A GENERALIZATION OF MODULATION SPECTRA*

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I. Introduction. A general theory of modulation spectra may be developed by the use of Fourier analysis. It may be applied to frequency as well as to amplitude modulation and is particularly valuable in the study of modulation products resulting when nonlinear devices such as rectifiers are used as modulators. In all cases it shows that the modulation products are harmonics of the highest common factor among the carrier and the modulating frequencies. Also, this approach yields some new results and some clarification of concepts.

Of course the methods of Fourier can only be used where there is an integral relationship between the carrier and each modulating frequency so that the modulated wave may be treated as a periodic function. When this is not strictly true Bohr's method for almost periodic functions may be used.

II. Outline of Theory.

1. *Modulation products.* When two or more waves are combined in a nonlinear circuit such as a diode rectifier, a reactance-tube oscillator, or the human ear, new frequencies appear as a result of some characteristic (such as amplitude or frequency) of one wave being modified by another. Mathematically the process may be expressed as

$$e = F(e_1, e_2, \cdots, e_n). \tag{1}$$

The new waves, which may include waves of the same frequency as the original waves, are the modulation products.

The principal ways in which modulation may be achieved for the simple case of two input waves are:

(a) Mixing in a nonlinear circuit whose characteristic is representable by a finite number of terms of a power series.

$$F_1(e_1, e_2) = \sum_{m=0}^{m=k} (ae_1 + be_2)^n. \tag{2}$$

(b) Mixing in a nonlinear circuit consisting of a biased ideal rectifier whose forward characteristic is representable by a finite number of terms of a power series.

$$F_2(e_1, e_2) = 1[ae_1 + be_2 - E] \sum_{m=0}^{m=k} (ae_1 + be_2 - E)^m. \tag{3}$$

(Here 1[ ] is the Heaviside unit function. The summation is zero unless the term in square brackets is positive.)

(c) Amplitude Modulation

$$F_3(e_1, e_2) = \sum_{m=m_1}^{m_k} \sum_{n=n_1}^{n_h} (a_m + b_m e_1)^m (a_n + b_n e_2)^n. \tag{4}$$

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(d) Angle Modulation

$$F_4(e_1\,,e_2) = K \begin{matrix}\sin\\ \text{or}\\ \cos\end{matrix} \{f_1(e_1\,,e_2) + f_2(e_1\,,e_2)\}. \tag{5}$$

Particular values of the constants in these expressions reduce them to ones which are more familiar in engineering practice. Thus if in (4) $a_n = 0$, $m = n = 1$, one has ordinary amplitude modulation.

$$F_3(e_1\,,e_2) = a_m b_n(1 + ke_1)e_2\,, \tag{6}$$

where $K = b_m a_n$ is the modulation factor.

In (5) if $f_1(e_1\,,e_2) = k_1 \sin^{-1} e_1/\mid e_1 \mid \equiv k_1\omega t$, and $f_2(e_1\,,e_2) = k_2\, e_2$ then one has ordinary phase modulation, or

$$F_4(e_1\,,e_2) = K \begin{matrix}\sin\\ \text{or}\\ \cos\end{matrix} (k_1\omega t + k_2 e_2). \tag{7}$$

If the frequencies of the waves which are combined in a nonlinear circuit are commensurable (in the language of electrical engineering) or contain a common factor or factors, then the modulation products will have a common period which is given by the highest common factor among the original frequencies. In this case ordinary Fourier analysis may be used to find the spectral components.

It is possible that the frequencies of the combined waves may be incommensurable or have no common factor. In such cases (1) may be treated as an "almost periodic function", the theory of which was first advanced by H. Bohr in 1925 [1]. Such a wave would never repeat itself exactly, but for any small quantity ϵ there is always an approximate period τ at the beginning and end of which the amplitudes of the wave differ by less than ϵ. There are actually infinitely many such periods.

The expansion of the almost periodic function is called a generalized Fourier series whose coefficients are found by a limiting process as follows:

$$a_n = \lim_{T\to\infty} \frac{2}{T} \int_0^T f(t) \cos \lambda_n t\, dt, \tag{8}$$

$$b_n = \lim_{T\to\infty} \frac{2}{T} \int_0^T f(t) \sin \lambda_n t\, dt. \tag{9}$$

It is to be noted that it is no longer necessary to ascertain λ_n beforehand. If one replaces λ_n in (8) by some variable x, the limit will be in general zero. The λ_n's are then the values of x which could render these limits not identically zero.

Therefore, to sum up, we see that the modulation products can always be analyzed into systematic spectral components by finding the Fourier or generalized Fourier coefficients. The result is often more revealing than the conventional trigonometrical expansion used in engineering.

