TWO NEW NON-LINEARIZED CONICAL FLOWS*

BY

J. H. GIESE (*Ballistic Research Laboratories, Aberdeen Proving Ground, Md.*)
AND H. COHN (*Wayne University*)

1. Introduction. A steady compressible non-viscous flow is *conical* if it contains a *vertex* P, such that on every half line through P the velocity components, pressure, density, and entropy are constant. Various linearized conical flows have been discussed by numerous authors. However, only three examples of non-linearized conical potential flow fields are known to exist mathematically: Prandtl-Meyer flow around an edge [4][1] with or without sweep; Taylor-Maccoll flow about a non-yawing circular cone [5]; and an axisymmetric flow through a convergent nozzle discussed by Busemann [1]. In this paper the construction of two new examples will be considered. Both contain two regions of swept Prandtl-Meyer flow. In the first, the boundary has been chosen to prevent them from interacting, and the hodograph is one-dimensional. From it can be obtained a flow, with attached plane shock, over an object resembling an airplane with a swept-forward wing of positive dihedral and with a thick fin. In the second, the boundary has been chosen to permit interaction and the hodograph is two-dimensional. It was studied originally in the hodograph space by one of the authors [2]. In the present treatment, the need to consider possible difficulties in constructing the flow field from a knowledge of the hodograph has been avoided by confining the discussion to the physical space. It should also be remarked that in both examples the second order partial differential equation for conical potential flow is of mixed type.

A pair of these examples could conceivably be used to study a particular, atypical case of wing-body interference. Numerical results can easily be calculated, if necessary, with the aid of a characteristics table and by means of standard techniques for numerical integration of ordinary differential equations and numerical solution of characteristic initial value problems for second order hyperbolic partial differential equations in two independent variables.

2. Fundamental Ideas. The velocity potential function of a steady irrotational non-viscous isentropic flow satisfies

$$(a^2\delta_{ij} - u_iu_j)\partial^2\varphi/\partial x_i\,\partial x_j = 0 \tag{2.1}$$

where x_i ($i = 1, 2, 3$) are rectangular coordinates,

$$u_i = \partial\varphi/\partial x_i \tag{2.2}$$

is the velocity component parallel to the x_i-axis in units of the maximum speed of flow,

$$a^2 = 1/2(\gamma - 1)(1 - u_iu_i) = 1/2(\gamma - 1)(1 - q^2) \tag{2.3}$$

is the square of the velocity of sound, Kronecker's delta $\delta_{ij} = 1(0)$ if $i = (\neq)\, j$, and the convention has been adopted that repeated indices imply summation over their range. If $q^2 = u_iu_i > a^2$, the flow is supersonic, and there exist real characteristic surfaces, which are envelopes of the Mach cones

$$[u_i(x_i^* - x_i)]^2 = (q^2 - a^2)(x_i^* - x_i)(x_i^* - x_i) \tag{2.4}$$

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[1]Numbers in brackets designate references listed at the end of the paper.

where x_i^* are running coordinates. At these surfaces the partial derivatives of φ of the second or higher order may have discontinuities while φ and $\partial\varphi/\partial x_i$ remain continuous, as may occur when two solutions are patched together.

The image of a flow under the mapping $x_i \to u_i$ is defined to be its hodograph. In general, three-dimensional flows have three-dimensional hodographs, but not the examples in the following sections. Flows with one (two)-dimensional hodographs have been called *simple* (*double*) waves. The relevant properties of simple waves, discussed in [3], will be summarized briefly. For some function $\mu(x)$, $u_i = u_i(\mu)$. Now (2.1) implies

$$a^2 u_i' u_i' = (u_i u_i')^2 = (qq')^2 \tag{2.5}$$

where $u_i' = du_i/d\mu$. The hodograph is any curve obtainable from a Prandtl-Meyer epicycloid by deforming its plane into a cone with vertex at $u_i = 0$. To specify the curve completely an additional equation and initial conditions are required. For example, in a swept Prandtl-Meyer flow the velocity component parallel to a fixed unit vector λ_i must be constant, that is,

$$u_i \lambda_i = \text{constant}. \tag{2.6}$$

In the physical space the *prototype* of the hodograph point $u_i(\mu)$, a surface $\mu =$ constant which also bears constant pressure, density, and entropy, is a plane normal to u_i'. The exact locations of the prototype planes in the physical space are as yet undetermined. If all are forced to pass through a common point, the simple wave will be a conical flow. Finally, note that a prototype plane is a characteristic surface, all of whose Mach cones are congruent and have parallel axes.

