INFINITE MATRICES ASSOCIATED WITH DIFFRACTION BY AN APERTURE*

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1. Introduction and summary. As an example of their "variational method", LEVINE and SCHWINGER [1] investigated a boundary value problem which arises from the diffraction of a plane scalar (acoustical) wave by a plane screen with a circular aperture. It is equivalent to the problem of finding the field of a freely vibrating circular disk. A full discussion of the physical problems was given by Bouwkamp [2]. Let z, ρ, θ be cylindrical coordinates and let $z = 0$ be the plane occupied by the screen. Let $z = 0$, $0 \leqq \rho < a$ define the aperture (or the vibrating disk). The diffracted field is given by a function u which satisfies $\nabla^2 u + k^2 u = 0$ (with a constant k) everywhere except for $z = 0$ and at infinity satisfies a Sommerfeld radiation condition. For $z = 0$, u must satisfy the "mixed" boundary conditions $u = 0$ for $\rho > a$ and $\partial u/\partial z = v_0$ with a given constant value v_0 for $0 \leqq \rho < a$. These conditions determine u uniquely. For $z = 0$, $0 \leqq \rho < a$, $u = \Phi(\rho)$ becomes a function of ρ only, and if $\Phi(\rho)$ is known or even if only $C_0\Phi(\rho)$ with an undetermined constant factor C_0 is known, u can be determined everywhere; see formulas (A.1), (A.2), (A.3) in [1].

Levine and Schwinger [1] show that the ratio of the energy transmitted through the aperture to the energy incident on the aperture is the imaginary part of the complex transmission coefficient T^*, which is a quotient of two integrals involving $\Phi(\rho)$ quadratically. As a functional of $\Phi(\rho)$, T^* becomes stationary for the correct function Φ which determines u. Levine and Schwinger find approximate values for T^* by expanding first $\Phi(\rho)$ in an infinite series of auxiliary functions (see 3.1 and 3.2) with coefficients D_m. Then T^* becomes a linear form in the D_m (see 3.10), and the unknowns D_m are determined by an inhomogeneous system of infinitely many linear equations with a coefficient matrix L (see 3.4, 3.5). In [1], these equations are solved "section wise", using the first $l = 1, 2, 3, \cdots$ equations to determine the first l unknowns. All quantities D_m, T^*, L are power series in $\beta = ka/2$, and Levine and Schwinger compute the first coefficients of the expansion of T^* in a power series in β which were determined independently by Bouwkamp [2], who used spheroidal wave functions.

It will be shown that the algebraic properties of the matrix L make it possible not only to find approximate values for T^* as in [1] but also to determine $\Phi(\rho)$. This is due to the fact that L factorizes in a product $L^{(0)}S$, where $L^{(0)}$ is the matrix for the static case $k = 0$ and where S can be inverted by solving finite recurrence relations. The details are stated in lemma 1 and theorem 1 of section 3. Lemma 2 gives additional algebraic relations. Problems of convergence and uniqueness are settled in section 5. These depend largely on an investigation of the properties of $L^{(0)}$ which is carried through in section 4. There it is shown that in the limiting cases $k = 0$ and $k = \infty$ the matrices $L^{(0)}$ and $L^{(\infty)}$ of the linear equations also arise from a problem of moments. This also makes it possible to prove that the variational method for the calculation of the transmission

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coefficient will work even for $k = \infty$ where the linear equations for the D_m do not have any solution at all.

2. Notations. The elements of (infinite) matrices are denoted by subscripts n, $m = 0, 1, 2, \cdots$ where n denotes the rows and m denotes the columns. A vector with components x_m is denoted by $\{x_m\}$. We also use the notations

$$(a)_n = \Gamma(a + n)/\Gamma(a) = a(a + 1) \cdots (a + n - 1); \qquad (a)_0 = 1, \tag{2.1}$$

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \tag{2.2}$$

where Γ denotes the gamma function and F denotes the hypergeometric series. For results needed here see Whittaker and Watson [3] and Bailey [4].