An important theorem on Fourier coefficients known as Parseval's theorem [2] will be of use in the development of energy changes due to modulation. In its simplest form, this theorem states that if a function $f(x)$ has its square summable in $(-\pi, \pi)$ and if its

Fourier coefficients are $a_0/2$, a_1 , a_2 , $\cdots$ b_1 , b_2 , b_3 , $\cdots$ then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \tag{10}$$

This theorem fits our problem because it is well known that the average power or energy of an electric wave is proportional to the sum of the squares of the amplitudes of its Fourier components.

Bohr's "Fundamental Theorem" has proved that the generalized Fourier series for almost periodic functions still satisfies Parseval's formula, providing again a theoretical basis for considering the energy changes in modulation.

III. Application of the Method and New Results. The method will now be applied to the analysis of several typical examples of modulated waves whose conventional sideband expansions are known. The new findings and clarification of concepts will be evident where they occur.

1. *Modulation products with pure period.*

a. *Simple Amplitude-Modulated Wave.* This is usually given in the form

$$e = A(1 + m\cos\omega_a t)\cos\omega_c t, \tag{11}$$

where A is the carrier amplitude,
ω_a is the modulation frequency,
ω_c is the carrier frequency,
m is the degree of modulation.

By a trigonometrical identity (11) can be written as

$$e = A\cos\omega_c t + (mA/2)\cos(\omega_c + \omega_a)t + (mA/2)\cos(\omega_c - \omega_a)t. \tag{12}$$

It can be shown that these sidebands are actually harnonics of a certain fundamental in (11) as follows. Let the highest common factor between ω_c and ω_a be ω_0 , such that $\omega_c = n_c\omega_0$, $\omega_a = n_a\omega_0$. Then $T = 2\pi/\omega_0$, and

$$\begin{aligned} a_n &= \frac{4}{T}\int_0^{T/2} A(1 + m\cos n_a\omega_0 t)\cos n_c\omega_0 t\cos n\omega_0 t\,dt, \\ b_n &= 0. \end{aligned} \tag{13}$$

The cosine coefficient in (13) gives

$$\begin{aligned} a_n = \frac{2}{\pi}\int_0^{\pi} A\cos n_c\omega_0 t\cos n\omega_0 t\,d\omega_0 t \\ + \frac{1}{\pi}\int_0^{\pi} mA[\cos(n_c + n_a)\omega_0 t + \cos(n_c - n_a)\omega_0 t]\cos n\omega_0 t\,d\omega_0 t. \end{aligned} \tag{14}$$

Thus a_n is not zero only when $n = n_c$ and when $n = n_c + n_a$ giving

$$a_{n_c} = A, \qquad a_{n_c \pm n_a} = mA/z. \tag{15}$$

Therefore the three terms in (12) are the (n_c)th, the $(n_c + n_a)$th and the $(n_c - n_a)$th harmonics of a wave of period $2\pi/\omega_0$ whose fundamental and other harmonics are zero.

b. *Simple Frequency-Modulated Wave.* Assume again the simplest form with conventional notation,

$$e = A \sin (\omega_c t + m_f \sin \omega_a t). \tag{16}$$

Since this is an odd function, one can safely ignore all the cosine coefficients. Then if ω_0 is again the highest common factor between ω_c and ω_a ,

$$b_n = \frac{2}{\pi} \int_0^{\pi} A \sin (n_c\omega_0 t + m_f \sin n_a\omega_0 t) \sin n\omega_0 t \, d\omega_0 t. \tag{17}$$

If we use the identities,

$$\cos (m_f \sin x) = J_0(m_f) + 2 \sum_{k=1}^{\infty} J_{2k}(m_f) \cos 2kx, \tag{18a}$$

$$\sin (m_f \sin x) = 2 \sum_{k=0}^{\infty} J_{2k+1}(m_f) \sin (2k + 1)x, \tag{18b}$$

then (17) can be expanded and integrated to give

$$b_n = A[J_{s_1}(m_f) - J_{s_2}(m_f)](-1)^{s_1} \tag{19}$$

where $s_1 = (n_c - n)/n_a$ has values which are positive or negative integers including zero and $s_2 = (n_c + n)/n_a$ has values which are positive integers and $J_s(m_f)$ is the Bessel coefficient of the first kind of order s and argument m_f . When s is negative,

$$J_s(m_f) = (-1)^{-s} J_{-s}(m_f). \tag{20}$$

There are special values for n_a for which (19) will actually involve two terms as given but otherwise there will be but one term. This can be seen as follows: n_c and n_a are prime to each other and therefore both are odd numbers or one is odd and the other even. Because $n_c = n_a(s_1 + s_2)/2$ where s_1 and s_2 are integers then for n_a and n_c both odd it is only possible to have $n_a = 1$. If n_a is even and n_c is odd it is only possible to have $n_a = 2$ while if n_a is odd and n_c even it is again only possible to have $n_a = 1$.