Now let

$$X_\alpha = x_\alpha / z (\alpha = 1, 2), \qquad z = x_3 , \qquad w = u_3 \tag{2.7}$$

Later it will also be convenient to use $X = X_1$, $Y = X_2$, $u = u_i$, and $v = u_2$. Note that to a point (curve) in the X_1X_2-plane there corresponds in the x_i-space a line through (cone with vertex at) the origin. Consider a general conical flow with vertex at the origin. Since $u_i = \partial\varphi/\partial x_i$ are homogeneous of degree zero in the x, s, φ may be assumed with no loss of generality to be homogeneous of degree one. Then for some Φ

$$\varphi = z\Phi(X_1 , X_2) \tag{2.8}$$

$$u_\alpha = \partial\varphi/\partial x_\alpha = \partial\Phi/\partial X_\alpha , \qquad w = \partial\varphi/\partial z = \Phi - X_\alpha u_\alpha \tag{2.9}$$

$$z\, \partial^2\varphi/\partial x_\alpha\, \partial x_\beta = \partial^2\Phi/\partial X_\alpha\, \partial X_\beta , \qquad z\, \partial^2\varphi/\partial x_\alpha\, \partial z = -X_\beta\, \partial^2\Phi/\partial X_\alpha\, \partial X_\beta$$

$$z\, \partial^2\varphi/\partial z^2 = X_\alpha X\, \partial^2\Phi/\partial X_\alpha\, \partial X \tag{2.10}$$

and (2.1) becomes

$$[a^2(\delta_{\alpha\beta} + X_\alpha X_\beta) - (u_\alpha - wX_\alpha)(u_\beta - wX_\beta)]\, \partial^2\Phi/\partial X_\alpha\, \partial X_\beta = 0. \tag{2.11}$$

Characteristics for (2.11) are curves $X_\alpha = X_\alpha(t)$ in the X_1X_2-plane on which the partial derivatives of Φ of second or higher order may have discontinuities, while Φ and $\partial\Phi/\partial X_\alpha$ remain continuous. To these curves there correspond cones $x_\alpha/z = X_\alpha(t)$ on which, by (2.10) or its analogs the partial derivatives of φ of second or higher order will have corresponding discontinuities, while by (2.8) and (2.9) φ and $\partial\varphi/\partial x_i$ remain continuous.

Thus $x_\alpha/z = X_\alpha(t)$ are conical characteristic surfaces for (2.1). A plane tangent to one of them along the ray $x_\alpha/z = X_\alpha$ must be tangent to the Mach cone (2.4) based on $u_i = u_i(X_1, X_2)$ with vertex at the origin and must correspond to the tangent to the characteristic $X_\alpha = X_\alpha(t)$. Hence the characteristic directions for (2.11) must be those of the lines through (X_1, X_2) tangent to the conic

$$(u_\alpha X_\alpha^* + w)^2 = (q^2 - a^2)(X_\alpha^* X_\alpha^* + 1) \tag{2.12}$$

with running coordinates X_α^*. The type of (2.11) is hyperbolic, parabolic, or elliptic accordingly as (X_1, X_2) is outside, on, or inside (2.12). For a conical simple wave (2.11) is of hyperbolic type (except possibly on a curve on which the type is parabolic), with straight characteristics. Furthermore, a region of hyperbolic type adjacent to a region of uniform flow must be a conical simple wave.

3. Flow along a conical wall. It is well known that swept Prandtl-Meyer flow over a dihedral angle can be generalized into simple wave flow past a curved cylindrical wall. Now a generalization to simple wave flow in a trough with boundaries composed of plane and conical segments will be discussed.