3. Algebraic properties of the linear equations. Let

$$\Phi(\rho) = -\frac{1}{2} aC_0 \sum_{m=0}^{\infty} x_m(1 - \rho^2/a^2)^{m+1/2} \tag{3.1}$$

be the expansion of the field $\Phi(\rho)$ in the aperture in terms of powers of $1 - \rho^2/a^2$. Here C_0 denotes an undetermined constant and

$$-\tfrac{1}{2}ax_m = D_m \tag{3.2}$$

where the D_m are the unknowns used by Levine and Schwinger [1]. The linear equations for the x_m as obtained from the variational method can be written as follows:

Let $p, q = 0, 1, 2, \cdots$ and let $L^{(2p)}, L^{(2q+3)}$ be infinite matrices with elements $l^{(2p)}_{n,m}$, $l^{(2q+3)}_{n,m}$ defined by

$$l^{(2p)}_{n,m} = (-1)^p \pi^{1/2} A(n, m, p)/B(n, m, p), \tag{3.3}$$

$$l^{(2q+3)}_{n,m} = i(-1)^q \pi^{1/2} A(n, m, q + 3/2)/B(n, m, q + 3/2), \tag{3.4}$$

where, for any values of n, m, t

$$A(n, m, t) = \Gamma(n + 3/2)\Gamma(m + 3/2)\Gamma(n + m + 2t + 1),$$

$$B(n, m, t) = 4\Gamma(t + 1)\Gamma(n + t + 1)\Gamma(m + t + 1)\Gamma(n + m + t + 5/2).$$

Let L be the matrix

$$L = \sum_{p=0}^{\infty} \beta^{2p} L^{(2p)} + \sum_{q=0}^{\infty} \beta^{(2q+3)} L^{(2q+3)}, \tag{3.5}$$

the general element $l_{n,m} = l_{n,m}(\beta)$ of which is a power series in $\beta = \frac{1}{2}ka$. Then

$$\sum_{m=0}^{\infty} l_{n,m} x_m = (n + 3/2)^{-1}. \tag{3.6}$$

Let ξ denote the vector with the components x_m and let $\xi^{(r)}$, $r = 0, 1, \cdots$ be the vector with the components $x^{(r)}_m$ where

$$x_m = \sum_{r=0}^{\infty} \beta^r x^{(r)}_m. \tag{3.7}$$

Let $\eta^{(0)}$ denote the vector with the components $1/(m + 3/2)$. Comparing the coefficients of β^r, $r = 0, 1, \cdots$, on both sides of (3.6) we find

$$L^{(0)}\xi^{(0)} = \eta^{(0)}, \qquad L^{(0)}\xi^{(1)} = 0. \tag{3.8}$$

and, for $r = 2, 3, 4, \cdots$:

$$L^{(0)}\xi^{(r)} + L^{(2)}\xi^{(r-2)} + \cdots + L^{(r)}\xi^{(0)} = 0. \tag{3.9}$$

If

$$T^* = \sum_{m=0}^{\infty} x_m/(m + 3/2), \tag{3.10}$$

the transmission coefficient T becomes

$$T = \beta/2 \text{ Im } T^* \tag{3.11}$$

where Im denotes the imaginary part. We shall now show that $L^{(0)}$ is a common left hand factor of all the matrices $L^{(2p)}$, $L^{(2q+3)}$, such that the right hand factor is a bounded matrix.

Lemma 1. *Let* $p = 1, 2, 3, \cdots$ *and* $q = 0, 1, 2, \cdots$, *and let* $S^{(2p)} = (s_{n,m}^{(2p)})$ and $S^{(2q+3)} = (s_{n,m}^{(2q+3)})$ *be the matrices defined by*

$$\left.\begin{aligned} s_{n,m}^{(2p)} &= 0 \quad \text{if} \quad n > p + m \\ s_{n,m}^{(2q+3)} &= 0 \quad \text{if} \quad n > q \end{aligned}\right\} \tag{3.12}$$

and otherwise

$$s_{n,m}^{(2p)} = (-1)^p G(n, m, p)/H(n, m, p), \tag{3.13}$$

$$s_{n,m}^{(2q+3)} = i(-1)^q G(n, m, q + 3/2)/H(n, m, q + 3/2), \tag{3.14}$$

where, for any values of n, m, t

$$G(n, m, t) = (-t + 3/2)_n \Gamma(2t - n + m)\Gamma(m + 3/2),$$

$$H(n, m, t) = \Gamma(t + 1)\Gamma(t)\Gamma(t + m - n + 1)\Gamma(t + m + 3/2)(3/2)_n .$$

