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ON AXIALLY SYMMETRIC FLOW AND THE METHOD OF GENERALIZED ELECTROSTATICS*

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1. Introduction. In his recent work in Generalized Axially Symmetric Potential Theory, A. Weinstein [1] has pointed out that the flow about an axially symmetric body in ordinary three-dimensional space may be obtained from the electrostatic potential of the five-dimensional body of revolution possessing the same meridian profile. This method of solution is referred to as the method of Generalized Electrostatics. It has been recently used by L. E. Payne and A. Weinstein [2] in deriving a relationship between capacity and virtual mass and has been employed by A. Weinstein [3] in solving certain torsion problems.

In this paper the method of Generalized Electrostatics is used to obtain the flow about a spindle and a lens. The flow about a spindle seems not to have been treated in the literature despite its obvious importance. The lens problem however was treated in 1947 by M. Shiffman and D. C. Spencer [4] who applied an ingenious and difficult procedure involving the method of images in a multi-sheeted Riemann-Sommerfeld space. The solution given in this paper is a straightforward generalization of results given already in 1868 by F. G. Mehler [5]. Our formulas are considerably simpler than those of Shiffman and Spencer but there is no obvious computational method of showing that their solution may be reduced to ours or vice versa. However, the identity of these two solutions is guaranteed by a uniqueness theorem. In an oral communique Professor D. C. Spencer has pointed out the fact that the problem of the spindle would be difficult if not impossible to solve by the method of images in a Riemann-Sommerfeld space. It will be seen that generally speaking little difficulty is encountered in extending to an arbitrary odd-dimensional space the known solutions of three-dimensional electrostatics problems. Hence by the method of Generalized Electrostatics we obtain the flows about axially symmetric profiles almost immediately from the electrostatic solutions.

In this paper we are concerned chiefly with bodies of revolution in three- and five-dimensional spaces; but since spaces of other dimensions have various applications we shall first obtain the solution in a general n-dimensional space. The ordinary three-dimensional terminology will be retained throughout. Later we shall assign the particular value to n which is demanded by the physical problem.

2. Basic Equations. We shall restrict ourselves in this paper to profiles of revolution

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in a uniform stream. We shall assume further that the fluid is incompressible and that at infinity it is travelling at uniform velocity U parallel to the axis of symmetry. Let xy be the meridian plane of an n-dimensional body which is symmetric about the x-axis. We shall define all functions in the meridian half plane $y \geqq 0$.

An axially symmetric potential function $\varphi\{n\}$ in a space of n-dimensions is a solution of the partial differential equation

$$\frac{\partial}{\partial x}\left(y^{n-2}\frac{\partial\varphi\{n\}}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^{n-2}\frac{\partial\varphi\{n\}}{\partial y}\right) = 0, \qquad n \geqq 2. \tag{1}$$

The associated stream function $\psi\{n\}$ is defined with the aid of the generalized Stokes-Beltrami equations

$$y^{n-2}\frac{\partial\varphi\{n\}}{\partial x} = \frac{\partial\psi\{n\}}{\partial y}, \qquad y^{n-2}\frac{\partial\varphi\{n\}}{\partial y} = -\frac{\partial\psi\{n\}}{\partial x}. \tag{2}$$

Let $\Psi\{n\} = Uy^{n-1}(n-1)^{-1} - \psi\{n\}$ be the stream function describing our flow. We assume $\Psi\{n\}$ to vanish on the profile and along the x-axis.* We make use of a correspondance relationship [1], namely $\psi\{n\} = Uy^{n-1}(n-1)^{-1}\varphi\{n+2\}$, to obtain the fundamental equation

$$\Psi\{n\} = Uy^{n-1}(n-1)^{-1}(1 - \varphi\{n+2\}). \tag{3}$$

This equation relates the stream function $\Psi\{n\}$ for an n-dimensional body of revolution B to an electrostatic potential function $\varphi\{n+2\}$ in a space of $n+2$ dimensions. The potential $\varphi\{n+2\}$ assumes the value unity on the profile boundary and vanishes at infinity.