In view of this, equation (19) can be written as follows:

If $n_a = 1$, whether n_c is even or odd,

$$b_n = A[J_{n_c-n}(m_f) - J_{n_c+n}(m_f)](-1)^{(n_c-n)/2}. \tag{21}$$

If $n_a = 2$, and n_c is odd,

$$b_n = (A/2)[J_{n_c-n}(m_f) - J_{n_c+n}(m_f)](-1)^{(n_c+n)/2}. \tag{22}$$

If $n_a = 1$ or 2, n_c either even or odd,

$$b_n = (A/n_a)[J_{n_c-n}(m_f)](-1)^{(n_c-n)/n_a}. \tag{23}$$

To see what these coefficients really mean, take $n_c = 26$, $n_a = 7$. If $n = 1$, (i.e. consider the fundamental of the wave), then $(n_c \pm n)/n_a = (26 \pm 1)/7$ is not an integer indicating that the frequency component at ω_0 is zero.

Next let $n = 2$; then $(n_c - n)/n_a = (26 - 2)/7 \neq$ an integer, and $(n_c + n)/n_a = (26 + 2)/7 = 4$ and $(-1)^4 = 1$ indicating that the second harmonic at $2\omega_0$ of magnitude $-AJ_4(m_f)$ exists.

A continuation of this process will show that the fifth harmonic exists and is of magnitude $-AJ_3(m_f)$. The ninth harmonic exists and is of magnitude $AJ_5(m_f)$ and the twelfth harmonic exists and is of magnitude $AJ_2(m_f)$, and etc. This result is shown in the diagram of Fig. 1.

It is interesting to note that this is the same spectrum as if in the ordinary expansion those side-bands of negative frequencies were reflected at the zero frequency axis with signs reversed. This diagram also shows that in general there will be additional frequency components sandwiched between the ordinary sideband spaces, for example, those between the carrier J_0 and the first sidebands J_1 .

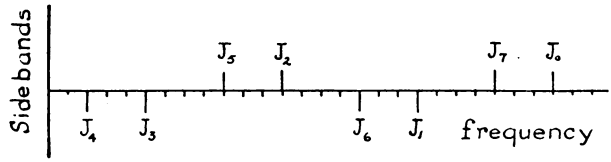

FIG. 1. FM sidebands, $n_c = 26$, $n_a = 7$, (magnitudes not to scale).

Another defect of the ordinary expansion appears in the special case when $n_a = 1$ or when $n_a = 2$ and n_c is an odd number. Thus, if $n_a = 1$, $n_c = 9$, say, then application of (19) will show that the fundamental is of magnitude

$$A[J_8(m_f) - J_{10}(m_f)],$$

and the second harmonic is of magnitude

$$-A[J_7(m_f) - J_{11}(m_f)].$$

And if $n_a = 2$, $n_c = 9$, the application of (19) will show that even harmonics do not exist and the odd harmonics each involve two terms. These are shown in Fig. 2.

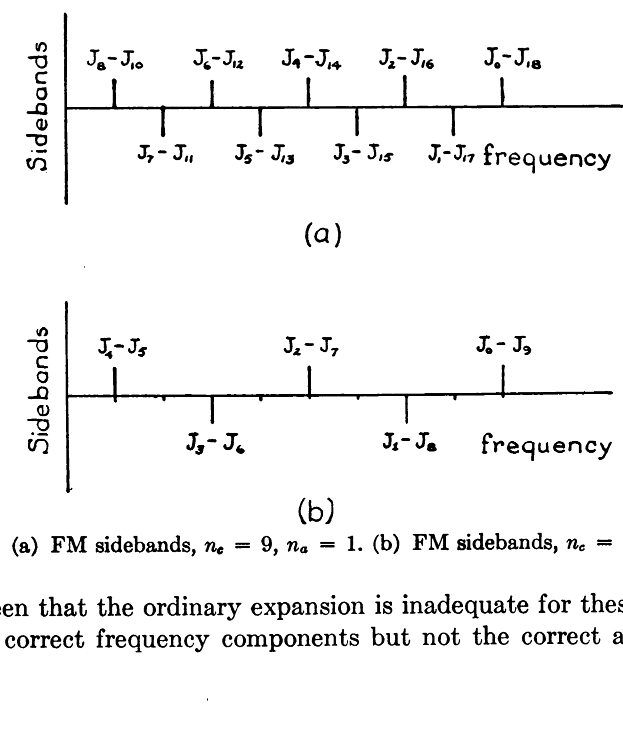

FIG. 2. (a) FM sidebands, $n_c = 9$, $n_a = 1$. (b) FM sidebands, $n_c = 9$, $n_a = 2$.

It may be seen that the ordinary expansion is inadequate for these particular cases, for it gives the correct frequency components but not the correct amplitude for each component.

c. *Multi-tone Modulation.* The same procedure applied to multi-tone modulation shows that for an AM wave of the form

$$e = A\left(1 + \sum_{s=1}^{k} m_s \cos \omega_s t\right) \cos \omega_c t, \tag{24}$$

there exist the Fourier coefficients

$$a_n = \frac{2A}{\pi} \int_0^{\pi} \left(1 + \sum_{s=1}^{k} m_s \cos \omega_{st}\right) \cos \omega_c t \cos n\omega_0 t \, d(\omega_0 t). \tag{25}$$

where ω_0 again is the highest common factor among ω_1, $\omega_2 \cdots \omega_k$, and each sideband can be identified as one of these Fourier harmonics.