Then

$$L^{(2p)} = L^{(0)}S^{(2p)}, \qquad L^{(2q+3)} = L^{(0)}S^{(2q+3)}. \tag{3.15}$$

Proof: The element in the n-th row and m-th column of $L^{(0)}S^{(2p)}$ is

$$\frac{\sqrt{\pi}}{4} \frac{(-1)^p}{p!(p-1)!} \frac{\Gamma(n + 3/2)}{n!} \frac{\Gamma(m + 3/2)}{\Gamma(m + p + 3/2)} \sum \tag{3.16}$$

where, because of (2.1) and simple properties of the Gamma function

$$\sum_{n,m} = \sum_{r=0}^{p+m} \frac{(n + r!)}{r!} \frac{\Gamma(r + 3/2)}{\Gamma(n + r + 5/2)} \frac{(-p + 3/2)_r}{(3/2)_r} \frac{\Gamma(2p + m - r)}{\Gamma(p + m - r + 1)} \tag{3.17}$$

$$= \frac{n!\Gamma(2p + m)\Gamma(3/2)}{(p + m)!\Gamma(n + 5/2)} \sum_{r=0}^{p+m} \frac{(n + 1)_r}{r!} \frac{(3/2 - p)_r}{(n + 5/2)_r} \frac{(-p - m)_r}{(1 - m - 2p)_r}. \tag{3.18}$$

The sum in (3.18) can be computed by using Saalschuetz's formula (cf. Bailey [4] for a simple proof) which can be written in the form

$$\sum_{r=0}^{k} \frac{(a)_r(b)_r(-k)_r}{r!(c)_r(1 + a + b - c - k)_r} = \frac{(c - a)_k(c - b)_k}{(c)_k(c - a - b)_k}. \tag{3.19}$$

$(k = 0, 1, 2, \cdots ; c \neq 0, -1 - 2, \cdots \quad -k - 1; 1 + a + b - c \neq 1, 2, \cdots , k)$

Taking $a = n + 1$, $b = -p + 3/2$, $c = 5/2 + n$, $k = p + m$, (3.19) gives for $\sum_{n,m}$ in (3.17)

$$\sum_{n,m} = \frac{n!\Gamma(2p + m)\Gamma(3/2)}{(p + m)!\Gamma(n + 5/2)} \frac{(3/2)_{m-p}(n + p + 1)_{m-1}}{(n + 5/2)_{m+p}(p)_{m+p}}. \tag{3.20}$$

From (3.20) and (3.16) it follows that $L^{(2p)} = L^{(0)}S^{(2p)}$. The proof of $L^{(2p+3)} = L^{(0)}S^{(2p+3)}$ follows by the same method.

The elements of the matrices $S^{(2q-3)}$ are zero except for those in the first q rows. This is not true for the $S^{(2p)}$ but the following lemma shows that $S^{(2p)}$ is a polynomial in $S^{(2)}$ apart from right hand factors which are either the identity or of the type of the $S^{(2q+3)}$.

We have:

Lemma 2. *Let p, t* $= 1, 2, 3, \cdots$ *and let* $R^{(t)}$ *be the matrix for which the element in the first row and m-th column is*

$$\frac{(-1)^{t+1}}{(t - 1/2)t!(t - 1)!} \frac{\Gamma(m + 3/2)\Gamma(2t + m + 1)}{\Gamma(m + t + 3/2)(t + m + 1)!} \tag{3.21}$$

all other elements of $R^{(t)}$ *being zero. Then*

$$S^{(2)}S^{(2t)} - \frac{t + 1}{1 - 2t} S^{(2t+2)} = R^{(t)}, \tag{3.22}$$

$$S^{(2t+2)} = \sum_{\mu=0}^{t} (-2)^{\mu+1}[(-t + 1/2)_{\mu+1}/(-1 - t)_{\mu+1}]\{S^{(2)}\}^{\mu}R^{(t-\mu)}, \tag{3.23}$$

where, for $\mu = t$, $R^{(0)}$ *denotes* $S^{(2)}$. *In general,*

$$S^{(2p)}S^{(2t)} - \frac{(t + p)!}{p!t!} \frac{\Gamma(3/2)\Gamma(-t - p + 3/2)}{\Gamma(-p + 3/2)\Gamma(-t + 3/2)} S^{(2p+2t)} \tag{3.24}$$

is a matrix in which all elements are zero except those in the first p rows.

