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A GENERAL SOLUTION FOR THE RECTANGULAR AIRFOIL IN SUPERSONIC FLOW*

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Summary. The potential on a rectangular airfoil due to an arbitrarily prescribed motion at its surface is obtained by an operational solution of the linearized equations and subsequent comparison with the known solution in steady flow. It is shown that the result can be extended to more general planforms with the aid of the Lorentz transformation. Other methods of solution are noted.

1. Introduction. The problem of unsteady motion of a rectangular airfoil in supersonic flow has been treated in closed form by Goodman [1, 2], Miles [3, 4], Rott [5], Stewartson [6], and Stewart and Li [7, 8]. The results of refs. 1-6 are in mutual agreement, but those of Stewart and Li are believed to be incorrect due to their (in our opinion†) erroneous conclusion that Evvard's "equivalent area" concept is applicable to non-stationary flow, the derivation [9] of which has been criticized elsewhere [10].

The reason for the addition of the present paper to this already voluminous literature is to present a solution that is valid for an arbitrarily prescribed motion of the airfoil. In principle, the solution of ref. 3 is sufficiently general in virtue of Fourier's theorem (with reference to the time dependence) and the possibility of expanding velocity distributions of practical interest in powers of y^n, but the following solution has the advantage of exhibiting the dependence of the potential on the prescribed velocity in a more explicit form. Moreover, the end result affords an immediate extension to planforms with oblique edges. Finally, the result permits a precise extension of the equivalent area concept to non-stationary motion of a rectangular wing, albeit this interpretation does not permit further generalization to other planforms.

2. Formulation of the problem. We consider a quarter infinite airfoil whose projection on the plane $z = 0$ is bounded by $X = -Ut/l$ and $y > o$ in the *fixed*, right handed, dimensionless (with l as the characteristic length), Cartesian coordinates (X, y, z), U being the flight velocity and t the time. On the basis of the usual assumptions, the linearized equation for the velocity potential is the wave equation

$$\phi_{XX} + \phi_{yy} + \phi_{zz} = \phi_{TT} \tag{2.1}$$

$$T = at/l \tag{2.2}$$

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†We have discussed this with Prof. Stewart, who also is now of this belief.

where a is the sonic velocity and T a dimensionless time. In the end results we shall exhibit ϕ as a function of the *moving* coordinates (x, y, z), related to the fixed coordinates (X, y, z) by the Gallilean transformation

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ lT/a \end{pmatrix} \tag{2.3}$$

but in carrying out the analysis it is expedient to introduce the modified Lorentz transformation

$$\begin{pmatrix} \xi \\ \tau \end{pmatrix} = \beta^{-1} \begin{pmatrix} 1 & M \\ M & 1 \end{pmatrix} \begin{pmatrix} X \\ T \end{pmatrix}, \qquad \beta = (M^2 - 1)^{1/2} \tag{2.4}$$

$$\psi(\xi, y, z, \tau) = \phi(x, y, z, t) \tag{2.5}$$

Transforming the original wave equation (1), we obtain the new wave equation (with the positions of the axial and time variables reversed)

$$\psi_{\tau\tau} + \psi_{yy} + \psi_{zz} = \psi_{\xi\xi} \tag{2.6}$$

which is conveniently regarded as the canonical form for non-stationary flow at supersonic speeds. [If in (4) we take $\beta = (1 - M^2)^{1/2}$, $M < 1$, we have the classical Lorentz transformation, and (1) remains invariant with respect to the position of the space and time coordinates. The application to subsonic airfoil theory has been discussed by Kussner [11].]

The boundary condition to be invoked at the airfoil is

$$\phi_z = -w(x, y, t); \qquad x > 0, y > 0, z = 0 \tag{2.7}$$

where w is the prescribed velocity, *positive down*. In the transformed coordinates, we write

$$\psi_z = -v(\xi, y, \tau); \qquad \xi > 0, y > 0, z = 0 \tag{2.8}$$

$$v(\xi, y, \tau) = w(x, y, t) = w[\beta\xi, y, (l/\beta a)(M\xi - \tau)] \tag{2.9}$$

In consequence of the symmetry of this boundary condition with respect to $z = 0$, the potential itself is anti-symmetric thereto, and we may restrict our consideration to the half space $z \geq 0$, provided that we pose the additional boundary condition

$$\psi = 0; \qquad \xi > 0, y < 0, z = 0 \tag{2.10}$$

In addition to the local boundary conditions (8) and (10), it is necessary to impose the requirement that ψ behave asymptotically as a disturbance originating at the wing and vanishing identically forward of the zone of action of the wing, toward which end it suffices to write

$$\psi = 0; \qquad \xi < 0 \tag{2.11}$$

together with the Sommerfeld finiteness and radiation conditions in the original coordinates (X, y, z, T), although the latter may not be necessary in all cases.