We may by introducing the substitution $\chi\{n\} = y^{(n-2)/2}\varphi\{n\}$ reduce Eq. (1) to the form

$$\nabla^2\chi - [(n-2)(n-4)/4y^2]\chi = 0 \tag{4}$$

where ∇^2 denotes the Laplacian operator. It is obvious that for $n = 2$ or $n = 4$, χ is harmonic. C. Snow [6] in a discussion of non-axially symmetric potential problems in three dimensions has treated solutions of Eq. (4) for odd values of n.

Under a transformation $x + iy = z = f(\zeta) = f(\xi + i\eta)$ Eq. (4) takes the form

$$\chi_{\xi\xi} + \chi_{\eta\eta} - [(n-2)(n-4)/4h^2y^2]\chi = 0 \tag{5}$$

where $h^2 = |f'(\zeta)|^{-2}$. Similarly under such a transformation Eq. (1) becomes

$$\frac{\partial}{\partial\xi}\left(y^{n-2}\frac{\partial\varphi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(y^{n-2}\frac{\partial\varphi}{\partial\eta}\right) = 0. \tag{6}$$

If y is of the form $f(\xi)\cdot g(\eta)$ the solution of (6) is readily obtained by separation of variables. If on the other hand $(h^2y^2)^{-1} = f_1(\xi) + g_1(\eta)$ as in the cases considered here, we find it more convenient to use Eq. (5).

Payne and Weinstein [2] have derived a relationship between the n-dimensional virtual mass $M\{n\}$ (uniform flow in the x-direction) and the $n+2$-dimensional capacity $C\{n+2\}$ which for $n = 3$ is given by

$$M\{3\} + V\{3\} = (2\pi/3)C\{5\}. \tag{7}$$

*This implies that the profile and the x-axis form a single streamline.

In (7) $V\{3\}$ denotes the volume of the body of revolution (the density of the fluid has been taken as unity). The capacity $C\{n\}$ is determined from the equation

$$C\{n\} = (n - 2) \lim_{r \to \infty} r^{n-2} \varphi\{n\}, \qquad r^2 = x^2 + y^2 \tag{8}$$

where $\varphi\{n\}$ is the electrostatic potential of the body. We shall make use of (7) and (8) in determining the virtual mass of the spindle and the lens.

3. Flow About a Spindle. A curve $\xi = \xi_0$ in dipolar coordinates defines the profile of a a spindle. The dipolar transformation is given by

$$x + iy = ic \cot \tfrac{1}{2}(\xi + i\eta) \tag{9}$$

where c is a positive constant. The range of coordinates is chosen as $-\infty < \eta < +\infty$, $0 < \xi \leqq \pi$. The boundary of the spindle is given by $\xi = \xi_0 < \pi$ and the exterior region is defined by $0 < \xi < \xi_0$. We may choose as particular solutions of Eq. (1) functions determined with the aid of (5) which are of the form

$$[(s - t)/(1 - t^2)^{1/2}]^{q+1/2} Q_{q-1/2}^{im}(i\lambda) \cos m\eta \tag{10}$$

where $2q = n - 3$, $s = \cosh \eta$, $t = \cos \xi$ and $\lambda = \cot \xi$. The Q function represents a generalized spherical harmonic of the second kind as defined by E. Hobson [7]. The functions considered in (10) obviously vanish at infinity ($\xi = 0$, $\eta = 0$). We are primarily interested in odd integral values of n and in this case (10) is considerably simplified. Particular solutions of (1) are then given by

$$(s - t)^{q+1/2} K_m^{(q)}(t) \cos m\eta \tag{11}$$

where (q) denotes the qth partial derivative with respect to the argument and $K_\alpha(t)$ is a Legendre function of complex degree commonly called a conal function [7, p. 445]. It is defined as

$$K_\alpha(t) = \left(\frac{2}{\pi}\right) \cosh \alpha\pi \int_0^\infty [2 \cosh u + 2 \cos \xi]^{-1/2} \cos \alpha u \, du. \tag{12}$$

If we replace ξ by $(\pi - \xi)$ in (12) we obtain

$$K_\alpha(-t) = \left(\frac{2}{\pi}\right) \cosh \alpha\pi \int_0^\infty [2 \cosh u - 2 \cos \xi]^{-1/2} \cos \alpha u \, du. \tag{13}$$