The proof of lemma 2 follows again from Saalschuetz's formula. We have now:

Theorem 1. *If the equations*

$$L^{(0)}\xi^{(0)} = \eta \tag{3.25}$$

have a solution, then all the vectors $\xi^{(m)}$ *are determined by* $\xi^{(0)}$ *and by the relations* $\xi^{(1)} = 0$ *and the recurrence relations*

$$\xi^{(r)} = -S^{(2)}\xi^{(r-2)} - S^{(3)}\xi^{(r-3)} - \cdots - S^{(r)}\xi^{(0)}. \tag{3.26}$$

In the particular case where

$$\eta = \eta^{(0)} = (2/3, 2/5, 2/7, \cdots), \tag{3.27}$$

we have

$$\xi^{(0)} = (8/\pi, 0, 0, 0, \cdots), \tag{3.28}$$

and at most the first $r + 1$ *components of* $\xi^{(r)}$ *are different from zero.* $\xi^{(0)}, \cdots, \xi^{(r)}$ *are the solutions of the original system* (3.6), *if we use the first* $r + 1$ *equations for determining the first* $r + 1$ *unknowns and thereby neglect all terms involving the higher powers of* β *from the r-th power onwards.* $\xi^{(0)}, \cdots, \xi^{(r)}$ *also determine the exact values of the first* $r + 1$ *coefficients of the expansion of* T^* *in powers of* β.

The proof of theorem 1 follows immediately from lemma 1 and in particular from the fact that the $S^{(2p)}$, $S^{(2q+3)}$ involve many vanishing elements. The uniqueness of the $\xi^{(r)}$, and the existence of the x_m (at least for sufficiently small values of β) will be proved in section 5.

4. Limiting cases for the matrix L. Let

$$P(t) = \Gamma(t + 3/2)/\Gamma(t + 1), \qquad Q(t) = \Gamma(t + 5/2)/\Gamma(t + 1). \tag{4.1}$$

Then Theorem 1 states that the equations

$$\sum_{m=0}^{\infty} l_{n,m}(\beta)x_m = h_n \qquad (n = 0, 1, 2, \cdots) \tag{4.2}$$

can be solved by formal (i.e. not necessarily convergent) power series in β if the equations

$$4L^{(0)}\xi \equiv \left\{\pi^{1/2}P(n) \sum_{m=0}^{\infty} x_m P(m)/Q(n + m)\right\} = \{4h_n\} \tag{4.3}$$

have a solution $x_m = x_m^{(0)}$. We shall investigate (4.3) together with the limiting case $\beta \to \infty$. Levine and Schwinger [1] have shown that then (4.2) tends towards the system of linear equations

$$L^{(\infty)}\xi \equiv \left\{\sum_{m=0}^{\infty} x_m/(n + m + 2)\right\} = \mu\{h_n\}, \qquad (n = 0, 1, 2, \cdots) \tag{4.4}$$

where μ is a constant.

We have to define first the linear space of admissible solutions x_m from the nature of the problem. Since (3.1) is supposed to define the field in the aperture, and since the field cannot have a singularity in the center of the aperture, we must assume that

$$\lim_{\epsilon \to 0} \sum_{m=0} x_m(1 - \epsilon)^m \tag{4.5}$$

exists. Since the original system (3.6) was set up merely in order to define the transmission coefficient, we shall assume that

$$\sum_{m=0}^{\infty} x_m/(m + 3/2) \tag{4.6}$$

converges. This implies, that

$$\sum_{m=0}^{\infty} x_m z^m \tag{4.7}$$

converges for $|z| < 1$ and therefore that the x_m actually define the field in the aperture. Then we prove first:

Lemma 3. *If the vector ξ with the components x_m satisfies* (4.5) *and* (4.6), *then the operators $L^{(0)}$ and $L^{(\infty)}$ are defined for ξ in the sense that the sums in* (4.3), (4.4) *converge for* $n = 0, 1, 2, \cdots$