For the FM case

$$e = A \sin\left(\omega_c t + \sum_{s=1}^{k} m_s \cos \omega_s t\right). \tag{26}$$

The Fourier coefficients,

$$a_n = \frac{2A}{\pi} \int_0^{\pi} \sin\left(n_c\omega_0 t + \sum_{s=1}^{k} m_s \cos n_s\omega_0 t\right) \cos n\omega_0 t \, d(\omega_0 t), \tag{27}$$

$$b_n = \frac{2A}{\pi} \int_0^{\pi} \sin\left(n_c\omega_0 t + \sum_{s=1}^{k} m_s \cos n_s\omega_0 t\right) \sin n\omega_0 t \, d(\omega_0 t), \tag{28}$$

can be obtained and one has,

$$a_n = \left[\sum_{k_{s2}}^{\pm\infty} \left(\prod_{s=1}^{K} J_{k_{s2}}(m_s)(-1)^{k_b - 1/2}\right) + \sum_{k_{s1}}^{\pm\infty} \left(\prod_{s=1}^{K} J_{k_{s1}}(m_s)(-1)^{(k_b-1)/2}\right)\right] A, \tag{29}$$

where

$$\left| \sum_{s=1}^{K} k_{s1} n_s \right| = | n_c + n |, \qquad \left| \sum_{s=1}^{K} k_{s2} n_s \right| = | n_c - n |, \tag{29a}$$

and

$$k_b = \sum_{s=1}^{K} k_{s1} \quad \text{or} \quad \sum_{s=1}^{K} k_{s2}, \tag{29b}$$

when they are odd numbers.

$$b_n = \left[\sum_{k_{s1}}^{\pm\infty} \left(\prod_{s=1}^{K} J_{k_{s1}}(m_s)(-1)^{k_a/2}\right) - \sum_{k_{s2}}^{\pm\infty} \left(\prod_{s=1}^{K} J_{k_{s2}}(m_s)(-1)^{k_a/2}\right)\right] A \tag{30}$$

where

$$\left| \sum_{s=1}^{K} k_{s1} n_s \right| = | n_c - n |, \qquad \left| \sum_{s=1}^{K} k_{s2} n_s \right| = | n_c + n |, \tag{30a}$$

and

$$k_a = \sum_{s=1}^{K} k_{s1} \quad \text{or} \quad \sum_{s=1}^{K} k_{s2}, \tag{30b}$$

when they are even numbers.

Therefore, we have

$$e = \sum_n a_n \cos n\omega_0 t + \sum_n b_n \sin n\omega_0 t,$$

which is considerably different from the ordinary result,

$$e = \sum_{k_s=0}^{\pm\infty} \left\{ \prod_{s=1}^{K} J_{k_s}(m_s) \right\} \cos \left(\sum_{s=1}^{K} k_s \omega_s t \right). \tag{31}$$

d. *Modulation Products from a Linear Rectifier.* The subject of heterodyne detection has been investigated by many. Engineering practice assumes that detector can follow the envelope ideally so that higher harmonics can be neglected and the difference frequency taken as the fundamental. W. R. Bennett [3] gives a double Fourier series development of the output of such a rectifier for any amplitude and frequency ratio which seems to be the only exact analysis which has so far appeared. The present method of analysis appears to be an interesting and useful alternative.

Express Heaviside's Unit Function by,

$$1(t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin \omega t}{\omega}\, d\omega. \tag{32}$$

Assume an input wave of the form

$$e = A \cos \omega_1 t + B \cos \omega_2 t. \tag{33}$$

From (3) and (32) the output wave from a zero-bias linear rectifier is

$$\begin{aligned} e_0 &= \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin \omega e}{\omega}\, d\omega \right\} e \\ &= \frac{1}{2} (A \cos \omega_1 t + B \cos \omega_2 t) \\ &\qquad + \frac{A \cos \omega_1 t + B \cos \omega_2 t}{\pi} \int_0^\infty \frac{\sin \omega (A \cos \omega_1 t + B \cos \omega_2 t)}{\omega}\, d\omega. \end{aligned} \tag{34}$$

Let $\omega_0 t = x$, where ω_0 is again the highest common factor between ω_1 and ω_2, such that $\omega_1 = n_1 \omega_0$, $\omega_2 = n_2 \omega_0$. Then the Fourier series of (34), of cosine terms only, can be specified completely by

$$\begin{aligned} a_n = \frac{2}{\pi} \int_0^\pi \Bigg[&\frac{1}{2} (A \cos n_1 x + B \cos n_2 x) \\ &+ \frac{A \cos n_1 x + B \cos n_2 x}{\pi} \int_0^\pi \frac{\sin (A \cos n_1 x + B \cos n_2 x)}{\omega}\, d\omega \Bigg] \cos nx\, dx. \end{aligned} \tag{35}$$

