ON A "PARADOX" IN BEAM VIBRATION THEORY*

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1. Introduction. Timoshenko[1] discusses the problem of a constant vertical point force P moving with constant velocity v across a uniform elastic beam of length l simply supported at its ends at the same level as shown in Figure 1. The beam is considered to be at rest and in equilibrium when the force commences to move across the span at $t = 0$. The force sets the beam in vertical oscillation, more or less violent depending on the magnitude of the force and its speed of traverse. In general the beam will be in oscillation, involving both kinetic and elastic strain energy, when the force reaches the right hand support. The "paradox" arises concerning the source of this energy, since the vertical force undergoes no net vertical displacement and so might be considered to do no net work.

Timoshenko's explanation consists in considering a different but related problem. He states that we should consider the force to act through a frictionless constraint, so that the force must always be directed normally to the instantaneous direction of the beam at the point of action. This introduces a horizontal component of force on the beam which is prescribed when the motion due to the vertical force has been determined. This horizontal component does work, and Timoshenko shows that in the case of resonance, when the traverse time is half the fundamental period of the beam, this work is almost equal to the energy in the fundamental mode of oscillation, the discrepancy being attributed to the higher modes of oscillation.

Although this explanation seems reasonable from a purely mechanical standpoint, it remains puzzling as to why one can't specify the dynamical problem in terms of a vertical force. The work of Lagrange showed how useful the concept of frictionless constraints can be in dynamics, but this does not imply that one must use them in specifying a problem instead of prescribing forces. The careful discussion of energy relations given below shows that the paradox does not in fact exist even with a prescribed vertical force. Since the same type of difficulty arises occasionally in other problems, it was thought worthwhile to write this note to clarify the issue.

2. Energy relations. This discussion is concerned with the usual linear theory of small lateral vibration of beams. We shall use Timoshenko's notation, which in addition to the quantities specified in Fig. 1 specifies: Young's modulus E, second moment of

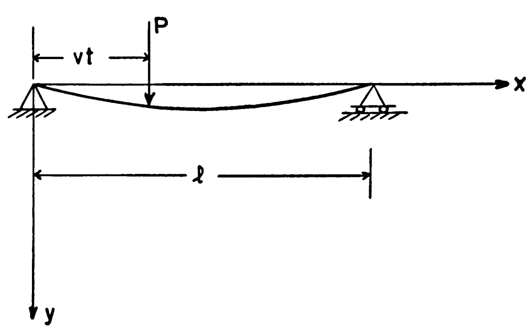

Fig. 1.

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[1]S. Timoshenko, *Vibration problems in engineering*, Van Nostrand, New York, 1937, p. 355.

the section 1, section area A, density γ, and acceleration due to gravity g.

When a force acts on a moving body, the rate of work done on the body is equal to the scalar product of the force and the particle velocity of the body at the point of application. In the present example, therefore, the rate of work done by the vertical force P is $P\ \partial y/\partial t$. However, the vertical component of the velocity of the point of application of the force which moves along the beam is given by a convected derivative:

$$\left(\frac{dy}{dt}\right)_{x=vt} = \frac{\partial y}{\partial t} + v\,\frac{\partial y}{\partial x} \tag{1}$$

Since the force leaves the beam at the level at which it entered the span

$$\int_0^{l/v} \left(\frac{dy}{dt}\right)_{x=vt} dt = 0 \tag{2}$$

The total work done on the beam by the constant vertical force P is:

$$P \int_0^{l/v} \left(\frac{\partial y}{\partial t}\right)_{x=vt} dt \tag{3}$$

which in general will not be equal to zero. In fact it follows directly from (1) and (2) that the work (3) done by the vertical force is equal to the work done by the horizontal component of the constraint force in Timoshenko's modified problem. Thus:

$$P \int_0^{l/v} \left(\frac{\partial y}{\partial t}\right)_{x=vt} dt = -P \int_0^{l/v} \left(\frac{\partial y}{\partial x}\right)_{x=vt} v\, dt. \tag{4}$$

