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## INDUCED MASS WITH VARIABLE DENSITY*

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The concept of induced mass, and some of its properties, are extended to the case of an incompressible fluid of variable density. The proofs parallel closely those given recently by the author, for the case of free boundaries[1].

**1. Minimum principle.** Let the fluid, supposed originally at rest and occupying a region $R$, be given an acceleration $\mathbf{a}$, by the motion of a wall $W$ bounding the fluid internally. By continuity,

$$a_n = f(\mathbf{x}) \text{ on } W, \tag{1}$$

where $f(\mathbf{x})$ is the normal acceleration of $W$.

**THEOREM 1.** The acceleration kinetic energy

$$T = \frac{1}{2} \int_R \rho(\mathbf{a} \cdot \mathbf{a}) \, dR \tag{2}$$

is minimized, relative to all other volume conserving flows satisfying (1).

*Proof.* Let $\mathbf{a} + \mathbf{b}$ be any other volume-conserving initial acceleration satisfying (1). Then Div $\mathbf{b} = 0$, and, by (1), $b_n = 0$ on $W$. Consider next the expansion

$$T = \frac{1}{2} \int_R \rho(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \, dR$$

$$= T_0 + \int_R (\rho \mathbf{a}) \cdot \mathbf{b} \, dR + \frac{1}{2} \int_R \rho(\mathbf{b} \cdot \mathbf{b}) \, dR.$$

Since the last term is positive unless $\mathbf{b}$ vanishes identically, it is sufficient to show that the middle integral is zero. But, since Div $\mathbf{b} = 0$ and (by the equations of motion, neglecting gravity[2]) $\rho \mathbf{a} = -\nabla p$,

$$\text{Div}\,(p\mathbf{b}) = p \, \text{Div}\, \mathbf{b} + (\nabla p) \cdot \mathbf{b} = -\rho \mathbf{a} \cdot \mathbf{b},$$

where $p$ is the scalar pressure. Hence, by the Divergence Theorem, letting $b_n$ denote the outward normal component of $\mathbf{b}$,

$$\int_R (\rho \mathbf{a}) \cdot \mathbf{b} \, dR = \int_W p b_n \, dS = 0,$$

since $b_n = 0$ on $W$. (The same conclusion will hold if the boundary is partly "free", since then we can take $p = 0$.) This completes the proof.

The case of a free boundary (say, of an internal cavity) should not, of course, be confused with the boundary of an incompressible region of zero density.

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[1] "Induced mass with free boundaries," this *Quarterly* **10**, 81–86 (1952).

[2] Gravity can be neglected, for very rapid ("impulsive") accelerations; viscosity is without effect in the case of initial acceleration from rest.

**2. Induced mass tensor and momentum.** Letting $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^3$ and $\mathbf{a}^4$, $\mathbf{a}^5$, $\mathbf{a}^6$ be the acceleration fields associated with *unit* infinitesimal translations and rotations, one can define, as previously[1], the symmetric $6 \times 6$ induced mass tensor (matrix) $\| T_{hk} \|$ by

$$T_{hk} = \int_R \rho \mathbf{a}^h \cdot \mathbf{a}^k \, dR = T_{kh} . \tag{3}$$

Since the diagonal components $T_{hh}$ satisfy Theorem 1, we obtain immediately

**COROLLARY 1.** The diagonal components of induced mass are increased, if the fluid density is increased in any region (and left unchanged elsewhere).

Also, much as before, one can prove

**COROLLARY 2.** Let $\| T_{hk} \|$ and $\| T'_{hk} \|$ be associated with solids, of which the second is obtained from the first by replacing a mass $\Delta m$ of fluid by solid. Then the (scalar) components of translation induced mass $T_{hh}$ satisfy the inequality

$$T_{hh} \geqq T'_{hh} - \Delta m. \tag{4}$$

Again, letting $p_h$ denote the initial pressure required to produce the initial acceleration $\mathbf{a}^h$, we have

$$-T_{hk} = \int_R (\nabla p_h) \cdot \mathbf{a}^k \, dR = \int_R \text{Div} \, (p_h \mathbf{a}^k) \, dR,$$

since $\rho \mathbf{a}^h = \nabla p_h$ by the equations of motion, and

$$\text{Div} \, (p_h \mathbf{a}^k) = p_h \cdot \text{Div} \, \mathbf{a}^k + \nabla p_h \cdot \mathbf{a}^k = \nabla p_h \cdot \mathbf{a}^k,$$

since the flow is incompressible (Div $\mathbf{a}^k = 0$).

By the Divergence Theorem,

$$-T_{hk} = \int_W p_h a_n^k \, dS, \tag{5}$$

where $a_n^k$ is the normal component of $\mathbf{a}^k$, and hence (by continuity) the $k^{th}$ direction cosine (for translation), and a similar moment producing factor for rotation if $k = 1$, 2, 3. Thus, in this case, $T_{hk}$ is the total *thrust* in the $k$-direction, produced by the initial $h$-acceleration. Similarly, if $k = 4, 5, 6$, $T_{nk}$ is the *moment* about the appropriate axis, produced by the $h$-acceleration. We conclude

**THEOREM 2.** The tensor component $T_{hk}$ represents the total $h$-component of initial *pressure force* required to produce a unit $k$-acceleration of the missile.

The arguments given before[1] apply without change to prove also, if gravity is negligible, and if an "$h$-curve" is defined as before[1] to be a cylinder of finite cross-section bounded by parallels to the $h$-axis, in case $h = 1, 2, 3$, and as a circle perpendicular to, and centered on, the $(h - 3) =$ axis if $h = 4, 5, 6$,

**THEOREM 3.** Let a rigid solid be given a unit $k$-acceleration from rest, in an incompressible liquid of variable density. Then the initial rate of increase in the $h$-component of liquid momentum is $T_{hk}$ , in any region $C$ bounded by $h$-curves which contains the solid.

It is worth noting that, if the density $\rho(\lambda)$ is constant on each sphere of a concentric family, or on each ellipsoid of a confocal family, then the acceleration fields can be expressed as solutions of an ordinary differential equation. The details will be published elsewhere.