Now if $0 < \xi < \pi$, Eq. (13) may be differentiated q times with respect to $\cos \xi$ the order of differentiation and integration being interchanged on the right. For $0 < \xi_0 < \pi$ the term $(\cosh \eta - \cos \xi_0)^{-(q+1/2)}$ satisfies the conditions of the Fourier integral theorem and may be expanded as follows

$$(\cosh \eta - \cos \xi_0)^{-(q+1/2)}$$
$$= \left(\frac{2}{\pi}\right) \int_0^\infty \cos \alpha\eta \left\{ \int_0^\infty [\cosh \eta' - \cos \xi_0]^{-(q+1/2)} \cos \alpha\eta' \, d\eta' \right\} d\alpha. \tag{14}$$

We notice that the term in braces is simply $K_\alpha^{(q)}(-t)$ multiplied by a function of α. We have used t_0 to represent $\cos \xi_0$. We understand by the superscript (q) in $K_\alpha^{(q)}(-t)$ the qth partial derivative with respect to $\cos \xi$ and not with respect to $-\cos \xi$.

We now choose for the electrostatic potential $\varphi\{n\}$ a function of the form

$$\varphi\{n\}(s - t)^{-(q+1/2)} = \int_0^\infty A(\alpha) K_\alpha^{(q)}(t) \cos \alpha\eta \, d\alpha. \tag{15}$$

The condition that $\varphi = 1$ for $\xi = \xi_0$ determines with the aid of (14) the function $A(\alpha)$ since (15) must be satisfied identically in η. The n-dimensional electrostatic potential $\varphi\{n\}$ is found for odd n to be

$$\varphi\{n\} = (2\pi)^{1/2}\Gamma^{-1}(q + 1/2)(s - t)^{q+1/2} \int_0^\infty \frac{K_\alpha^{(q)}(-t_0)K_\alpha^{(q)}(t) \cos \alpha\eta}{K_\alpha^{(q)}(t_0) \cosh \alpha\pi} \, d\alpha. \tag{16}$$

This integral converges for all η and for $0 \leqq \xi < \xi_0$.

If n is not an odd integer the electrostatic potential function becomes much more complicated in general. However, in the special case $n = 4$ the potential may be easily determined with the aid of (5) as

$$\varphi\{4\} = 2(s - t)(1 - t^2)^{-1/2} \int_0^\infty \frac{\sinh \alpha(\pi - \xi_0) \sinh \alpha\xi \cos \alpha\eta}{\sinh \alpha\pi \sinh \alpha\xi_0} \, d\alpha. \tag{17}$$

For odd values of n the stream function $\Psi\{n\}$ representing the flow about a spindle may now be obtained from (16) with the aid of (3). We have

$$\Psi\{n\} = \frac{U(c \sin \xi)^{2(q+1)}}{2(q + 1)(s - t)^{2(q+1)}} \cdot \left[1 - \frac{(2\pi)^{1/2}(s - t)^{q+3/2}}{\Gamma(q + 3/2)} \int_0^\infty \frac{K_\alpha^{(q+1)}(-t_0)K_\alpha^{(q+1)}(t) \cos \alpha\eta}{K_\alpha^{(q+1)}(t_0) \cosh \alpha\pi} \, d\alpha\right]. \tag{18}$$

Thus when $n = 3$ $(q = 0)$ we obtain the stream function corresponding to a uniform flow about a three-dimensional spindle. It is given by

$$\Psi\{3\} = \frac{Uc^2 \sin^2 \xi}{2(s - t)^2}\left[1 - (2)^{3/2}(s - t)^{3/2} \int_0^\infty \frac{K_\alpha^{(1)}(-t_0)K_\alpha^{(1)}(t) \cos \alpha\eta}{K_\alpha^{(1)}(t_0) \cosh \alpha\pi} \, d\alpha\right]. \tag{19}$$

If $\xi_0 = \pi/2$ the spindle becomes a sphere and the well known stream function for the sphere is obtained.