Proof: Let $Q(t)$ be defined as in (4.1) and let

$$\tau_m = Q(m)/Q(n + m), \qquad \sigma_m = \sum_{r=0}^{m} x_r/(r + 3/2). \tag{4.8}$$

Then the partial sums of the series in (4.3) are

$$\sum_{r=0}^{m} \tau_r x_r/(r + 3/2) = \sum_{r=0}^{m-1} (\tau_r - \tau_{r+1})\sigma_r + \tau_m \sigma_m \tag{4.9}$$

where

$$2\tau_{r+1} - 2\tau_r = 3nP(r+1)/\{Q(n+r)[n+r+5/2]\}. \tag{4.10}$$

Since the $|\sigma_n|$ are bounded and $\sum_r |\tau_r - \tau_{r+1}|$ converges, the sums in (4.3) also converge. The proof for the convergence of the sums in (4.4) is even simpler.

Theorem 2. *If the equations* $L^{(0)}\xi = \{h_n\}$ *or* $L^{(\infty)}\xi = \{h_n^*\}$ *have a solution* $\xi = \{x_m^{(0)}\}$ *or* $\xi = \{x_m^{(\infty)}\}$ *satisfying* (4.5) *and* (4.6), *then the integral equations*

$$\int_0^1 f(v)(1-v)^{1/2}(1-vz)^{-1}\,dv = 4\pi^{-1/2}\sum_{n=0}^{\infty} z^n h_n n!/(3/2)_n\,, \tag{4.11}$$

$$\int_0^1 f^*(v)v(1-vz)^{-1}\,dv = \sum_{n=0}^{\infty} h_n^* z^n, \tag{4.12}$$

have analytic solutions

$$f(v) = \sum_{m=0}^{\infty} v^m x_m^{(0)}\Gamma(m+3/2)/m!, \qquad f^*(v) = \sum_{m=0}^{\infty} x_m^{(\infty)} v^m. \tag{4.13}$$

The solutions are unique and they also solve the problems of moments

$$\int_0^1 f(v)(1-v)^{1/2}v^n\,dv = 4\pi^{-1/2}h_n n!/(3/2)_n\,, \qquad \int_0^1 f^*(v)v^{n+1}\,dv = h_n^*\,. \tag{4.14}$$

The integrals in (4.11) (4.12) are defined by

$$\int_0^1 = \lim_{\epsilon\to+0}\int_0^{1-\epsilon}. \tag{4.15}$$

Since a formal expansion of the left hand sides of (4.11) and (4.12) leads to the linear equations $L^{(0)}\xi = \{h_n\}$ and $L^{(\infty)}\xi = \{h_n^*\}$, it has only to be shown that, under the assumptions made about the x_m, such an expansion is legitimate. It suffices to prove that

$$\lim_{\epsilon\to 0}\int_0^{1-\epsilon} f(v)(1-v)^{1/2}v^n\,dv = 8\pi^{-1}h_n n!/\Gamma(n+3/2) \tag{4.16}$$

where now $f(v)$ is defined by (4.13) and h_n by $L^{(0)}\xi = \{h_n\}$. Since it follows from the assumption (4.5) about the x_m that $f(v)$ converges absolutely and uniformly for $0 \leqq v \leqq 1 - \epsilon$, we may integrate term by term in (4.16). Putting $Y_m = x_m^{(0)}\Gamma(m+3/2)/m!$ this gives (with $v = (1-\epsilon)W$)

$$\sum_{m=0}^{\infty} Y_m \int_0^{1-\epsilon} v^{n+m}(1-v)^{1/2}\,dv$$
$$= \sum_{m=0}^{\infty} Y_m(1-\epsilon)^{n+m+1}\int_0^1 W^{n+m}[1-(1-\epsilon)W]^{1/2}\,dW \tag{4.17}$$

$$= \sum_{m=0}^{\infty} Y_m(1-\epsilon)^{n+m+1}(n+m+1)^{-1}F(-1/2, n+m+1, n+m+2; 1-\epsilon) \tag{4.18}$$

$$= \sum_{m=0}^{\infty} Y_m(1-\epsilon)^{n+m+1}(n+m+1)^{-1}F(-1/2, n+m+1; n+m+2; 1)$$
$$+ \sum_{m=0}^{\infty} Y_m(1-\epsilon)^{n+m+1}(n+m+1)^{-1}\{F(\cdots;1-\epsilon) - F(\cdots;1)\}. \tag{4.19}$$

