tribution given by (14), but the results are not yet satisfactory. This means probably that the form of $\phi(x, y)$, chosen above, does not describe the physical situation sufficiently closely.

The author is indebted to Professor J. S. Marshall, of McGill University, for suggesting the above problem, and to Dr. Walter Hitschfeld, of McGill University, for helpful suggestions concerning the physical model used.

REFERENCES

J. O. Laws and D. A. Parsons, *The relation of raindrop-size to intensity*, Trans. Amer. Geophys. Union, **24**, part II, 452 (1943).

J. S. Marshall and W. McK. Palmer, *The distribution of raindrops with size*, J. Meteor, **5**, 4, 165 (1948).

T. E. W. Schumann, *Theoretical aspects of the size distribution of fog particles*, Q. J. Roy. Met. Soc., **66** (1940).

D. Sinclair, *Handbook on Aerosols*. Chapter 5, Atomic Energy Commission, Washington (1950).

Whytlaw-Grey, *Smoke*. E. Arnold and Co., London (1932).

VARIATION OF COEFFICIENTS OF SIMULTANEOUS LINEAR EQUATIONS[1]

By JOHN E. BROCK (*Midwest Piping Supply Co., Inc.*)

1. Introduction. In dealing with sets of simultaneous linear equations it sometimes becomes necessary to modify one or more of the coefficients (to correct errors or for other reasons) after the solution has been obtained. Methods of obtaining the corresponding modifications in the solution with a minimum amount of computation have been treated by B. L. Weiner[2], and in a discussion of Weiner's paper (which bears the same title as the present paper), I. F. Morrison[3] indicates a matrix formulation of the analysis. In the present paper, we develop three particular prodedures each of which appears to be quite simple and quite useful.

Matrices are used throughout in these developments. Both matrices and scalars will be denoted by ordinary italic letters. Letters with subscripts will denote matrix elements (scalars) and other scalars will be easily distinguished from matrices. A symbol denoting a matrix or matrix product when enclosed within parentheses and followed by subscripts will denote the appropriate element of the matrix, thus $(ab)_{ij} = c_{ij}$ where $ab = c$. Superscripts will be used both to denote exponents and to serve as distinguishing indices; there should be no difficulty in distinguishing between these usages.

It is presumed that we know the set of inverse coefficients for the system, that is, the matrix which is inverse to (reciprocal to) the original coefficient matrix. This is not a severe demand since the reciprocal matrix is frequently already known or can be computed without much trouble making use of calculations already performed in obtaining the original solution.

So that the operations will not be obscured by voluminous calculations, the order, n, of the systems in the examples chosen for illustrative purposes is small. However, it is obvious that the advantage afforded by the procedures described here increases as n increases.

[1]Received June 15, 1952.

[2]B. L. Weiner, *Variation of coefficients of simultaneous linear equations*, Trans. ASCE, **113**, 1349 (1948) (Paper No. 2358).

[3]I. F. Morrison, ibid., p. 1379.

2. Symmetrical Systems. Suppose a is a symmetrical, non-singular matrix and that we know $\alpha = a^{-1}$, its inverse, and the solution, x, a column matrix, to the equation

$$ax = c. \tag{1}$$

Now let a be changed to

$$a^* = a + b, \tag{2}$$

where all elements of b are zero except only

$$b_{pq} = b_{qp} = \beta; \tag{3}$$

that is, the equal and symmetrically disposed elements a_{pq} and a_{qp} are increased by the amount β. Our problem is to find a new solution

$$x^* = x + y \tag{4}$$

such that

$$a^*x^* = c. \tag{5}$$

In the derivation,[4] let the matrices b^{rs} be defined by

$$(b^{rs})_{ij} = \delta^r_i \delta^s_j \tag{6}$$

where the (Kronecker) deltas are zero unless the upper and lower indices are the same in which case their value is unity. In establishing the relation (14) below, we use the "summation convention" which implies a summation from 1 to n inclusive over all repeated indices except p and q which are fixed. Then

$$(\alpha b^{rs})_{ij} = \alpha_{ir}\delta^s_j , \tag{7}$$

$$(b^{mn}\alpha b^{rs})_{ij} = \alpha_{nr}\delta^m_i\delta^s_j , \tag{8}$$

$$b = \beta(b^{pq} + b^{qp}), \tag{9}$$

$$(\alpha b)_{ij} = \beta(\alpha_{ip}\delta^q_j + \alpha_{iq}\delta^p_j), \tag{10}$$