Consider uniform supersonic flow along a plane wall on which l is a line inclined with respect to the direction of flow by more than the Mach angle. Let P be any point on l. Extend the boundary beyond l as a cone through l with vertex P, and eventually join the cone to another plane segment. Attempt to fit a conical simple wave to this boundary. Let the origin of coordinates be at P. Describe the boundary by

$$x_i = rv_i(\mu) \tag{3.1}$$

where r and μ are independent. With no loss of generality assume

$$v_i v_i = 1, \qquad v_i' v_i' = 1. \tag{3.2}$$

Then

$$v_i v_i' = v_i' v_i'' = 0, \qquad v_i v_i'' = -1. \tag{3.3}$$

Assume that the prototype planes pass through the rulings of (3.1). Then

$$u_i' v_i = 0 \tag{3.4}$$

and the boundary condition on the cone becomes $u_i = Av_i + Bv_i'$ for some scalar functions $A(\mu)$ and $B(\mu)$. By (3.3) and (3.4) $B = A'$, so

$$u_i(\mu) = A(\mu)v_i(\mu) + A'(\mu)v_i'(\mu). \tag{3.5}$$

Now (2.5) becomes $a^2 A'^2(v_i'' v_i'' - 1) = (A'^2 - a^2)\,(A + A'')^2$, where

$$a^2 = \frac{1}{2}(\gamma - 1)(1 - A^2 - A'^2). \tag{3.6}$$

For any curve on the unit sphere the curvature $v_i''\, v_i'' \geqq 1$. Also, as in Prandtl-Meyer flow with leading edge $v_i(\mu)$, the normal component of velocity $A' \geqq a \geqq 0$. Assume that the simple wave is an expansion. Then $A'\,(A + A'') \geq 0$, and

$$aA'(v_i'' v_i'' - 1)^{1/2} = (A + A'')(A'^2 - a^2)^{1/2}. \tag{3.7}$$

The initial values of A and A' depend, of course, on the original velocity and orientation of l.

The prototype planes through the origin satisfy

$$u_i'(\mu)x_i = 0. \tag{3.8}$$

Join the velocity field (3.5), (3.8) to uniform flows at both ends. Let s be any streamline not on (3.1) which does not intersect the envelope of the prototype planes, defined by

$$x_i = rn_i(\mu), \qquad n_in_i' = n_iu_i'' = 0, \qquad n_in_i = 1. \tag{3.9}$$

Such streamlines exist if (3.1) and (3.9) do not intersect. As the second wall of the trough, also composed of plane and conical segments, choose the cone through s with vertex P. A special example of this type of flow can easily be derived from a Prandtl-Meyer flow. Retain as one side of the trough the plane walls of the original boundary, and use a conical stream sheet for the other wall.

4. A conical simple wave. Insert a half plane into a uniform supersonic flow at a moderate angle of attack, and make the angle between the leading edge and the undisturbed velocity greater than the Mach angle. On one side there will be a swept Prandtl-Meyer flow around the leading edge. On the other there will be an attached plane shock wave behind which there will be uniform flow (supersonic if the angle of attack is not too large) parallel to the half plane. Throughout the entire flow the component of velocity parallel to the leading edge is constant. Introduce a coordinate system with origin on the leading edge, z-axis parallel to the undisturbed velocity, yz-plane parallel to the uniform flow behind the shock, and hence normal to the shock. Discard that part of the flow on the side of the yz-plane that contains the downstream half of the leading edge. Reflect the remainder with respect to the yz-plane. So far the boundary, shown in Fig. 1 together