The capacity $C\{n\}$ of an n-dimensional spindle is found according to (8) as

$$C\{n\} = 2^{q+1}(2q + 1)\pi^{1/2}c^{2q+1}\Gamma^{-1}(q + 1/2) \int_0^\infty \frac{K_\alpha^{(q)}(-t_0)K_\alpha^{(q)}(1)}{K_\alpha^{(q)}(t_0) \cosh \alpha\pi} \, d\alpha. \tag{20}$$

The case $n = 3$ has been given by G. Szegö [8]. We obtain the virtual mass $M\{3\}$ with the aid of (7).

$$M\{3\} = -\left(\frac{2}{3}\right)\pi c^3\left\{2 + 3 \cot^2 \xi_0 + 3\xi_0 \cot \xi_0 \csc^2 \xi_0 + 3 \int_0^\infty \frac{(4\alpha^2 + 1)K_\alpha^{(1)}(-t_0)}{K_\alpha^{(1)}(t_0) \cosh \alpha\pi} \, d\alpha\right\}. \tag{21}$$

It is easily verified that for $\xi_0 = \pi/2$ we have the well known virtual mass of the sphere.

4. Flow About a Lens. Let us introduce the peripolar transformation, $x + iy = -c \cot \frac{1}{2}(\xi + i\eta)$, where c is a positive constant. The profile of a lens is defined by two curves $\xi = \xi_1$, and $\xi = \xi_2$. We shall assume that $0 < \xi_1 < \xi_2 < 2\pi$. The external region is chosen as $\eta > 0$, $\xi_2 - 2\pi < \xi < \xi_1$. Particular solutions of (1) are given by functions of the type

$$(s - t)^{q+1/2}(s^2 - 1)^{-q/2}K_m^q(s) \cosh m\xi \tag{22}$$

and those obtained by replacing $\cosh m\xi$ by $\sinh m\xi$. In (22) $K_m^q(s)$ is a Legendre function of the type discussed by Mehler, see [7, p. 451]. As in the case of the spindle the lens problem is solved much more easily and the solution is given in a much simpler form whenever n is an odd integer. Particular solutions of (1) are in this case given by

$$(s - t)^{q+1/2} K_m^{(q)}(s) \cosh m\xi. \tag{23}$$

solutions containing $\sinh m\xi$ being understood as before. The function $K_\alpha(s)$ is defined [7, p. 451] as

$$K_\alpha(s) = \left(\frac{2}{\pi}\right) \cosh \alpha\pi \int_0^\infty [2 \cosh u + 2 \cosh \eta]^{-1/2} \cos \alpha u \, du. \tag{24}$$

We have also the known expansion [7, pp. 452, 453] valid for $0 < \xi < 2\pi$

$$(s - t)^{-1/2} = 2^{1/2} \int_0^\infty \cosh \alpha(\xi - \pi) \operatorname{sech} \alpha\pi \, K_\alpha(s) \, d\alpha. \tag{25}$$

Clearly $K_\alpha^{(q)}(s)$ is a well defined function obtained from (24) by a permissible exchange of order of integration and differentiation. Also for $0 < \xi < 2\pi$ Eq. (25) may be differentiated q times with respect to s the order of integration and differentiation being interchanged on the right. We choose as the electrostatic potential $\varphi\{n\}$ of the lens

$$\varphi\{n\}(s - t)^{-(q+1/2)} = \int_0^\infty [A(\alpha) \cosh \alpha\xi + B(\alpha) \sinh \alpha\xi] K_\alpha^{(q)}(s) \, d\alpha. \tag{26}$$

The functions $A(\alpha)$ and $B(\alpha)$ may be chosen in such a way that $\varphi\{n\} = 1$ for $\xi = \xi_1$ and $\xi = \xi_2$. We need only differentiate (25) q times with respect to s evaluate for $\xi = \xi_1$ and $\xi = \xi_2$ and insert in (26). Since (26) must be satisfied identically in η the functions $A(\alpha)$ and $B(\alpha)$ are easily determined. The electrostatic potential of an n-dimensional lens (n odd) is thus given by

$$\varphi\{n\} = (-1)^q (2\pi)^{1/2} \Gamma^{-1}(q + 1/2)(s - t)^{q+1/2} \int_0^\infty F(\alpha, \xi) \operatorname{sech} \alpha\pi \, K_\alpha^{(q)}(s) \, d\alpha \tag{27}$$

where

$$\begin{aligned} &\sinh \alpha(2\pi - \xi_2 + \xi_1) F(\alpha, \xi) \\ &\quad = \sinh \alpha(\xi_1 - \xi) \cosh \alpha(\pi - \xi_2) + \cosh \alpha(\pi - \xi_1) \sinh \alpha(2\pi - \xi_2 + \xi). \end{aligned} \tag{28}$$

Equation (27) is valid for all positive η and for all ξ in the interval $\xi_2 - 2\pi < \xi < \xi_1$. The case $n = 3$ was given by F. G. Mehler [5] in 1868.