The first part of (35) gives

$$a_{n_1} = A/2, \qquad a_{n_2} = B/2. \tag{36}$$

The second part can be written as

$$\begin{aligned} I_2 = \frac{2}{\pi^2} \int_0^\infty \int_0^\pi \Bigg[&(A \cos n_1 x + B \cos n_2 x) \Bigg\{ \left(\sum_{K=0}^{\infty} \mathrm{E}_{2K+1} J_{2K+1}(A\omega)(-1)^K \cos (2K+1) n_1 x \right) \\ &\cdot (\mathrm{E}_{2K} J_{2K}(B\omega)(-1)^K \cos 2Kn_2 x) + \left(\sum_{K=0}^{\infty} \mathrm{E}_{2K} J_{2K}(A\omega)(-1)^K \cos 2Kn_1 x \right) \\ &\cdot \left(\sum_{K=0}^{\infty} \mathrm{E}_{2K+1} J_{2K+1}(B\omega)(-1)^K \cos (2K+1) n_1 x \right) \Bigg\} \Bigg] \cos nx\, dx\, \frac{d\omega}{\omega}, \end{aligned} \tag{37}$$

by using the expansions,

$$\cos (m \cos x) = \sum_{K=0}^{\infty} \mathrm{E}_{2K} J_{2K}(m)(-1)^K \cos 2Kx, \tag{38a}$$

$$\sin (m \cos x) = \sum_{K=0}^{\infty} \mathrm{E}_{2K+1} J_{2K+1}(m)(-1)^K \cos (2K + 1)x, \tag{38b}$$

where E_i is the Neumann E-factor defined as,

$$\begin{aligned} \mathrm{E}_i &= 1, \qquad i = 0, \\ \mathrm{E}_i &= 2, \qquad i \neq 0. \end{aligned} \tag{39}$$

There are in (37) infinitely many terms in the integrand, each term involving four cosine functions multiplied together associated with a product of two Bessel coefficients which are constant if we integrate with respect to x first. Integration with respect to x shows that each term in the integrand is not zero only when the sum and difference of two of the four cosine angles are equal respectively to the sum or difference of the other two angles. Following this the formula below is obtained.

If n_1 is even, and n_2 is odd, then for n odd

$$a_n = \frac{A}{\mathrm{E}_n \pi} \int_0^{\infty} \sum_{S_1, K=0}^{\infty} \mathrm{E}_{2S_1+1} (-1)^{K+S_1} J_{2K}(A\omega) J_{2S_1+1}(B\omega) \frac{d\omega}{\omega}$$

$$+ \frac{B}{\mathrm{E}_n \pi} \int_0^{\infty} \sum_{S_2, K=0}^{\infty} \mathrm{E}_{2K+1} (-1)^{K+S_2} J_{2K+1}(A\omega) J_{2S_2}(B\omega) \frac{d\omega}{\omega} \tag{40a}$$

where

$$2S_1 + 1 = \left| \frac{\pm n \pm n_1 \pm 2Kn_1}{n_2} \right|, \tag{40b}$$

$$2S_2 = \left| \frac{\pm n \pm n_2 \pm (2K + 1)n_1}{n_2} \right|, \tag{40c}$$

and for n even

$$a_n = \frac{A}{\mathrm{E}_n \pi} \int_0^{\infty} \sum_{S_1, K=0}^{\infty} \mathrm{E}_{2K+1} (-1)^{K+S_1} J_{2K+1}(A\omega) J_{2S_1}(B\omega) \frac{d\omega}{\omega}$$

$$+ \frac{B}{\mathrm{E}_n \pi} \int_0^{\infty} \sum_{S_2, K=0}^{\infty} \mathrm{E}_{2S_2+1} (-1)^{K+S_2} J_{2K}(A\omega) J_{2S_2+1}(B\omega) \frac{d\omega}{\omega}, \tag{41a}$$

where,

$$2S_1 = \left| \frac{\pm n \pm n_1 \pm (2K + 1)n_1}{n_2} \right|, \tag{41b}$$

$$2S_2 + 1 = \left| \frac{\pm n \pm n_2 \pm 2Kn_1}{n_2} \right|. \tag{41c}$$

If n_1 and n_2 are both odd, all odd harmonics are missing in the output because then in (42) there will be no possibility of combining the angles such that the whole integral is not zero. For the even harmonics a_n is given by (41a) with (41b) and (41c) replaced by,

$$2S_1 = \left| \frac{\pm n \pm n_1 \pm (2K+1)n_1}{n_2} \right|. \tag{42a}$$

and

$$2S_2 + 1 = \left| \frac{\pm n \pm n_2 \pm 2Kn_1}{n_2} \right|. \tag{42b}$$

In these formulas, the summations should run over all possible integral values of S and K that may satisfy the Diophantine equations in absolute value form.