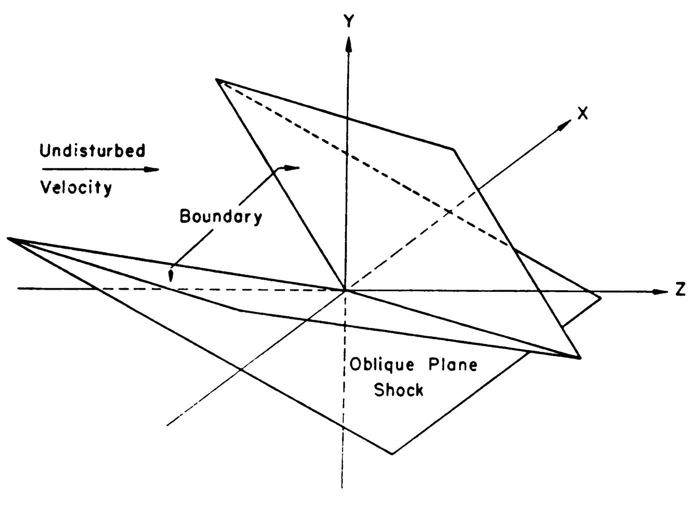

Figure I Shock and Tentative Boundary

with the shock, consists of a dihedral angle with congruent sectors removed from each face. Since the two halves of the shock are coplanar, the uniform flows on the compression side join continuously. On the expansion side u_1 is double valued in the yz-plane. This difficulty can be avoided by modifying the upper side of the boundary as follows.

First, examine the hodograph of the swept Prandtl-Meyer flow. In Fig. 2 let ON^*

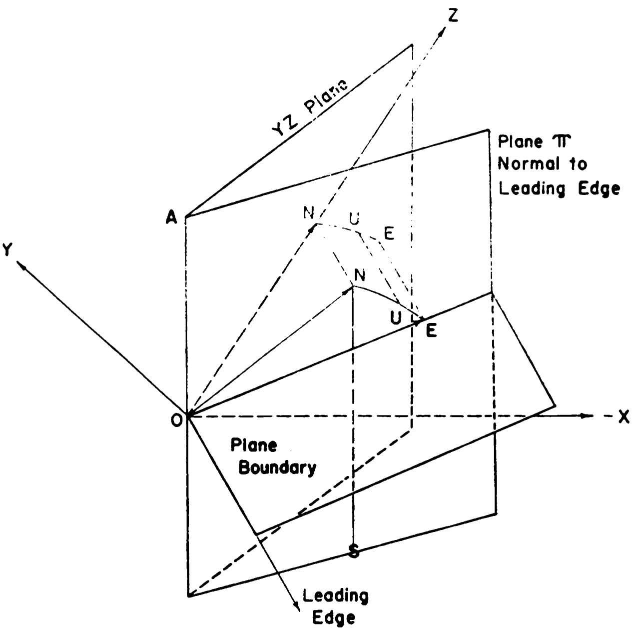

Figure 2 Hodograph of Swept Prandtl-Meyer Flow

be the undisturbed velocity in the yz-plane, and let N be the projection of N^* onto the plane Π that passes through the origin and is normal to the leading edge. During the expansion, the component of velocity normal to Π remains equal to NN^* while the component parallel to Π traces an arc NE of a Prandtl-Meyer epicycloid, shrunk by a factor $(1 - NN^{*2})^{1/2}$. Fig 1 shows that the x-axis is under the plane boundary, so the angle AOE is acute. Hence the epicycloidal arc NE cannot have another intersection with the line NS, parallel to OA. Now construct the hodograph by subjecting every point of NE to the displacement $NN.^*$ Clearly, during the expansion from ON^* to OE^* the angle between the velocity vector OU^* and the yz-plane (Π) steadily increases (decreases).

Now let Λ be the intersection of $x = 0$ and the plane Mach surfaces that pass through the leading edges and are based on the undisturbed velocity. Let P be any point on Λ. From the nature of the hodograph it is clear that the streamline through P for the simple wave in $x \geqq 0$ turns immediately into $x > 0$ and stays there. Hence an entire conical stream sheet through Λ bends into $x > 0$. Thus it is possible to separate the two regions of swept Prandtl-Meyer flow by means of a symmetrical conical fin, the thickness of which increases with increasing sweep. Note that at the junctions of the fin and the original boundary u_i is parallel to x_i. Accordingly, near the corresponding points X_α (2.11) is of elliptic type. On the other hand, for very large values of X_α near the boundary (2.11) is of hyperbolic type.

