where v' and v approximate

$$\Omega(x, \gamma; E, \eta, \tau) \qquad \text{and} \qquad \chi(x, \gamma; E, \eta', \tau'),$$

respectively.

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A FORM OF NEWTON'S METHOD WITH CUBIC CONVERGENCE*

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For obtaining an approximation to a root of a transcendental equation $f(x) = 0$ Newton's method may well be unsatisfactory because it is only quadratically convergent, thus requiring considerable interpolation if the functions involved are scantily tabulated. On the other hand, the formulas of Stewart [1], Hamilton [2], Bodewig [3], and others, which offer cubic or higher convergence have the serious drawback of requiring the evaluation of second or higher derivatives. Formula (4) below, based on the generalized Taylor expansion of Hummel and Seebeck [4], offers cubic convergence in terms of $f(x)$ and $f'(x)$ evaluated at points on each side of the root.

Taking $n = m$ in the Hummel-Seebeck expansion,

$$f(x) = f(a) + (x - a)\,\frac{f'(a) + f'(x)}{2} + (x - a)^2\,\frac{f''(a) - f''(x)}{12} + \cdots, \tag{1}$$

we obtain two approximations to a root,

$$x - a = \frac{-2f(a)}{f'(a) + f'(x)} \qquad \text{and} \qquad x - b = \frac{-2f(b)}{f'(b) + f'(x)}. \tag{2}$$

We choose a and b so $f(a)$ and $f(b)$ are opposite in sign, $f'(a)$ and $f'(b)$ same sign. Elimination of $f'(x)$ in equations (2) yields

$$\frac{f(b)}{x - b} - \frac{f(a)}{x - a} + \frac{f'(b) - f'(a)}{2} = 0 \tag{3}$$

or, by an obvious procedure,

$$x = \frac{b + a}{2} - \frac{f(b) - f(a)}{f'(b) - f'(a)} \pm \left\{\left[\frac{b - a}{2} + \frac{f(b) - f(a)}{f'(b) - f'(a)}\right]^2 - \frac{2(b - a)f(b)}{f'(b) - f'(a)}\right\}^{1/2}, \tag{4}$$

where choice of the ambiguous sign is quite obvious.

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Proof of cubic convergence of the method closely follows the more general discussion of Bodewig. We take A as the true value of the first order root,

$$f(x) = (x - A)g(x), \qquad g(A) \neq 0, \tag{5}$$

and expand $g(x)$ in a generalized Taylor series, powers of $(x - A)$. Setting $x = a_n$, b_n in the series of equations (2) and carrying out the indicated division we obtain

$$\begin{aligned} \frac{x_{n+1} - a_n}{a_n - A} &= -\left[1 - (x_{n+1} - A)\,\frac{g(A) + g(x_{n+1})}{2g(A)} + P_1(x - A) + \cdots\right], \\ \frac{x_{n+1} - b_n}{b_n - A} &= -\left[1 - (x_{n+1} - A)\,\frac{g(A) + g(x_{n+1})}{2g(A)} + P_2(x - A) + \cdots\right], \end{aligned} \tag{6}$$

or, finally,

$$x_{n+1} - A = \frac{(a_n - A)(b_n - A)}{b_n - a_n}\,P(x - A), \tag{7}$$

where $P(x - A)$ represents a power series in $(a_n - A)$, $(b_n - A)$, $(x_{n+1} - A)$, quadratic terms and higher.

As a numerical example consider the first root of $x \tan x - 1 = 0$. Taking $a = 0.8$, $b = 0.9$ equation (4) yields the tabulated value 0.8603. Two or more applications of Newton's method will involve interpolation if one has only a two place table at hand. Estimates of the magnitude of error involved in interpolation by means of (4) have been found by Hummel and Seebeck [5].

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THE SECOND FUNDAMENTAL THEOREM OF ELECTRICAL NETWORKS*

By CHARLES SALTZER (*Case Institute of Technology*)

1. Introduction. This paper will deal with an extension of the work of W. H. Ingram and C. M. Cramlet[1] as discussed by J. L. Synge.[2] In addition it will be shown how their theories fit into a unified theory. The terminology of Synge's paper will be used.

A network may be represented by its Thévenin representation, i.e. by regarding its branches as consisting of impedances in series with constant voltage sources; or, it may be represented by its Norton representation, i.e. by regarding the branches as

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