Without loss of generality, we can assume that $A > B$ (The case $A = B$ will be discussed later). Then (40) and (41) can be integrated as a special case of the infinite discontinuous integral of Weber and Schafheitlin [4]. Thus if n_1 is even and n_2 odd, then for n odd,

$$\begin{aligned} a_n = {} & \frac{A}{E_n\pi} \sum_{S_1,K=0}^{\infty} \left(\frac{B}{A}\right)^{2S_1+1} \frac{\Gamma\left(\frac{2K+2S_1+1}{2}\right)(-1)^{K+S_1}}{\Gamma\left(\frac{2K-2S_1+1}{2}\right)\Gamma(2S_1+2)} \\ & \cdot F\left\{\frac{2K+2S_1+1}{2}, \frac{-2K+2S_1+1}{2}, 2S_1+2, \frac{B^2}{A^2}\right\} \\ & + \frac{B}{E_n\pi} \sum_{S_2,K=0}^{\infty} \left(\frac{B}{A}\right)^{2S_2} \frac{\Gamma\left(\frac{2K+2S_1+1}{2}\right)(-1)^{K+S_2}}{\Gamma\left(\frac{2K-2S_2+3}{2}\right)\Gamma(2S_2+1)} \\ & \cdot F\left\{\frac{2K+2S_2+1}{2}, \frac{-2K+2S_2-1}{2}, 2S_2+1, \frac{B^2}{A^2}\right\}. \end{aligned} \tag{43}$$

Here equations (40b) and (40c) have to be satisfied by $2S_2 + 1$ and $2S_2$.
For n even

$$\begin{aligned} a_n = {} & \frac{A}{E_n\pi} \sum_{S_1,K=0}^{\infty} \left(\frac{B}{A}\right)^{2S_1} \frac{\Gamma\left(\frac{2K+2S_1+1}{2}\right)(-1)^{K+S_1}}{\Gamma\left(\frac{2K-2S_1+3}{2}\right)\Gamma(2S_1+1)} \\ & \cdot F\left\{\frac{2K+2S_1+1}{2}, \frac{-2K+2S_1-1}{2}, 2S_1+1, \frac{B^2}{A^2}\right\} \\ & + \frac{B}{E_n\pi} \sum_{S_2,K=0}^{\infty} \left(\frac{B}{A}\right)^{2S_2+1} \frac{\Gamma\left(\frac{2K+2S_1+1}{2}\right)(-1)^{K+S_2}}{\Gamma\left(\frac{2K-2S_2+1}{2}\right)\Gamma(2S_2+2)} \\ & \cdot F\left\{\frac{2K+2S_2+1}{2}, \frac{-2K+2S_2+1}{2}, 2S_2+1, \frac{B^2}{A^2}\right\} \end{aligned} \tag{44}$$

where $2S_1$ and $2S_2 + 1$ satisfy (41b) and (41c).

If n_1 and n_2 are both odd, only even harmonics exist in the output and these are given by (44) with the Diophantine equations replaced by those of (42a) and (42b).

In the foregoing equations, $\Gamma(x)$ is the gamma function of argument x and $F(a, b, c, x)$ is the hypergeometric function of parameters a, b, c and argument x.

If $A = B$, (40) and (41) still exist, but the hypergeometric functions simplify to gamma functions, i.e.,

$$F(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{45}$$

This Fourier series analysis of the output shows two interesting results. First, it shows that the output may have important components of frequency lower than the difference frequency. When the difference frequency is not too much smaller than the beating frequencies and if it is not the highest common factor, there will be beat tones of considerable amplitude at frequencies lower than the difference frequency. Secondly, for a particular frequency component in the output, these formulas give the amplitudes of all components provided the Diophantine equations are solved for all the possible S and K. For illustration, suppose $f_1 = 800, f_2 = 1400$ such that the highest common factor is 200, and $n_1 = 4$, $n_2 = 7$. Then instead of the difference frequency 600, one would have the series of frequencies of values 200, 400, 600, 800, 1,000 etc. in the output. The frequencies 200 and 400 are both lower than the difference 600. On the other hand, suppose $f_1 = 210, f_2 = 330$, so that the highest common factor is 30 and $n_1 = 7$, $n_2 = 11$. We would expect the series of frequencies 30, 60, 90, 120, 150 etc. in the output. Here, however, since n_1 and n_2 are both odd, the odd harmonics 30, 90, 150 etc. will be missing except 210 and 330 which will be present, (Eq. 36).

e. *Arbitrary Wave-Shape Modulation.* It is evident that the same analysis for frequency modulation by any arbitrary wave-shape would exhibit the same reflected side-band phenomena as the case of sinusoidal modulation. A particular case of rectangular wave modulation has been analyzed in detail and the result compared with the ordinary spectrum to verify this conclusion. The result, however, does not seem to deserve more space here.