To obtain from this boundary a finite obstacle resembling an airplane, symmetrically terminate the wing, as in Fig. 3, at a trailing edge which is supersonic with respect to the uniform flows adjacent to both sides of the wing. On the lower side there will be an expansion around the trailing edges. The flow field just described will be unaffected up to the first Mach surface in the expansion fan emanating from the trailing edges. In particular, the leading edge shock will cease to be plane at its intersection with this surface. On the upper side there will be a shock attached to the trailing edges, ahead of which

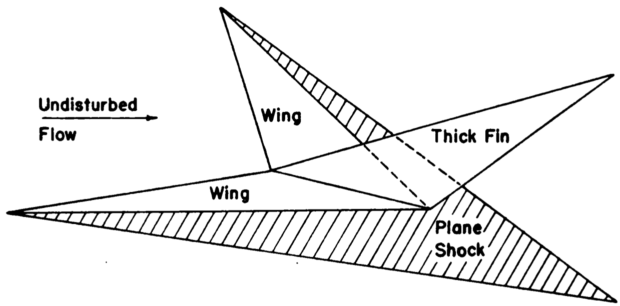

Figure 3 Schematic Representation of Finite Obstacle in Simple Wave Flow

the original flow will be unaltered. If the trailing edges are straight, as in Fig. 3, the shock and immediately following flow for either half of the wing will be conical with respect to the corresponding wing tip. Possible trailing edges for the fin would be its intersections with the trailing edge shocks. If a thick wing is desired, the upper surfaces need not be parallel to the lower surfaces. Finally the upper surfaces need not even be plane, but may be cylindrical or even conical, with vertices at the wing tips. However, after such changes the simple wave flows cease to be conical with respect to the origin.

5. Interaction of simple waves [2]. Return to the stage of the discussion at the end of the first paragraph of Section 3. Examination of Fig. 1 shows that cross sections by the planes $z = \pm 1$ would have the appearances of Figs. 4 and 5. E_R and E_L are traces

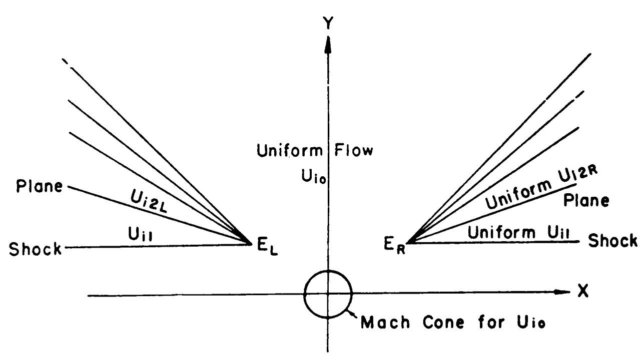

Figure 4 Trace by Z = -1

of the leading edges. The traces of the boundary and shock wave have been shown mainly to clarify the preceding example. They really have no immediate bearing on the interaction example, since it will be necessary to consider the possibility of having to reduce the amount of expansion and the extents of the interacting simple waves. The

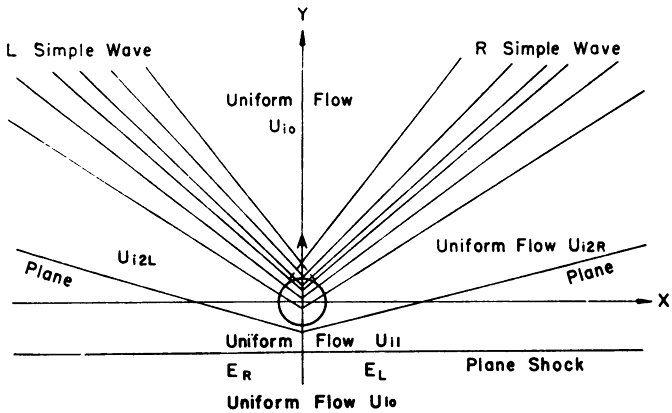

Figure 5 Trace of Tentative Flow by Z = + 1

boundary will actually be constructed later. The essential point is that the discussion starts from simple waves in which all of the straight characteristics pass through the centers E_R or E_L .