The equality between the work (3) and the energy of the motion induced in the beam follows directly from the differential equation of the motion, and the boundary and initial conditions:

$$\begin{aligned} &EI\,\frac{\partial^4 y}{\partial x^4} + \frac{\gamma A}{g}\,\frac{\partial^2 y}{\partial t^2} = P\ \delta(x/vt) \\ &y = \frac{\partial^2 y}{\partial x^2} = 0, \qquad x = 0,\ l, \qquad t \geq 0 \\ &y = \frac{\partial y}{\partial t} = 0, \qquad t = 0, \qquad 0 \leq x \leq l \end{aligned} \tag{5}$$

where $\delta(x/vt)$ is the Dirac delta function which is non zero at $x = vt$. Multiplying both sides by $\partial y/\partial t$ and integrating with respect to x from 0 to l we obtain, after integration by parts:

$$\frac{\partial}{\partial t}\left[\frac{EI}{2}\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx + \frac{\gamma A}{2g}\int_0^l \left(\frac{\partial y}{\partial t}\right)^2 dx\right] = P\left(\frac{\partial y}{\partial t}\right)_{x=vt} \tag{6}$$

Thus the rate of increase of kinetic plus strain energy is equal to the rate of work as defined in (3).

3. Conclusion. Thus when energy relations are written down carefully, the fact that the vertical force leaves the beam at the level at which it entered the span does not imply no net work done. The reason for misinterpretation is that in most mechanical problems the point of action of an applied force does move with the body on which it is acting. However, there are practical examples where this is not the case. Consider for example an induction motor. The magnetic force producing the torque moves with the synchronous angular velocity of the alternating current, whereas the rotor speed falls below this by the slip, so that the magnetic force is moving relative to the rotor on which it is producing a torque. The rate of work done on the rotor is the torque multiplied by the rotor speed, and not the torque multiplied by the synchronous speed. In many cases, as in this case, the difference in these two quantities may be lost as mechanical work; in this case it appears as eddy current loss. The energy balance at the point of application is a function of the detailed method of application of the force. A problem, in which a similar difficulty in the use of partial and convected derivatives arose, appeared in the discussion of the motion of a bar containing a hinge moving along its length.[2] In this case the wrong choice involved an apparent paradox: the failure of the momentum principle.

NOTE ON THE LEAST EIGENVALUE OF THE HILL EQUATION

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Let us consider the Hill equation

$$y'' + [\lambda + f(x)]y = 0, \qquad -\infty < x < \infty, \tag{1}$$

where $f(x)$ is a real-valued periodic function of period 1 with the Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \exp(2\pi inx), \qquad \bar{c}_n = c_{-n} .$$

Recently Wintner([5], Eq. (23)) deduced the following inequalities satisfied by the lower limit λ_0 of the spectrum of (1):

$$-c_0 \geqq \lambda_0 \geqq -c_0 - 2 \sum_{n=1}^{\infty} | c_n |^2. \tag{2}$$

Also the question is raised (Putnam [3], p. 314) whether the coefficient 2 on the right-hand side is the least possible value. In the present note we shall show that better estimates do exist. In particular, we shall show that

$$\lambda_0 \geqq -c_0 - \frac{1}{8} \sum_{n=1}^{\infty} | c_n |^2. \tag{3}$$

For this purpose we note that λ_0 is characterized as the least eigenvalue of (1) considered on the *finite* interval $0 \leqq x \leqq 1$ with the periodic boundary conditions

$$y(0) = y(1), \qquad y'(0) = y'(1) \tag{4}$$

(see e.g. Strutt [4], p. 15). Therefore, according to the Ritz variational principle, λ_0 is the minimum value of the expression

$$J[y] = \int_0^1 (y'^2 - fy^2)\, dx \Big/ \int_0^1 y^2\, dx,$$