To complete the formulation, we note that the perturbation pressure at the upper surface of the wing is given by

$$p = -\rho_0 l(\phi_t)_X = -\rho_0 a\beta^{-1}(M\psi_\xi + \psi_\tau); \qquad \xi > 0,\ y > 0,\ z = 0+ \tag{2.12}$$

We now pose the following problem: given w, find ϕ on $z = 0+$ or, equivalently, find a solution to (6) satisfying the boundary conditions (8), (10) and (11), together with the appropriate conditions at infinity.

3. Reduction to steady flow problem. Let Ψ be the transform obtained by posing the time dependence exp $(i\kappa\tau)$ and taking a Laplace transform with respect to ξ, *viz.*

$$\exp(i\kappa\tau)\Psi(s, y, z, \kappa) = \mathfrak{L}_\xi\{\psi\} = \int_0^\infty e^{-s\xi}\psi(\xi, y, z, \tau)\, d\xi \tag{3.1}$$

with a similar representative, V, for v. Transforming the boundary value problem of the preceding section, we have

$$\Psi_{yy} + \Psi_{zz} - \lambda^2\Psi = 0, \qquad \lambda^2 = s^2 + \kappa^2 \tag{3.2}$$

$$\Psi_z = -V; \qquad y > 0,\ z = 0 \tag{3.3}$$

$$\Psi = 0; \qquad y < 0,\ z = 0 \tag{3.4}$$

We now consider the "quasi-steady" problem obtained by setting $\lambda = s$ $(\kappa = 0)$ in (2) [but $\kappa \neq 0$ in (3)], which is tantamount to neglecting $\psi_{\tau\tau}$ in (2.6), thereby reducing the boundary value problem to one in steady flow. The solution to this reduced problem, which is obtained most directly by Evvard's method [12], is designated by ψ_0 and given at the wing (all subsequent potentials also are specified at $z = 0+$) by

$$\psi_0(\xi, y, 0+, \tau) = \frac{1}{\pi}\int_0^\xi d\mu \int_{|(\xi-\mu)-y|}^{(\xi-\mu)+y} [(\xi - \mu)^2 - (y - \eta)^2]^{-1/2} v(\mu, \eta, \tau)\, d\eta \tag{3.5}$$

the integration being carried out over the trapezoid bounded by the Mach lines passing through (ξ, y), the reflection of one of these $[\eta = y - (\xi - \mu)]$ in the side edge, and the leading edge.

Taking the Laplace transform of ψ_0, we have, by a slight extension of the Faltung heorem,

$$\Psi_0 = \mathfrak{L}_\xi\{\psi_0\} = \mathfrak{L}_\xi\mathfrak{L}_\mu\left\{\frac{1}{\pi}\int_{|\xi-y|}^{\xi+y} [\xi^2 - (y - \eta)^2]^{-1/2} v(\mu, \eta, \tau)\, d\eta\right\} \tag{3.6}$$

Carrying out the indicated operation on v and substituting λ for s in the operation with respect to ξ, we have

$$\Psi(s, y, 0+, \kappa) = \int_0^\infty \exp[-(s^2 + \kappa^2)^{1/2}\xi]\left\{\frac{1}{\pi}\int_{|\xi-y|}^{\xi+y} [\xi^2 - (y - \eta)^2]^{-1/2} V(s, \eta, \kappa)\, d\eta\right\} d\xi \tag{3.7}$$

where the phase of $(s^2 + \kappa^2)^{1/2}$ is chosen to ensure a positive real part.