In Fig. 4 the simple waves have not begun to interact so the boundary need not be altered for $z < 0$. In Fig. 5 they begin to influence each other along the characteristics AB_R and AB_L , shown in a magnified view in Fig. 6. Since the type of (2.11) is hyperbolic

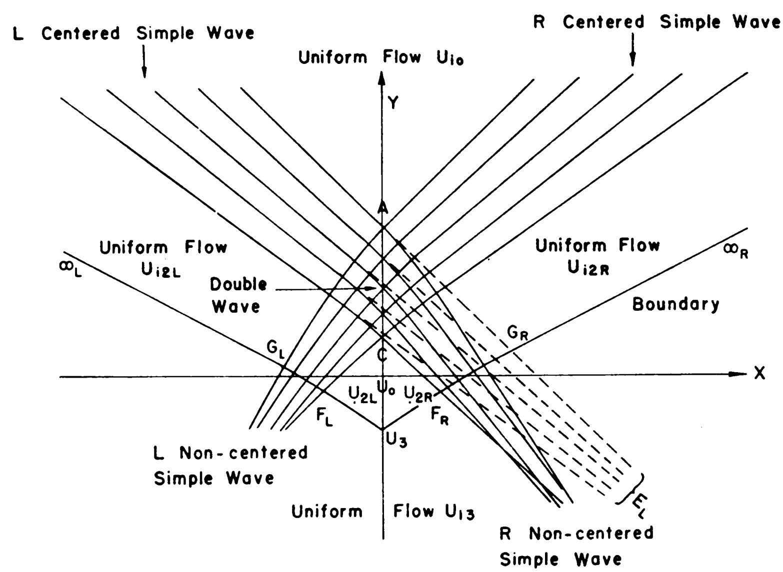

Figure 6 Interaction of Simple Waves

at A, the characteristics actually extend some distance beyond A as shown. In the simple waves u_α and w are known on AB_R and AB_L. Hence Φ and $\partial\Phi/\partial X_\alpha$ are known by (2.9). The characteristic initial value problem for (2.11) with these initial data will have a unique solution in some sufficiently small characteristic quadrilateral AB_RCB_L, where C may be assumed to be on the Y-axis. Since the initial data are symmetrical with respect to the Y-axis, so is Φ. Extend Φ beyond B_RC as the velocity potential of a conical simple wave. In general, the corresponding straight characteristics will not be centered. However, if the double wave AB_RCB_L is kept small enough, their envelope and intersections can be kept arbitrarily close to E_L, well beyond the boundary to be constructed later. Between the two simple waves in $X > 0$ there falls a region of uniform flow with the velocity u_{i2R} of B_R. Extend the definition of Φ symmetrically with respect to the Y-axis. Finally, between the two non-centered simple waves there falls a region of uniform flow with the velocity u_{i3} of C.

It remains to choose a boundary in $z > 0$. First let U_0, U_{2R}, U_{2L}, and U denote the *points* where rays from (0, 0, 0) parallel to u_{i0}, u_{i2R}, u_{i2L}, and u_{i3} intersect $z = 1$. Let $G_R\infty$, starting from the right edge of the right hand non-centered simple wave, be on the line E_RU_{2R}. Then u_{i2R} is tangent to the corresponding plane through the origin. Let F_RG_R be an integral curve of the equation $dX/(u - wX) = dY/(v - wY)$ of conical stream sheets. The final part of the boundary in $X > 0$ consists of the segment F_RU_3 which corresponds to a plane to which u_i is tangent. Extend the boundary symmetrically into $X < 0$.

Note that near U_3 (2.11) is of elliptic type.

It is interesting to observe that an alternation of double waves, simple waves, and regions of uniform flow similar to that in Fig. 6 also appears in an intersection of simple waves which occurs when uniform plane supersonic flow expands into an infinite sector. This is shown schematically in Fig. 7.

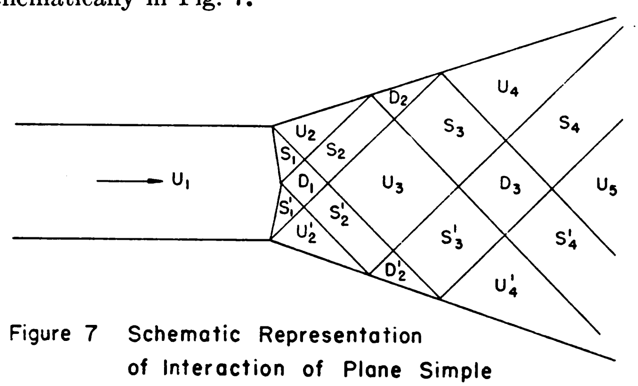

Figure 7 Schematic Representation of Interaction of Plane Simple Waves in Divergent Channel

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