We note again that in case $n = 4$ the electrostatic potential may be easily obtained with the aid of (5). We have

$$\varphi\{4\} = 2[(s - t)/(s^2 - 1)^{1/2}] \int_0^\infty F(\alpha, \xi) \operatorname{csch} \alpha\pi \sin \alpha\eta \, d\alpha. \tag{29}$$

The stream function $\Psi\{n\}$ representing the flow about an odd-dimensional lens is obtained from (27) with the aid of (3). Thus for $n = 3$

$$\Psi\{3\} = [Uc^2(s^2 - 1)/2(s - t)^2] \left[1 + 2^{3/2}(s - t)^{3/2} \int_0^\infty F(\alpha, \xi) \operatorname{sech} \alpha\pi \, K_\alpha^{(1)}(s) \, d\alpha\right]. \tag{30}$$

By an appropriate choice of ξ_1 and ξ_2 electrostatic potentials and stream functions for a spherical bowl, symmetrical lens, hemisphere, *etc.*, may be determined.

Because of the invariance of form of the Stokes-Beltrami equations, i.e.

$$y^{n-2}\Phi_\xi = \Psi_\eta , \qquad y^{n-2}\Phi_\eta = -\Psi_\xi \tag{31}$$

it is a simple matter to determine from Ψ the velocity potential Φ to which Ψ is associated· It remains to be shown that this soluion is unique.

The problem of establishing the uniqueness of the stream function Ψ defined in the infinite region outside the profile in the xy plane and satisfying prescribed boundary conditions is equivalent to the problem of establishing uniqueness of this function in an infinite half strip in the $\xi\eta$ plane. An application of Green's formula would demand a knowledge of the behavior of the derivatives of Ψ at infinity in the $\xi\eta$ plane. Hence we find it more convenient to make use of an eigen value method due to A. Weinstein [9] which requires only a knowledge of the behavior of Ψ at infinity. By this method we can show that there is only one stream function Ψ which satisfies the prescribed conditions on the lens profile. If the stream function is unique the potential Φ is also unique up to an arbitrary constant which must be zero in order that the potential vanish at infinity.

The electrostatic capacity of an n-dimensional lens is determined for integral values of q (n odd) with the aid of Eq. (8). It is given by

$$C\{n\} = \frac{(-1)^q 2^{q+1}\Gamma(1/2)(2q+1)c^{2q+1}}{\Gamma(q+1/2)} \cdot \int_0^\infty \frac{\sinh \alpha\xi_1 \cosh \alpha(\pi-\xi_2) + \cosh \alpha(\pi-\xi_1)\sinh \alpha(2\pi-\xi_2)}{\sinh \alpha(2\pi+\xi_1-\xi_2)\cosh \alpha\pi} K_\alpha^{(q)}(1)\,d\alpha. \tag{32}$$

The case $n = 3$ has been given by G. Szego [8]. The virtual mass $M\{3\}$ is obtained with the aid of [7] and given by

$$M\{3\} = 2\pi c^3 \int_0^\infty \frac{\sinh \alpha\xi_1 \cosh \alpha(\pi-\xi_2) + \cosh \alpha(\pi-\xi_1)\sinh \alpha(2\pi-\xi_2)}{\sinh \alpha(2\pi+\xi_1-\xi_2)\cosh \alpha\pi} \cdot (4\alpha^2+1)\,d\alpha - V\{3\} \tag{33}$$

where

$$V\{3\} = (\pi c^3/6)\{(2-\cos\xi_1)\cot\xi_1/2\csc^2\xi_1/2 - (2-\cos\xi_2)\cot\xi_2/2\csc^2\xi_2/2\}.$$

Equation (33) is a much simpler expression for the virtual mass than that given by Shiffman and Spencer. Several special cases may be obtained easily from (33). In particular the virtual mass of a hemisphere is given by

$$M\{3\} = (2\pi c^3/81)[135 - 59(3)^{1/2}] = 2.545c^3 \tag{34}$$

where c is the radius of the hemisphere.

