NOTE ON THE ELASTIC DISTORTION OF A CYLINDRICAL HOLE BY TANGENTIAL TRACTIONS ON THE INNER BOUNDARY.*

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Introduction. Elastic distortion of a cylindrical hole by localized hydrostatic pressure has been discussed by H. M. Westergaard[1] and C. J. Tranter.[2]

In this paper a few problems of elastic distortion of a cylindrical hole by tangential traction on its inner boundary are discussed. The first case considered is concerned with the problem of an infinite elastic plate having finite thickness with a cylindrical hole acted on by localized tangential traction on the inner boundary, the two faces of the plate being free. In the second case one face of the plate is supposed to be fixed and the other free, while the hole is acted on by uniform tangential traction throughout. In the last case a localized tangential traction is supposed to act over a narrow band on the inner boundary of an infinitely long cylindrical hole in an infinite elastic solid.

The solutions in the first two cases are obtained in terms of infinite series while in the last case it is expressed as an infinite integral.

1. Method of solution. We take the axis of the cylindrical hole as the axis of z. Using cylindrical co-ordinates and assuming $u = w = 0$ and v to be independent of θ, we have

$$e = 0, \qquad 2w_r = -\frac{\partial v}{\partial z}$$
$$2w_\theta = 0, \qquad 2w_z = \frac{1}{r}\frac{\partial}{\partial r}(rv) \tag{1.1}$$

and

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0,$$
$$\tau_{\theta z} = G\frac{\partial v}{\partial_z}, \qquad \tau_{r\theta} = G\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = Gr\frac{\partial}{\partial r}\left(\frac{v}{r}\right) \tag{1.2}$$

Two equations of equilibrium are identically satisfied and the third takes the form

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = 0 \tag{1.3}$$

One particular solution suitable for the problem is

$$v = \frac{A_0}{r}. \tag{1.4}$$

Also substituting

$$v = V\cos kz \qquad \text{or} \qquad V\sin kz \tag{1.5}$$

we get

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \left(k^2 + \frac{1}{r^2}\right)V = 0 \tag{1.6}$$

where V is a function of r only.

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[1]H. M. Westergaard, Kármán Anniversary Volume, 1941, p. 154

[2]C. J. Tranter, Quart. of Appl. Math., 4, 298 (1946).

As a solution to the above we take

$$V = c_k K_1(kr) \tag{1.7}$$

where K_1 is the modified Bessel function of the second kind of degree one.

We now consider the following cases.

2. The cylindrical hole in a large plate of finite thickness with a distribution of tangential traction over a narrow band. We take the plate to be of thickness $2L$ and the surfaces defined by $z = \pm L$. Both ends of the plate are free and the tangential traction is localized within the zone $z = \pm h$.

The boundary conditions to be satisfied are

1) $$\tau_{\theta z} = G\frac{\partial v}{\partial z} = 0, \quad \text{when} \quad z = \pm L \tag{2.1}$$

and 2) $$\tau_{r\theta} = -S_1, \quad \text{when} \quad |z| < h$$
$$= 0, \quad \text{when} \quad |z| > h \tag{2.2}$$

Assume

$$v = \frac{A_0}{r} + \sum_{k=1}^{\infty} c_k K_1(kr) \cos kz \tag{2.3}$$

Then from (1.2) we have

$$\tau_{r\theta} = -\frac{2GA_0}{r^2} - \sum_{k=1}^{\infty} c_k k G K_2(kr) \cos kz \tag{2.4}$$

where K_2 is modified Bessel function of the second kind of the degree two.

From the condition (2.1) we get

$$k = \frac{n\pi}{L} \tag{2.5}$$

where n is any integer.

Also the conditions given by (2.2) will be satisfied if

$$A_0 = \frac{a^2}{2G}\cdot\frac{S_1 h}{L}$$

and

$$C_k = \frac{2LS_1 \sin(n\pi h/L)}{n^2\pi^2 G K_2(n\pi a/L)} \tag{2.6}$$

Hence

$$v = \frac{S_1 a^2 h}{2GLr} + \sum_{n=1}^{\infty} \frac{2LS_1 \sin(n\pi h/L) K_1(n\pi r/L) \cos(n\pi z/L)}{n^2\pi^2 G K_2(n\pi a/L)} \tag{2.7}$$

which is evidently convergent as for large values of x

$$\frac{K_1(x)}{K_2(x)} \approx 1.$$

3. The cylindrical hole with uniform tangential traction in a large plate one of whose faces is fixed and the other free. Here we take the plate to be of thickness L. The free surface is given by $z = L$, the face given by $z = 0$, being fixed. The uniform tangential traction is supposed to act throughout the hole.