2. *Modulation Products With No Exact Period.* When a common factor among the component frequencies present in the modulation products does not exist, the wave as a whole is not periodic. The theory of almost periodic function then relieves us of any possible logical confusion. Here a definite spectrum still exists; the frequency components, however, are no longer related by a multiple of a common component but are determined by certain characteristic values as explained in connection with (8) and (9).

To illustrate the principle, consider again the amplitude-modulated wave $e = A (1 + m \operatorname{Cos} \omega_a t) \operatorname{Cos} \omega_c t$. If ω_c and ω_a have no factor in common, then the conventional Fourier analysis no longer applies. However by, (8),

$$\begin{aligned} a_n &= \lim_{T\to\infty} \frac{A}{T} \int_0^T (1 + m \cos \omega_a t) \cos \omega_c t \cos \lambda t \, dt \\ &= \lim_{T\to\infty} \frac{A}{T} \int_0^T [\cos (\omega_c + \lambda)t + \cos (\omega_c - \lambda)t] \, dt \\ &\qquad + \lim_{T\to\infty} \frac{mA}{2T} \int_0^T [\cos (\omega_c + \omega_a + \lambda)t + \cos (\omega_c + \omega_a - \lambda)t \\ &\qquad + \cos (\omega_c - \omega_a + \lambda)t + \cos (\omega_c - \omega_a - \lambda)t] \, dt. \end{aligned} \tag{46}$$

After integration each term in (46) will be of the form

$$\lim_{T\to\infty} \frac{A}{T}\cdot\frac{\sin (K \pm \lambda)t}{K \pm \lambda}\Bigg|_0^T, \tag{47}$$

and will be identically zero except when $\lambda = \pm K$. Therefore, since we are dealing with real or positive frequency only, the characteristic values are $\lambda_1 = \omega_c$, $\lambda_2 = \omega_c + \omega_a$, and $\lambda_3 = \omega_c - \omega_a$. It can easily be shown that

$$a_1 = A, \qquad a_2 = mA/2, \qquad a_3 = mA/2. \tag{48}$$

These are the same as the sideband amplitudes obtained in the periodic case.

The other kinds of modulation, which so far have been considered only for the case where the carrier and modulating frequencies have a common factor, may similarly be handled by this method for the almost periodic case. Also the results may be inferred for the case where ω_c and ω_m have a very small highest common factor and *almost* have a larger highest common factor. Here we would expect that the true modulation products would be large for the cases where they most nearly coincided with the products for the period based on the *large* "highest common factor".

IV. Findings on Energy Relationships. It is well known that in amplitude modulation the modulated wave has its energy increased by an amount corresponding to that in the sidebands. From the point of view of Fourier's series, this finding is nothing but an application of Parseval's theorem, since the energy or average power per cycle of an oscillation is proportional to the average square of the wave, and hence equal to the sum of the square of its Fourier coefficients. Thus, for the wave in equation (11) the energy E is,

$$E = \sum_{n=1}^{\infty} a_n^2 = A^2 + \frac{1}{4} A^2m^2 + \frac{1}{4} A^2m^2 = A^2 + \frac{1}{2} A^2m^2 \tag{49}$$

which agrees with the usual result.

For an FM wave, the coefficients obtained from (21), (i.e. when $n_a = 1$, n_c either even or odd) give

$$E = \sum_{n=1}^{\infty} b_n^2 = A^2 \sum_{n=1}^{\infty} [J^2_{n_c-n}(m_f) + J^2_{n_c+n}(m_f) - 2J_{n_c-n}(m_f)J_{n_c+n}(m_f)], \tag{50}$$

and the coefficients from (22), i.e. for $n_a = 2$, n_c odd, give

$$E = \sum_{n=1}^{\infty} b_n^2 = A^2 \sum_{K=0}^{\infty} [J^2_{n_c'-K}(m_f) + J^2_{n_c'+K+1}(m_f) - 2J_{n_c'-K}(m_f)J_{n_c'+K+1}(m_f)] \tag{51}$$

where $n_c' = (n_c - 1)/2$, and $2K + 1 = n$.

Equations (50) and (51) can be simplified (Appendix) to

$$E = A^2[1 - J_{2n_c}(2m_f)], \tag{52}$$

and

$$E = A^2[1 - J_{n_c}(2m_f)] \text{ respectively.} \tag{53}$$

The general case when $n_a \neq 1$ or 2 as in equation (23) can be easily analyzed by using the fact that as n runs from 1 to ∞, the Fourier coefficients $J_i(m_f)$ will ee twice when $i \neq 0$ and only once when $i = 0$. Therefore Parseval's formula gi the aid of the relation

$$J_0^2(m_f) + 2\sum_{n=1}^{\infty} J_n^2(m_f) = 1,$$

$$E = A^2\left[J_0^2(m_f) + 2\sum_{n=1}^{\infty} J_n^2(m_f)\right] = A^2.$$

Since the energy before modulation is A^2, (52) and (53) show that in tl particular cases frequency modulation decreases the wave energy because always positive if $x < K$. In the case of (54), the energy remains unchanged.