$$(b\alpha b)_{ij} = \beta^2(\alpha_{qp}\delta^p_i\delta^q_j + \alpha_{qq}\delta^p_i\delta^p_j + \alpha_{pp}\delta^q_i\delta^q_j + \alpha_{pq}\delta^q_i\delta^p_j), \tag{11}$$

$$\begin{aligned}(\alpha b\alpha b)_{ij} &= \beta^2(\alpha_{ip}\alpha_{qp}\delta^q_j + \alpha_{ip}\alpha_{qq}\delta^p_j + \alpha_{iq}\alpha_{pp}\delta^q_j + \alpha_{iq}\alpha_{pq}\delta^p_j)\\ &= \beta\alpha_{pq}(\alpha b)_{ij} + \beta^2(\alpha_{ip}\alpha_{qq}\delta^p_j + \alpha_{iq}\alpha_{pp}\delta^q_j),\end{aligned} \tag{12}$$

$$\begin{aligned}(b\alpha b\alpha b)_{ij} &= \beta\alpha_{pq}(b\alpha b)_{ij} + \beta^3(\alpha_{qp}\alpha_{qq}\delta^p_i\delta^p_j + \alpha_{pp}\alpha_{qq}\delta^p_i\delta^q_j\\ &\quad + \alpha_{pp}\alpha_{qq}\delta^q_i\delta^p_j + \alpha_{pq}\alpha_{pp}\delta^q_i\delta^q_j)\\ &= 2\beta\alpha_{pq}(b\alpha b)_{ij} + \beta^2(\alpha_{pp}\alpha_{qq} - \alpha^2_{pq})b_{ij} .\end{aligned} \tag{13}$$

[4]The reviewer has suggested adding at this point a remark concerning motivation. The general case, Section 4, came first to the writer's attention. Success in summing the series (37) in particular cases leads to the analyses presented in Sections 2 and 3. Essentially one attempts to determine powers of the matrix $m = \alpha b$, and in Section 2 is led to Equation (14) and in Section 3 to Equation (31); having these results, the summation is easily accomplished. However, in presenting the results in these two Sections (2 and 3), it is not necessary to exhibit the series. Once Equations (14) and (31) are established, the desired results follow simply by a little algebraic manipulation. Use of index notation, the Kronecker deltas, and the summation convention abbreviates the derivation of Equations (14) and (31).

Thus we have established the relation

$$bmm = Abm + Bb, \tag{14}$$

where

$$m = \alpha b = a^{-1}b, \tag{15}$$

$$A = 2\beta\alpha_{pq}\ , \tag{16}$$

$$B = \beta^2(\alpha_{pp}\alpha_{qq} - \alpha_{pq}^2). \tag{17}$$

We now prove the important formulas

$$y = \frac{m(mx) - (1 + A)(mx)}{1 + A - B}, \tag{18}$$

and

$$x^* = x + y = x + \frac{m(mx) - (1 + A)(mx)}{1 + A - B}. \tag{19}$$

To do this, form the product a^*x^* and in simplifying note that

$$am = aa^{-1}b = b. \tag{20}$$

We have

$$a^*x^* = (a + b)(x + y) = ax + bx + (a + b)y$$

$$= c + \left\{b + \frac{a + b}{1 + A - B}[mm - (1 + A)m]\right\}x$$

$$= c + [(1 + A)b - Bb + bm + bmm - (1 + A)b - bm - Abm]\frac{x}{1 + A - B}$$

$$= c + (bmm - Abm - Bb)\frac{x}{1 + A - B} = c, \tag{21}$$

making use of Equation (14). Q. E. D. (Of course we require that $B \neq 1 + A$.)

We may also easily obtain the corrected inverse $\alpha^* = (a^*)^{-1}$. Write Equation (19) in the form

$$x^* = \gamma x, \tag{22}$$

where

$$\gamma = I + \frac{mm - (1 + A)m}{1 + A - B}. \tag{23}$$

If we postmultiply this matrix in succession by the columns of α we obtain the corrected columns, i.e., the columns of α^*. In other words

$$\alpha^* = \gamma\alpha$$

Example:

$$a = \begin{bmatrix} 19 & 10 & 4 \\ 10 & 5 & 2 \\ 4 & 2 & 1 \end{bmatrix}; \quad \alpha = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -3 & -2 \\ 0 & -2 & 5 \end{bmatrix}; \quad x = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}; \quad c = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}. \tag{24}$$