The boundary conditions to be satisfied are.

$$1) \qquad v = 0, \qquad \text{when} \qquad z = 0 \tag{3.1}$$

$$2) \qquad \tau_{\theta z} = G\frac{\partial v}{\partial z} = 0, \qquad \text{when} \qquad z = L \tag{3.2}$$

$$\text{and } 3) \qquad \tau_{r\theta} = -S, \qquad \text{when} \qquad 0 < z < L \tag{3.3}$$

at the cylindrical surface $r = a$.

Assuming

$$v = \sum_{k=1}^{\infty} D_k K_1(kr) \sin kz, \tag{3.4}$$

we get from (1.2)

$$\tau_{r\theta} = -\sum_{k=1}^{\infty} D_k k G K_2(kr) \sin kz. \tag{3.5}$$

We find that the condition (3.1) is evidently satisfied and condition (3.2) will be satisfied if

$$k = \frac{(2n+1)\pi}{2L} \tag{3.6}$$

where n is any integer.

The condition (3.3) will be satisfied if

$$D_k = \frac{8SL}{(2n+1)^2\pi^2 G K_2((2n+1)\pi a/2L)} \tag{3.7}$$

Hence

$$v = \sum_{n=1}^{\infty} \frac{8SLK_1((2n+1)\pi r/2L) \sin((2n+1)\pi a/2L)}{(2n+1)^2\pi^2 G K_2((2n+1)\pi a/2L)} \tag{3.8}$$

which is evidently convergent.

4. The cylindrical hole in an infinite solid under a tangential traction over a narrow band. We now consider the case of an infinite solid having an infinitely long cylindrical hole acted on by a tangential traction which is operating over a narrow band of breadth $2h$.

The boundary conditions here are

$$\begin{aligned} \tau_{r\theta} &= -S_1 \qquad \text{when} \qquad |z| < h \\ &= 0 \qquad \text{when} \qquad |z| > h \end{aligned} \tag{4.1}$$

at the surface of the cylinder $r = a$.

The conditions (4.1) can be expressed in the form

$$(\tau_{r\theta})_{r=a} = \frac{-2S_1}{\pi} \int_0^\infty \frac{\sin ht \cos zt}{t} \, dt \tag{4.2}$$

We assume

$$v = \int_0^\infty c(t) K_1(tr) \cos zt \, dt \tag{4.3}$$

as a solution of the equation of equilibrium (1.3) where $c(t)$ is a function of t only.

The boundary condition (4.2) will be satisfied if

$$c(t) = \frac{2S_1 \sin ht}{t^2 \pi G K_2(ta)}. \tag{4.4}$$

Therefore

$$v = \int_0^\infty \frac{2S_1 \sin ht \cos zt K_1(tr) \, dt}{t^2 \pi G K_2(ta)}. \tag{4.5}$$

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ON CERTAIN SOLUTIONS OF A PENDULUM-TYPE EQUATION*

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Introduction. In the study of the oscillations of a synchronous motor around its average angular velocity, a differential equation of the following type, the so-called pendulum-type arises [1]:

$$\frac{d^2\theta}{dt^2} + f(\theta) \frac{d\theta}{dt} = g(\theta) \tag{1}$$

where $f(\theta)$ and $g(\theta)$ are functions of period 2π in θ.

It has been shown [2] that in the case where $f(\theta) = \alpha > 0$, a constant, and $g(\theta) = \beta - \sin\theta$, where β is a constant such that $0 < \beta < 1$, there exists a constant $\alpha_c = \alpha_c(\beta) > 0$ such that if $\alpha < \alpha_c$, eq. (1) will have a solution $\theta(t)$ such that if $\theta'(t) = y(\theta)$, then $y(\theta) = y(\theta + 2\pi)$ for all t, while if $\alpha \geq \alpha_c$, no such solutions exist. Following Vlasov [3] and Minorsky [4], we call any such solution of eq. (1) a periodic solution of the second kind. Physically, such a solution corresponds to a subsynchronous level of performance of the motor described by eq. (1). It is known also [5] that questions of stability of solutions of eq. (1) with respect to the points of equilibrium of (1) involve questions of existence of such solutions.

The purpose of this note is to exhibit a set of explicit conditions on $f(\theta)$ and $g(\theta)$ which insure the existence of periodic solutions of the second kind for (1). Since it has already been noted [5] that if $f(\theta) > 0$ and either $g(\theta) < 0$ or $g(\theta) > 0$ for all θ, there will exist such solutions for eq. (1), we restrict ourselves to the case where $f(\theta) > 0$ and the equation $g(\theta) = 0$ has a finite number of roots in $0 \leq \theta < 2\pi$.

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