Appendix

Equations (52) and (53) may be derived by starting with equation (50) in tl

$$E = A^2 \sum_{n=\infty}^{1} [J_{n_c-n}^2(m_f) + J_{n_c+n}^2(m_f) - RJ_{n_c-n}(m_f)J_{n_c+n}(m_f)].$$

Let $n_c + n = m$, then $n_c - n = 2n_c - m$.
When $n = 1$, $m = n_c + 1$, and when $n = \infty$, $m = \infty$; therefore,

$$E = A^2 \sum_{m=n_c+1}^{\infty} [J_{2n_c-m}^2(m_f) + J_m^2(m_f) - 2J_{2n_c-m}(m_f)J_m(m_f)].$$

Now

$$\sum_{m=n_c+1}^{\infty} J_{2n_c-m}^2(m_f) = \sum_{m=n_c+1}^{2n_c} J_{2n_c-m}^2(m_f) + \sum_{m=2n_c+1}^{\infty} J_{2n_c-m}^2(m_f)$$

$$= \sum_{m=0}^{n_c-1} J_m^2(m_f) + \sum_{m=-1}^{-\infty} J_m^2(m_f),$$

and

$$\sum_{m=n_c+1}^{\infty} J_m^2(m_f) = \sum_{m=0}^{\infty} J_m^2(m_f) - \sum_{m=0}^{n_c} J_m^2(m_f),$$

so that

$$\sum_{m=n_c+1}^{\infty} [J_{2n_c-m}^2(m_f) + J_m^2(m_f)] = \sum_{m=-\infty}^{\infty} J_m^2(m_f) - J_{n_c}^2(m_f).$$

Since $J_{-m}(m_f) = (-1)^m J_m(m_f)$, $\sum_{m=-\infty}^{\infty} J_m^2(m_f)$ can be written as

$$J_0^2(m_f) + 2\sum_{m=1}^{\infty} J_m^2(m_f) = 1.$$

Equation (5′) becomes, therefore,

$$\sum_{m=n_c+1}^{\infty} [J^2_{2n_c-m}(m_f) + J^2_m(m_f)] = 1 - J^2_{n_c}(m_f). \tag{7′}$$

The product sum

$$\sum_{m=n+1}^{\infty} 2J_{2n_c-m}(m_f)J_m(m_f)$$

can be treated as follows:

$$\sum_{m=n_c+1}^{\infty} 2J_{n_c-m}(m_f)J_m(m_f)$$

$$= \sum_{m=n_c+1}^{\infty} [J_{2n_c-m}(m_f)J_m(m_f)] + \sum_{m=n_c+1}^{\infty} J_{2n_c-n}(m_f)J_m(m_f)$$

$$= \left[\sum_{m=n+1}^{2n_c} J_{2n_c-m}(m_f)J_m(m_f) + \sum_{m=2n_c+1}^{\infty} J_{2n_c-m}(m_f)J_m(m_f)\right]$$

$$+ \left[\sum_{m=0}^{\infty} J_{2n_c-m}(m_f)J_m(m_f) - \sum_{m=0}^{n_c} J_{2n_c-m}(m_f)J_m(m_f)\right]$$

$$= \left[\sum_{m=0}^{n_c-1} J_m(m_f)J_{2n_c-m}(m_f) + \sum_{m=-1}^{-\infty} J_m(m_f)J_{2n_c-m}(m_f)\right]$$

$$+ \left[\sum_{m=0}^{\infty} J_{2n_c-m}(m_f)J_m(m_f) - \sum_{m=0}^{n_c} J_{2n_c-m}(m_f)J_m(m_f)\right]$$

$$= \sum_{m=-\infty}^{\infty} J_{2n_c-m}(m_f)J_m(m_f) - J^2_{n_c}(m_f). \tag{8′}$$

Combining (7′) and (8′) one has (1′) in the simple form

$$E = A^2\left[1 - \sum_{m=-\infty}^{\infty} J_{2n_c-m}(m_f)J_m(m_f)\right]. \tag{9′}$$

By the Addition Theorem of Neumann and Schlafli (4)

$$J_m(y + z) = \sum_{n=-\infty}^{\infty} J_m(y)J_{n-m}(z). \tag{10′}$$

If $y = z$, therefore, $J_n(2y) = \sum_{n=-\infty}^{\infty} J_m(y)J_{n-m}(y)$ and (9′) can be replaced by

$$E = A^2[1 - J_{2n_c}(2m_f)] \tag{11′}$$

which is equation (52).

By an entirely similar process, it can be shown that (51) can be simplified to

$$E = A^2[1 - J_{n_c}(2m_f)] \tag{12′}$$

which is equation (53).

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