4. Solution for harmonic time dependence. To effect the inverse Laplace transform of (3.7) we introduce the theorem [*cf.* p. 123 of ref. 13, a source hereafter denoted by MO]

$$\mathfrak{L}^{-1}\{F[(s^2 + \kappa^2)^{1/2}]\} = f(\xi) + \int_0^{\pi/2} f(\xi\cos\theta)\frac{\partial}{\partial\theta} J_0(\kappa\xi\sin\theta)\, d\theta \tag{4.1}$$

the application of which yields

$$\psi = \psi_0 + \frac{1}{\pi}\int_0^{\pi/2} d\theta \int_0^{\xi} \frac{\partial}{\partial\theta} J_0[\kappa(\xi - \mu)\sin\theta]\, d\mu \cdot \int_{|(\xi-\mu)\cos\theta - y|}^{(\xi-\mu)\cos\theta + y} [(\xi - \mu)^2 \cos^2\theta - (y - \eta)^2]^{-1/2} v(\mu, \eta, \tau)\, d\eta \tag{4.2}$$

where the second term appears as a correction on the "quasi-steady" solution.

An alternative expression for ψ that separates out the "strip theory" contribution

$$\psi_s = \int_0^{\xi} J_0[\kappa(\xi - \mu)] v(\mu, y, \tau)\, d\mu \tag{4.3}$$

can be obtained from (2) after integrating by parts and noting that

$$\lim_{\theta\to\pi/2} \int_{|(\xi-\mu)\cos\theta - y|}^{(\xi-\mu)\cos\theta + y} [(\xi - \mu)^2 \cos^2\theta - (y - \eta)^2]^{-1/2} v(\mu, \eta, \tau)\, d\eta = \pi v(\xi, y, \tau) \tag{4.4}$$

whence

$$\psi = \psi_s - \frac{1}{\pi}\int_0^{\pi/2} d\theta \int_0^{\xi} J_0[\kappa(\xi - \mu)\sin\theta]\, d\mu \cdot \frac{\partial}{\partial\theta}\int_{|(\xi-\mu)\cos\theta - y|}^{(\xi-\mu)\cos\theta + y} [(\xi - \mu)^2 \cos^2\theta - (y - \eta)^2]^{-1/2} v(\mu, \eta, \tau)\, d\eta \tag{4.5}$$

In the special case where v is assumed to be independent of y the η integration may be carried out explicitly, the result being

$$\psi = \psi_s - \frac{2}{\pi}\int_0^{\pi/2} d\theta \int_0^{\xi} J_0[\kappa(\xi - \mu)\sin\theta] v(\mu, \tau) \frac{\partial}{\partial\theta} \sin^{-1}[y/(\xi - \mu)\cos\theta]^{1/2}\, d\mu \tag{4.6}$$

where the arc sine is to be replaced by $\pi/2$ when its argument exceeds unity. Changing the variable of integration yields

$$\psi = \psi_s - \frac{2}{\pi}\int_0^{\xi} d\mu \int_0^{\cos^{-1}(y/\mu)^{1/2}} J_0[\kappa(\mu^2 - y^2 \sec^4\varphi)^{1/2}] v(\xi - \mu, \tau)\, d\varphi \tag{4.7}$$

The last result is essentially in the form given by Stewartson [6]. [We remark that it was the form of Stewartson's result that suggested the present approach, although the use of (8) in connection with supersonic airfoil theory is due to Magnaradze [14, 15] and has been applied previously to the rectangular wing by Galin [14], but without much progress, since he found it necessary to introduce Fourier series in ξ (*cf.* ref. 15).]

5. Arbitrary time dependence. Generalizing the result (4.2) we have, after Fourier transformation (MO119) and convolution,

$$\psi = \psi_0 + \frac{1}{\pi^2}\int_0^{\pi/2} d\theta \int_0^{\xi} d\mu \int_{|(\xi-\mu)\cos\theta - y|}^{(\xi-\mu)\cos\theta + y} [(\xi - \mu)^2 \cos^2\theta - (y - \eta)^2]^{-1/2}\, d\eta \cdot \frac{\partial}{\partial\theta}\int_{\tau-(\xi-\mu)\sin\theta}^{\tau+(\xi-\mu)\sin\theta} [(\xi - \mu)^2 \sin^2\theta - (\tau - \zeta)^2]^{-1/2} v(\mu, \eta, \zeta)\, d\zeta \tag{5.1}$$

In connection with the ζ limits of integration, we remark that the maximum and minimum values of ζ permitted in v are ∞ and $\tau - M(\xi - \mu)$, corresponding to $-\infty$ and present time after transformation back to the physical variables. Both of these limits being outside of the domain in which the inverse transform of $J_0\,[\kappa(\xi - \mu)\sin\theta]$, is non-vanishing, we choose the limits as shown, although v may vanish over some part of the latter region, as in transient problems.

