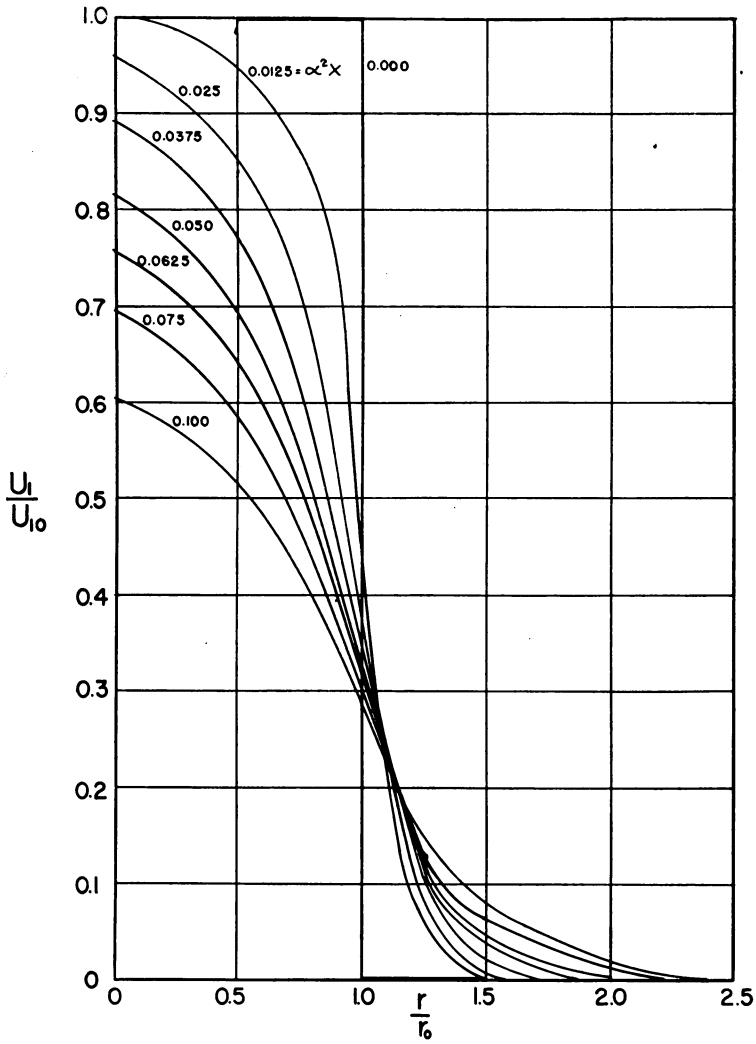

FIG. 2. Velocity distributions in axially symmetrical jet by numerical integration.

the jet decreases more rapidly for the case of the hot jet. These results is qualitatively the same as that in case of two-dimensional jet flow.[8]

The Turbulent Jet Mixing

5. Fundamental equations for turbulent jet mixing. The differential equation of motion for steady axially symmetrical flow of free turbulence, where viscous force can be completely neglected, is

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial v}{\partial r} = -\frac{\partial p}{\partial x} \tag{18}$$

If we put

$$\begin{aligned} u &= u_m + u', \qquad v = v_m + v', \\ \rho &= \rho_m + \rho', \qquad p = p_m + p', \end{aligned} \tag{19}$$

where u, v, p, ρ are the instantaneous values of the x and r components of velocity, pressure, and density, respectively, u_m, v_m, p_m, ρ_m, the corresponding mean values and u', v', p', ρ' the corresponding fluctuating values.

Substituting (19) into (18) and averaging it, one has

$$\begin{aligned} &\left[\rho_m u_m \left(\frac{\partial u}{\partial x}\right)_m + \rho_m v_m \left(\frac{\partial u}{\partial r}\right)_m\right] + \left[(\rho' u')_m \left(\frac{\partial u}{\partial x}\right)_m \right. \\ &\qquad \left. + u_m \left(\rho' \frac{\partial u'}{\partial x}\right)_m + \rho_m \left(\mu' \frac{\partial u'}{\partial x}\right)_m\right] + \left[(\rho' v')_m \left(\frac{\partial u}{\partial r}\right)_m \right. \\ &\qquad \left. + v_m \left(\rho' \frac{\partial u'}{\partial r}\right)_m + \rho_m \left(v' \frac{\partial u'}{\partial r}\right)_m\right] + \left[(\rho' u' \frac{\partial u'}{\partial x}\right)_m \\ &\qquad + \left(\rho' v' \frac{\partial u'}{\partial r}\right)_m\Bigg] = -\left(\frac{\partial p}{\partial x}\right)_m . \end{aligned} \tag{20}$$

where $(\ \ )_m$ denotes the mean value of the quantity in the parenthesis.

By comparison of the order of magnitude and using the ordinary boundary layer assumptions one has

$$\rho_m u_m \left(\frac{\partial u}{\partial x}\right)_m + \rho_m v_m \left(\frac{\partial u}{\partial r}\right)_m + (\rho' v')_m \left(\frac{\partial u}{\partial r}\right)_m + \rho_m \left(v' \frac{\partial u'}{\partial r}\right)_m = -\left(\frac{\partial p}{\partial x}\right)_m \tag{21}$$

In order to find the relation between $\partial u'/\partial r$ and the mean flow, we use Taylor's modified vorticity transport theory,[11] i.e.

$$\frac{\partial u'}{\partial r} = \Delta\omega = l \frac{1}{r} \frac{\partial}{\partial r}\left[r\left(\frac{\partial u}{\partial r}\right)_m\right] \tag{22}$$

where $\Delta\omega$ is the change of vorticity of mean flow in r direction and l is the mixing length.