Now suppose

$$a^* = \begin{bmatrix} 19 & 10 & 4 \\ 10 & 5 & 0 \\ 4 & 0 & 1 \end{bmatrix}. \quad \text{Then} \quad b = (-2)\cdot\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and}$$

we calculate: $\beta = -2$, $p = 2$, $q = 3$, $A = (2)(-2)(-2) = 8$,

$$B = (-2)^2[(-3)(5) - (-2)^2] = -76; \qquad 1 + A - B = 85, \qquad 1 + A = 9,$$

$$m = \begin{bmatrix} 0 & 0 & -4 \\ 0 & 4 & 6 \\ 0 & -10 & 4 \end{bmatrix}; \qquad mx = \begin{bmatrix} 4 \\ -18 \\ 26 \end{bmatrix};$$

$$y = \frac{[m - (1 + A)I](mx)}{1 + A - B} = \begin{bmatrix} -9 & 0 & -4 \\ 0 & -5 & 6 \\ 0 & -10 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ -18 \\ 26 \end{bmatrix} \div 85 = \begin{bmatrix} -140 \\ 246 \\ 50 \end{bmatrix} \div 85;$$

$$x^* = x + y = \begin{bmatrix} 30 \\ -9 \\ -35 \end{bmatrix} \div 85.$$

This is the easiest arrangement of the calculations. If the new inverse matrix is required, as would be the case, for example, if it were necessary to apply a second set of corrections, we would compute:

$$\gamma = \begin{bmatrix} 85 & 40 & 20 \\ 0 & 5 & -6 \\ 0 & 10 & 5 \end{bmatrix} \div 85;$$

$$\alpha^* = \gamma\alpha = \begin{bmatrix} -5 & 10 & 20 \\ 10 & -3 & -40 \\ 20 & -40 & 5 \end{bmatrix} \div 85; \qquad x^* = \gamma x = \begin{bmatrix} 30 \\ -9 \\ -35 \end{bmatrix} \div 85.$$

3. Single Corrections in Non-Symmetrical Systems. Suppose again that α, x, and C satisfy Equation (1), that a is modified as in Equation (2), and that we desire to find x^* given in Equation (4) so as to satisfy Equation (5). However, now we do not require that a be symmetrical and we presume b has only one non-vanishing element

$$\beta = b_{pq} .$$

Thus, using a notation introduced earlier, we can write

$$b = \beta b^{pq}, \tag{26}$$

$$m_{ij} = (\alpha b)_{ij} = \beta \alpha_{ip} \delta_j^q , \tag{27}$$

$$m_{qq} = \beta \alpha_{qp} , \tag{28}$$

$$m_{qq} b_{ij} = \beta^2 \alpha_{qp} \delta_i^p \delta_j^q , \tag{29}$$

$$(bm)_{ij} = \beta^2 \alpha_{qp} \delta_i^p \delta_j^q . \tag{30}$$

Thus, we have proved that

$$bm_{qq} = bm. \tag{31}$$

Now we argue that

$$y = \frac{-mx}{1 + m_{qq}}, \tag{32}$$

$$x^* = \gamma x, \tag{33}$$

where

$$\gamma = I - \frac{m}{1 + m_{qq}}, \tag{34}$$

for

$$a^* x^* = ax + bx - \frac{(a + b)m}{1 + m_{qq}} \cdot x$$

$$= c + [b + bm_{qq} - am - bm] \cdot \frac{x}{1 + m_{qq}} = c, \tag{35}$$

making use of Equations (20) and (31). Also, as before

$$\alpha^* = \gamma \alpha. \tag{36}$$

Example:

$$a = \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & -1 & 3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & -2 & 5 & 1 \end{bmatrix}; \quad \alpha = \begin{bmatrix} -5 & 15 & 19 & -8 \\ 9 & 17 & 1 & -12 \\ 4 & 10 & -2 & 2 \\ 3 & -31 & -7 & 18 \end{bmatrix} \div 44; \quad x = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}; \quad c = \begin{bmatrix} 2 \\ 6 \\ 4 \\ 14 \end{bmatrix}.$$

Suppose now that $a_{31} = 3$ is changed to 5. Then $p = 3$, $q = 1$, $\beta = 2$,

$$m = \alpha b = \begin{bmatrix} 19 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -7 & 0 & 0 & 0 \end{bmatrix} \div 22; \qquad m_{qq} = \frac{19}{22}$$

$$\gamma = I - \frac{44m}{63} = \begin{bmatrix} 22 & 0 & 0 & 0 \\ -1 & 41 & 0 & 0 \\ 2 & 0 & 41 & 0 \\ 7 & 0 & 0 & 41 \end{bmatrix} \div 41$$

$$x^* = \gamma x = \begin{bmatrix} 22 \\ -42 \\ 84 \\ 48 \end{bmatrix} \div 41; \quad \alpha^* = \gamma\alpha = \begin{bmatrix} -5 & 15 & 19 & -8 \\ 17 & 31 & 1 & -22 \\ 7 & 20 & -2 & 3 \\ 4 & -53 & -7 & 31 \end{bmatrix} \div 82.$$

4. General Case. Both the preceding procedures which give exact corrections in convenient finite form were originally motivated by the analysis which follows and evolved from it through success in actually summing the matrix series which appears below.