A more convenient expression for ψ, obtained by introducing a trigonometric variable in place of ζ, is given by

$$\psi = \psi_0 + \frac{1}{\pi^2}\int_0^{\pi/2} d\theta \int_0^{\pi} d\chi \int_0^{\xi} d\mu \int_{|(\xi-\mu)\cos\theta - y|}^{(\xi-\mu)\cos\theta + y} [(\xi - \mu)^2 \cos^2\theta - (y - \eta)^2]^{-1/2} \cdot \frac{\partial}{\partial\theta} v[\mu, \eta, \tau + (\xi - \mu)\sin\theta\cos\chi]\, d\eta \tag{5.2}$$

Numerous, alternative forms may be obtained by additional changes of variable and by integration by parts.

Finally, upon substituting the physical variables x, t, ϕ and w from (2.2, 2.3, 2.4, 2.5, 2.9) we obtain

$$\phi_0 = \frac{1}{\pi}\int_0^{x} d\mu \int_{|\beta^{-1}(x-\mu) - y|}^{\beta^{-1}(x-\mu) + y} [(x - \mu)^2 - \beta^2(y - \eta)^2]^{-1/2} \cdot w[\mu, \eta, t - (Ml/\beta^2 a)(x - \mu)]\, d\eta \tag{5.3}$$

$$\phi = \phi_0 + \frac{1}{\pi^2}\int_0^{\pi/2} d\theta \int_0^{\pi} d\chi \int_0^{x} d\mu \int_{|\beta^{-1}(x-\mu)\cos\theta - y|}^{\beta^{-1}(x-\mu)\cos\theta + y} [(x - \mu)^2 \cos^2\theta - \beta^2(y - \eta)^2]^{-1/2} \cdot \frac{\partial}{\partial\theta} w[\mu, \eta, t - (l/\beta^2 a)(M + \sin\theta\cos\chi)(x - \mu)]\, d\eta \tag{5.4}$$

6. Oscillating wing. For the important special case of a wing undergoing the oscillating motion prescribed by

$$w(x, y, t) = Rl\{w(x, y)e^{i\omega t}\} \tag{6.1}$$

we have only to choose

$$\chi = kM/\beta, \qquad k = \omega l/U \tag{6.2}$$

in (4.3) and (4.5) and return to the original variables, whence

$$\phi_s = \beta^{-1} Rl\left\{\int_0^{x} \exp\,[i\omega t - i(kM^2/\beta^2)(x - \mu)] J_0[(kM/\beta^2)(x - \mu)] w(\mu, y)\, d\mu\right\} \tag{6.3}$$

$$\phi = \phi_s - \frac{1}{\pi} Rl\left\{\int_0^{\pi/2} d\theta \int_0^{x} \exp\,[i\omega t - i(kM^2/\beta^2)(x - \mu)] J_0[(kM/\beta^2)(x - \mu)\sin\theta]\, d\mu \cdot \frac{\partial}{\partial\theta}\int_{|\beta^{-1}(x-\mu)\cos\theta - y|}^{\beta^{-1}(x-\mu)\cos\theta + y} [(x - \mu)^2\cos^2\theta - \beta^2(y - \eta)^2]^{-1/2} w(\mu, \eta)\, d\eta\right\} \tag{6.4}$$

7. Extension of Evvard's "equivalent area" concept. It is of some interest to note that in (4.2) *et. seq.* the domain of the (μ, η) integration is bounded by the pseudo Mach lines $\eta = y \pm (\xi - \mu)\cos\theta$, together with the reflection of $\eta = y - (\xi - \mu)\cos\theta$, namely $\eta = (\xi - \mu) \cos \theta - y$, in the side edge. This interpretation furnishes an extension of Evvard's "equivalent area" concept to unsteady flow, but the extension is not valid for oblique or curved side edges, since, in general, the equation for the reflected Mach line would be of the form $\eta = f(\xi, \xi - \mu) - y$, rather than simply $f(\xi - \mu) - y$, and the Faltung theorem [cf. (3.6)] would not be applicable.