According to Gauss's formula (cf. Whittaker-Watson [3])

$$F(-1/2, n + m + 1; n + m + 2; 1) = (n + m + 1)!\Gamma(3/2)/\Gamma(n + m + 5/2), \quad (4.20)$$

and from Abel's lemma and from lemma 3 it follows that

$$\lim_{\epsilon \to 0} \sum_{m=0}^{\infty} Y_m(1 - \epsilon)^{n+m+1}\Gamma(3/2)(n + m)!/(\Gamma(n + m + 5/2) = 4\pi^{-1/2}h_n n!/(3/2)_n . \quad (4.21)$$

Now we have to show that the second sum in (4.19) tends towards zero as $\epsilon \to 0$. Because of (4.5) it suffices to show that

$$\begin{aligned} c_{m,n}(\epsilon) & \\ = \Gamma(m + 3/2)[m!]^{-1}[n + m + 1]^{-1}\{F(-1/2, n + m + 1; n + m + 2; 1 - \epsilon) & \quad (4.22) \\ - F(-1/2, n + m + 1; n + m + 2; 1)\} & \end{aligned}$$

$$\begin{aligned} = \Gamma(m + 3/2)(2m!)^{-1} \sum_{k=0}^{\infty} [1 - (1 - \epsilon)^{k+1}] & \\ \cdot (1/2)_k/\{(k + 1)!(n + m + k + 2)\} \to 0 & \quad (4.23) \end{aligned}$$

as $\epsilon \to 0$ uniformly in n, m. We can prove that $| c_{m,n}(\epsilon) | < \epsilon$ by observing that $1 - (1 - \epsilon)^{k+1} \leqq (k + 1)\epsilon$. This and (4.23) gives

$$\begin{aligned} | c_{m,n}(\epsilon) | &\leqq \epsilon\Gamma(m + 3/2)(2m!)^{-1} \sum_{k=0}^{\infty} (1/2)_k(n + m + k + 2)^{-1}\{k!\}^{-1} \\ &= \epsilon\Gamma(m + 3/2)\{2m!(n + m + 2)\}^{-1}F(1/2, n + m + 2; n + m + 3; 1) \\ &= \epsilon\Gamma(1/2)\Gamma(m + 3/2)(n + m + 1)!\{2m!\Gamma(n + m + 5/2)\}^{-1} \quad (4.24) \\ &= \epsilon \frac{\pi^{1/2}}{2} \frac{(m + 1)(m + 2) \cdots (m + n + 1)}{(m + 3/2)(m + 5/2) \cdots (m + n + 3/2)} \leqq \epsilon\pi^{1/2}/2 < \epsilon. \end{aligned}$$

The uniqueness of the solution follows from

Lemma 4: *If* $\sum_{m=0}^{\infty} x_m/(m + 3/2)$ *converges, then for* $0 \leqq v < 1$, $(1 - v)^{3/2} | f(v) |$ *is bounded.* The proof follows from summation by parts with the notation (4.8) and from the remark that

$$\sum_{m=0}^{\infty} \Gamma(m + 5/2) | \sigma_m | v^m/(m + 1)! \leqq C[(1 - v)^{-3/2} - 1]v^{-1}, \quad (4.25)$$

where c does not depend on v.

Now we can show that (4.3) cannot have a null solution. Because then the difference $\phi(v)$ of two solutions of (4.11) would satisfy

$$\int_0^1 \phi(v)(1 - v)^{1/2}v^n \, dv = 0, \qquad n = 0, 1, 2, \cdots , \quad (4.26)$$

and therefore:

$$\int_0^1 \phi(v)(1 - v)^{1/2}(1 - v)v^n \, dv = 0, \qquad n = 0, 1, 2, \cdots \quad (4.27)$$

But $\phi(v)$ $(1 - v)^{3/2}$ would be a function continuous in $0 \leqq v \leqq 1$ according to lemma 4 and therefore (4.27) shows that $\phi(v)(1 - v)^{3/2}$ would be identically zero.