Furthermore we put

$$\rho' = l\left(\frac{\partial \rho}{\partial r}\right)_m \tag{23}$$

Substituting (22) and (23) into (21) and put $(\partial p/\partial x)_m = 0$, one has

$$\rho_m u_m \left(\frac{\partial u}{\partial x}\right)_m + \rho_m v_m \left(\frac{\partial u}{\partial r}\right)_m = \epsilon \frac{1}{r}\frac{\partial}{\partial r}\left[\rho_m r \left(\frac{\partial u}{\partial r}\right)_m\right] \tag{24}$$

where $\epsilon = -(lv')_m$ is the exchange coefficient of the turbulent flow. According to Reichardt's assumption, we assume that ϵ is independent of r in free turbulence problem. Hence

$$\rho_m u_m \left(\frac{\partial u}{\partial x}\right)_m + \rho_m v_m \left(\frac{\partial u}{\partial r}\right)_m = \frac{1}{r}\frac{\partial}{\partial r}\left[\epsilon \rho_m r \left(\frac{\partial u}{\partial r}\right)_m\right] \tag{25}$$

From (25) we see that the turbulent shearing stress of compressible fluid in the present problem may be defined as

$$\tau_{xr} = \epsilon \rho_m \left(\frac{\partial u}{\partial r}\right)_m = -\rho_m (u'v')_m \tag{26}$$

where

$$u' = l\left(\frac{\partial u}{\partial r}\right)_m.$$

The expression of (26) is in agreement to that obtained by Frankl,[12] but we use different assumptions. In Frankl's analysis, he assumed that ρ' is small by comparison with ρ_m which is not a good assumption in high speed flow.

The energy equation for free turbulence case is (from hereon we drop the subscript m for mean values)

$$\rho u \frac{\partial (C_p T)}{\partial x} + \rho v \frac{\partial (C_p T)}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r C_p \epsilon \rho \frac{\partial T}{\partial r}\right) + \epsilon \rho \left(\frac{\partial u}{\partial r}\right)^2 \tag{27}$$

According to mixing length theory, the Prandtl number for free turbulence is equal to unity, equation (5) holds true for turbulent flow too.

6. Solutions of the equations. According to similar arguments for two-dimensional case,[8] we may write

$$\epsilon = \epsilon_0 \left(\frac{x}{r_0}\right)^n = \epsilon_0 x^{*n} \tag{28}$$

where ϵ_0 is a constant to be determined by experiments.

Equation (25) becomes

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial r} = \epsilon_0 x^{*n} \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r \frac{\partial u}{\partial r}\right) \tag{29}$$

Under the assumption of small perturbation equation (29) becomes

$$u_0 \frac{\partial u_1}{\partial x} = \epsilon_0 x^{*n} \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) \tag{30}$$

In terms of variables x^* and ψ, equation (29) becomes

$$\frac{\partial u^*}{\partial x^*} = \frac{\epsilon_0}{u_0 r_0} x^{*n} \frac{\partial}{\partial \psi}\left(\rho^{*2} u^* r^{*2} \frac{\partial u^*}{\partial \psi}\right) \tag{31}$$

Now we introduce the new independent variable

$$X = \frac{x^{*(n+1)}}{n+1} \tag{32}$$

for x^*, equations (30) and (31) become respectively

$$\frac{\partial u_1}{\partial X} = \frac{\epsilon_0}{u_0 r_0} \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial u_1}{\partial r^*} \right) \tag{33}$$

and

$$\frac{\partial u^*}{\partial X} = \frac{\epsilon_0}{u_0 r_0} \frac{\partial}{\partial \psi} \left(\rho^{*2} u^* r^{*2} \frac{\partial u^*}{\partial r} \right) \tag{34}$$

Equations (33) and (34) are identical to equations (9) and (14) respectively, provided that ϵ_0 is used for μ_0/ρ_0 , X for x^* and ρ^* for μ^*. Therefore the solutions of the laminar jet mixing can be applied to the problem of the turbulent jet mixing of the same boundary conditions, provided that proper characteristic constants ϵ_0 and n are chosen.

References

1. W. Tollmien, *Berechnung der turbulenten Ausbreitungsvorgänge*, ZAMM **4**, 468, (1926). Translated as NACA TM 1085, (1945).

2. H. Schlichting, *Laminare Strahlausbreitung*, ZAMM **13**, 260 (1933).

3. A. M. Kuethe, *Investigations of the turbulent mixing regions formed by jets*, J. Appl. Mech., **2**, 87-95 (1935).

4. H. B. Squire and J. Trouncer, *Round jets in a general stream*, R. & M. No. 1974, British A.R.C. 1944.

5. G. N. Abramovich, *The theory of a free jet of a compressible gas*, NACA TM NO. 1058, March, 1944.

6. H. Reichardt, *Über eine neue Theorie der freien Turbulenz*, ZAMM **21**, 257-264 (1941).

7. H. Görtler, *Berechnung von Aufgaben der freien Turbulenz auf Grund eines neuen Naherungsansatzes*, ZAMM **22**, 244-254 (1942).

8. S. I. Pai, *Two-dimensional jet mixing of a compressible fluid*, J. Aero. Sci., Vol. 16, No. 8, pp. 463-469 (1949).

9. P. Frank and R. von Mises, *Die Differential- und Integralgleichungen der Mechanik und Physik*, vol. II, pp. 544, Mary S. Rosenberg, New York, 1943.

10. A. Weinstein, *On axially symmetric flow*, Q. Appl. Math., **5**, 429-444 (1948).

11. S. Goldstein, *Modern developments in fluid dynamics*, vol. I, Oxford Press, 1938, p. 213.

12. F. Frankl, *Heat transfer in the turbulent boundary layer of a compressible gas at high speed*, NACA TM 1032, 1942.