8. Oblique edges. The extension of the foregoing results to wings having arbitrarily prescribed supersonic leading edges and streamwise side edges is trivial, since it is necessary only to circumscribe a fictitious rectangular wing and set $w = 0$ over those portions that are not included in the original planform. [However, it should be remarked that the Mach lines from the opposite corners may intersect on the wing only if it is rectangular (*cf.* ref. 3).] To extend the results to a straight, subsonic leading edge that is adjacent to a supersonic leading edge and does not interfere with any other subsonic edge, we may apply the Lorentz transformation

$$\begin{pmatrix} \xi' \\ y' \end{pmatrix} = (1 - m^2)^{-1/2} \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \begin{pmatrix} \xi \\ y \end{pmatrix}, \qquad m > 1 \tag{8.1}$$

under which (2.6) remains invariant. Unfortunately, it is then no longer possible to obtain results for the spanwise integrals of the potential that are comparable in simplicity to those for the rectangular wing (*cf.* refs. 3, 5, 6.).

The use of (1) in connection with supersonic wings in steady flow is well known (*cf.* refs. 16-18), but the application to unsteady flow problems seems not to have been noticed previously, perhaps due to the fact that the differential equation usually is written for $\phi(x, y, z, t)$, rather than $\psi(\xi, y, z, \tau)$.

9. Other methods. It is rather curious that of all the methods that have been applied to the problem of diffraction by a half plane, the classical counterpart of the supersonic rectangular wing, none [*e.g.*, Poincaré's original treatment, Sommerfeld's celebrated application of multivalued integrals, Lamb's use of parabolic coordinates, Magnus' solution of the integral equation, the Wiener-Hopf technique applied by Copson and Schwinger, *etc.*; *cf.* ref. 19] has proved as effective as the methods derived especially for the wing problem, notably Busemann's use of homogeneous solutions ("conical flows") and Evvard's method. (We remark that each of these methods had antecedents in the work of Bateman [20].) This is at least partially due to the different foci of interest in the two situations, *viz.* the potential on $z = 0$ in the wing problem and the solution at a distance (particularly near the boundary of the geometric shadow) in the diffraction problem, but it is of interest to note that the method of conical flows furnishes an elegant approach to the problem of pulse diffraction [21, 22, 23], while the application of Evvard's method to case of an arbitrary incident wave has received attention in a recent paper by Friedlander [24].

The foregoing remarks notwithstanding, Lamb's method has proved rather attractive for the special case of no spanwise variation of w, an application discussed recently by Rott [5]. This method is, in fact, applicable to more general spanwise distributions [25], but, in the form presented by Rott, it is much less direct than the present method, from which it differs fundamentally in prescribing the boundary data on the wing tip Mach cone, rather than the wing proper. However, if the potential is desired only at the wing the boundary data may be prescribed there. Thus, a solution to (3.2) in the polar coordinates (ρ, φ) that vanishes on $z = 0$, $y < 0$ $(\varphi = \pi)$, gives a null value of the normal derivative on the wing ($\varphi = 0$) and vanishes properly at infinity is given by

$$\Psi_y = F(s, \kappa)\rho^{-1/2}e^{-\lambda\rho} \cos (\varphi/2). \tag{9.1}$$

This solution evidently is appropriate to the case where v is independent of y. Moreover, at large distances from the edge Ψ must reduce to the two dimensional solution, *viz.* [*cf.* ref. 6 or (3.2) and (3.3) with $\Psi_{yy} = 0$]

$$\lim_{y\to\infty} \Psi = \Psi_s = \lambda^{-1}v(s, \chi)e^{-\lambda|z|}. \tag{9.2}$$

Integrating (1) subject to the conditions that Ψ must vanish at $y = 0$ and satisfy (2) asymptotically, we have on $\varphi = 0$

$$\Psi = \lambda^{-1}v(s, \chi) \text{ erf } [(\lambda y)^{1/2}] \tag{9.3}$$

the inversion of which yields the result (4.7).

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