5. Additional Results. We shall list here the electrostatic potentials of certain other n-dimensional bodies of revolution. With the aid of (3), (7) and (8) the corresponding flow problems can be completely solved. The results given in this section are valid for any positive real value of $q (n > 2)$. It will be noted that the results are simplified considerably whenever n is an odd integer.

I. *Sphere*: The potential of an n-dimensional sphere of radius a is given as

$$\varphi\{n\} = (a/r)^{n-2}, \qquad r = (x^2 + y^2)^{1/2}. \tag{35}$$

II. *Two separated spheres*: The lines $\eta = \alpha < 0$ and $\eta = \beta > 0$ in dipolar coordinates define two separated spheres. The potential in this case is given by

$$\varphi\{n\} = 2^{q+1/2}(s - t)^{q+1/2} \sum_{n=0}^{\infty} \frac{e^{-N\beta} \sinh N(\eta - \alpha) + e^{N\alpha} \sinh N(\beta - \eta)}{\sinh N(\beta - \alpha)} P_n(t/2q + 1) \tag{36}$$

where $N = n + q + \frac{1}{2}$ and $P(t \mid 2q + 1)$ represents the $2q + 1$-dimensional zonal spherical harmonic or more commonly the Gegenbauer polynomial.

III. *Prolate Spheroid*: A line $\xi = \xi_0$ defines a prolate spheroid under the transformation $z = c \cosh \zeta$. The electrostatic potential for such a spheroid is

$$\varphi\{n\} = (\rho_0/\rho)^q Q_q^q(s)/Q_q^q(s_0) \tag{37}$$

where $\rho = \sinh \xi$, $s = \cosh \xi$.

IV. *Oblate Spheroid*: Under the transformation $z = c \sinh \zeta$ an oblate spheroid is defined by a line $\xi = \xi_0$, and the potential is given as:

$$\varphi\{n\} = (s_0/s)^q Q_q^q(i\rho)/Q_q^q(i\rho_0). \tag{38}$$

V. *Disc*: If $\xi_0 = 0$ the oblate spheroid becomes a disc of radius c and the potential of such a disc is obtained from (38) as

$$\varphi\{n\} = -\left[2^{q-1} \exp\left\{\left(\frac{q+1}{2}\right)\pi i\right\}\Gamma\left(q + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)s^q\right]^{-1} Q_q^q(i\rho). \tag{39}$$

We have listed here only a few examples. Numerous others can be easily obtained.

6. Internal Problems. The method of Generalized Electrostatics is also useful in determining the flow induced in a fluid contained between two or more axially symmetric boundaries when one or more of the boundaries moves with respect to the others at uniform velocity parallel to the axis of symmetry. In this case we consider instead of Eq. (3) the equation

$$\Psi\{n\} = Uy^{n-1}(n - 1)^{-1}\varphi\{n + 2\}. \tag{40}$$

On a moving boundary $\Psi\{n\} = V_i y^{n-1}(n - 1)^{-1}$ where $V_i = c_i U$ (c_i is a constant possibly differing for each moving boundary). On a stationary boundary $\Psi\{n\} = 0$. This problem is reduced by (40) to the solution of a steady state heat flow problem in $n + 2$-dimensions. The boundaries in $n + 2$-space corresponding to the moving boundaries in n-space are maintained at temperatures c_i and those corresponding to the stationary boundaries are kept at temperature 0. This procedure applies in particular to the case in which the fluid is bounded by two eccentric spheres, two tori or to the case in which one portion of the boundary moves with respect to another portion in an infinite fluid.

7. Concluding Remarks. In this paper we have considered only three dimensional flow problems. It should be remarked, however, that a similar procedure may be employed in solving plane flow problems for profiles symmetric about the x-axis. In fact if the profile possesses symmetry with respect to both axes the plane problem may be solved for uniform flow in any direction.

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