Conclusions from theorem 1. The equivalence of the equations $L^{(0)}\xi = \{h_m\}$ and $L^{(\infty)}\xi = \{h_m^*\}$ to a problem of moments shows that these sets of linear equations are unstable in the following sense: Not only may these equations have no solution at all, but this is certain to happen if we start with a set $\{h_m\}$ of right hand sides for which a solution exists and then change a finite number of the h_m by an amount however small. In this case there does not even exist a continuous function $f(v)$ which satisfies (4.11) or (4.12) with the modified right hand sides.

The integral operators in (4.11), (4.12) are extensions of the linear operators defined by $L^{(0)}$ or $L^{(\infty)}$, since (4.11) or (4.12) may have a continuous solution $f(v)$ which is not analytic. Consequently, a quantity like the transmission coefficient

$$T^* = \int_0^1 f(v)v^{1/2}\,dv = \sum_{m=0}^{\infty} x_m/(m + 3/2) \tag{4.28}$$

can be defined even in cases where the x_m do not exist. An easy example is offered by the equations

$$\sum_{m=0}^{\infty} x_m/(n + m + 2) = \mu/(n + 3/2), \qquad (n = 0, 1, 2, \cdots) \tag{4.29}$$

which were also investigated by Levine and Schwinger. The corresponding integral equation is

$$\int_0^1 f(v)v(1 - vW)^{-1}\,dv = \mu \sum_{n=0}^{\infty} W^n/(n + 3/2) = \mu \int_0^1 v^{1/2}/(1 - vW)^{-1}\,dv \tag{4.30}$$

which gives

$$f(v) = \mu v^{-1/2}, \qquad T^* = \mu. \tag{4.31}$$

In this case no set of x_m satisfying (4.29) can exist. However, it is possible to find sequences of constants $Y_m^{(r)}$ such that

$$\sum_{m=0}^{\infty} Y_m^{(r)}(m + n + 2)^{-1} = \psi_n^{(r)} \tag{4.32}$$

exist and

$$\lim_{r\to\infty} \sum_{n=0}^{\infty} \{\psi_n^{(r)} - \mu/(n + 3/2)\}^2 = 0, \qquad \lim_{r\to\infty} \sum_{m=0}^{\infty} Y_m^{(r)}/(m + 3/2) = \mu. \tag{4.33}$$

For this purpose, we can choose the $Y_m^{(r)}$ from

$$\sum_{m=0}^{\infty} Y_m^{(r)} v^m = \sum_{k=0}^{r} (1 - v)^k (1/2)_k/k! \tag{4.34}$$

The right hand side in (4.34) is a polynomial which approximates $v^{-1/2}$, since it is the $(r + 1)$-th partial sum of $[1 - (1 - v)]^{-1/2}$. Clearly, the $Y_m^{(r)} \to \infty$ as $r \to \infty$.

5. Uniqueness and existence of the solution. Once a vector $\xi^{(0)}$ has been determined such that $L^{(0)}\xi^{(0)} = \eta$, where η is the vector of the right hand sides in the original equations $L\xi = \eta$, we can determine ξ from

$$M\xi = \xi^{(0)} \tag{5.1}$$

where, for all values of β, M is defined by

$$M = \mathcal{I} + \sum_{p=1}^{\infty} \beta^{2p} S^{(2p)} + \sum_{q=0}^{\infty} \beta^{2q+3} S^{(2q+3)} \tag{5.2}$$

Here $\mathcal{I}$ denotes the identity. We shall call a vector ξ bounded if $\sum |\xi_m|^2 < \infty$ and we shall call a matrix M bounded if there exists a constant $U > 0$ such that for all bounded vectors ξ:

$$\xi^* M'^* M \xi \leqq U^2 \sum |\xi_m|^2 \tag{5.3}$$

where M' is the transposed matrix of M and an asterisk denotes the conjugate complex quantity. U is called an upper bound for M. It is well known that, if U_r is an upper bound for $S^{(r)}(r = 1, 2, 3 \cdots)$, the matrix M in (5.2) has a bounded inverse M^{-1} if

$$\sum_{r=2}^{\infty} \beta^r U_r < 1 \tag{5.4}$$

M^{-1} can be obtained from a Neumann series. We can use this in order to prove:

Theorem 3. *Let L, M,* $\eta^{(0)}$, $\xi^{(0)}$ *be defined by* (3.5), (5.1), (3.27), (3.28). *Then* M^{-1} *exists and is bounded for sufficiently small values of* $|\beta| < \beta_0$ *and the equations* $L\xi = \eta^{(0)}$ *have exactly one solution* ξ *which satisfies* (4.5) *and* (4.6), *namely* $\xi = M^{-1}\xi^{(0)}$.