Suppose again that a, x, and c satisfy Equation (1), that a is modified as in Equation (2), and that we desire to find x^* given in Equation (4) so that Equation (5) is satisfied. We now make no restriction on b other than that the infinite matrix series

$$\gamma = I - m + m^2 - m^3 + \cdots \tag{37}$$

converge, and indeed in such a manner that its terms may be rearranged.

Then, as before, Equations (22) and (24) hold, for

$$a^*x^* = (a + b)\gamma x = c + (-a + a\gamma + b\gamma)x = c, \tag{38}$$

since the quantity in parentheses may be written as

$$\begin{aligned}(-a + a\gamma + b\gamma) &= -a + a - am + am^2 - am^3 + am^4 - \cdots \\ &\qquad + b - bm + bm^2 - bm^3 + \cdots \\ &= (b - am)(I - m + m^2 - m^3 + \cdots) = (b - am)\gamma, \end{aligned} \tag{39}$$

by reference to Equation (20).

In practice, to investigate the convergence and then to sum a sufficient number of terms of the series to assure the desired accuracy may prove prohibitively laborious. Therefore, it is suggested that the procedure implied in Equations (37), (22), and (24) be used only if it is clearly obvious that the corrections b are so small relative to a that not only is convergence assured but also adequate accuracy may be obtained from the first few terms of the series.

Example:

$$a = \begin{bmatrix} 36 & 16 & 4 \\ 15 & 9 & 3 \\ 6 & 4 & 2 \end{bmatrix}; \quad \alpha = \begin{bmatrix} 3 & -8 & 6 \\ -6 & 24 & -24 \\ 3 & -24 & 42 \end{bmatrix} \div 24; \quad x = \begin{bmatrix} 17 \\ -50 \\ 57 \end{bmatrix} \div 8; \quad c = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix};$$

$$a^* = \begin{bmatrix} 36.02 & 16 & 4 \\ 15 & 9 & 2.99 \\ 6 & 4 & 2 \end{bmatrix}; \quad b = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \div 100;$$

$$m = \alpha b = \begin{bmatrix} 3 & 0 & 4 \\ -6 & 0 & -12 \\ 3 & 0 & -12 \end{bmatrix} \div 1200 = \begin{bmatrix} .002500 & 0 & .003333 \\ -.005000 & 0 & -.010000 \\ .002500 & 0 & -.010000 \end{bmatrix};$$

$$m^2 = \begin{bmatrix} 21 & 0 & 60 \\ -54 & 0 & -168 \\ 45 & 0 & 156 \end{bmatrix} \div (1200)^2 = \begin{bmatrix} .000015 & 0 & .000042 \\ -.000037 & & -.000117 \\ .000031 & 0 & .000108 \end{bmatrix};$$

$$\gamma \approx I - m + m^2 = \begin{bmatrix} .997515 & 0 & -.003291 \\ .004963 & 1 & .009883 \\ -.002469 & 0 & .990108 \end{bmatrix};$$

$$x^* = \gamma x \approx \begin{bmatrix} 2.096271 \\ -6.169037 \\ 7.049273 \end{bmatrix}.$$

The new inverse matrix α^* may be calculated from Equation (24). A complete new solution gives the values

$$x^* = \begin{bmatrix} 2.096277 \\ -6.169052 \\ 7.049277 \end{bmatrix},$$

and it is seen that good accuracy is obtained with our method using only the first three terms of the series given in Equation (37).

ON A SOLUTION OF THE ENERGY EQUATION FOR A ROTATING PLATE STARTED IMPULSIVELY FROM REST*

By RONALD F. PROBSTEIN (*Princeton University*)

1. Introduction. The problem of the steady, laminar incompressible flow of a fluid over an infinite plate rotating at a constant velocity was first solved by von Kármán [1]. Recently the associated heat transfer problem for the von Kármán example was

*Received Nov. 3, 1952. This work has been supported by the United States Air Force, Air Research and Development Command, under Contract AF 33(038)-250.