Proof: Let $V^{(r)}$ be matrices such that

$$\left\{\mathcal{I} + \sum_{r=2}^{\infty} \beta^r S^{(r)}\right\}\left\{\mathcal{I} + \sum_{r=0}^{\infty} \beta^r V^{(r)}\right\} = \mathcal{I}. \tag{5.5}$$

It is easily seen that the $V^{(r)}$ can be obtained from the $S^{(r)}$ by recurrence formulas. Let $U^{(r)}$ be upper bounds for the $S^{(r)}$ and assume that there exist constants Ω_r such that

$$\left(1 - \sum_{r=2}^{\infty} \beta^r U_r\right)\left(1 + \sum_{r=2}^{\infty} \beta^r \Omega_r\right) = 1. \tag{5.6}$$

This is true if

$$1 - \sum_{r=2}^{\infty} \beta^r U_r \tag{5.7}$$

is convergent and positive for $0 \leqq \beta < \beta_0$. Then it can be shown that Ω_r is an upper bound for $V^{(r)}$. Since it can also be shown that x_m (the m-th component of $\xi = M^{-1}\xi^{(0)}$) is equal to the m-th component of

$$\left\{\sum_{r=m}^{\infty} \beta^r V^{(r)}\right\}\xi_0 \tag{5.8}$$

it follows that

$$|x_m| \leqq \sum_{r=m}^{\infty} \beta^r \Omega_r . \tag{5.9}$$

From this it can easily be shown that for $|\beta| < \beta_0$ condition (4.5) for the x_m is satisfied. This proves the existence of M^{-1} and of a bounded ξ satisfying (4.5), (condition (4.6) is always satisfied for bounded ξ) if we can find U_r which are sufficiently small. We have

Lemma 4. *The matrices*

$$\{S^{(2)}\}^t, \qquad R^{(t)}, \qquad S^{(2t+2)}, \qquad S^{(2q+3)} \tag{5.10}$$

have as upper bounds

$$\pi(\pi^2 - 8)^{1/2}/4, \qquad 2^{1/2}(\pi^2 - 8)^{1/2}/t!, \qquad (2\pi^2 - 16)^{1/2}2^{t+2}(1/2)_t/(t+1)!, \qquad 2^{q+1}(2\pi^2 - 16)^{1/2}/(q+1)! \tag{5.11}$$

The proof is elementary but laborious and will be omitted since the upper bounds are not the best possible ones.

In order to prove the uniqueness of the solution $\xi = M^{-1}\xi^{(0)}$ we observe first that $(M - \mathfrak{I})\xi$ is bounded for every ξ merely satisfying (4.5); provided that β is so small that (5.4), with the U_r from Lemma 4, converges. This can be proved by an elementary investigation of the $S^{(r)}$. Now if there is a ξ^* satisfying (4.5) and (4.6) such that $L\xi^* = 0$, we would have $M\xi^* = \xi^* + \zeta$ where ζ is bounded and $L^{(0)}\xi^* + L^{(0)}\zeta = 0$. Now it follows from the equivalence of the operator $L^{(0)}$ to the operator of a moment problem (cf. Theorem 2) that $\xi^* + \zeta = 0$. Therefore ξ^* is bounded, and since M^{-1} is bounded, ξ^* must be zero since $M\xi^* = \xi^* + \zeta = 0$.

No numerical values for the permissible ranges of β are given since it is entirely possible that the inverse M^{-1} exists for all values of β. This seems to be indicated by a result of Sommerfeld and Perron [5] who showed that for the related problem of the freely vibrating disc the real part of a resulting set of linear equations can be solved explicitly and without restrictions.

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