SOME REMARKS ON A CLASS OF EIGENVALUE PROBLEMS WITH SPECIAL BOUNDARY CONDITIONS*

BY
G. W. MORGAN
Brown University

1. Introduction. The study of a variety of physical problems frequently leads one to an eigenvalue problem. One very common eigenvalue problem, the Sturm-Liouville problem, consists of finding non-identically vanishing solutions of the differential equation

$$L[y(x)] + \lambda wy \equiv (py')' - qy + \lambda wy = 0, \qquad x_0 \le x \le x_1 \tag{1}$$

subject to the homogeneous boundary conditions

$$\begin{aligned} a_0 y'(x_0) - b_0 y(x_0) &= 0, \\ a_1 y'(x_1) + b_1 y(x_1) &= 0. \end{aligned} \tag{2}$$

Here p, q, w are functions of x which are positive in the domain and possess the required number of derivatives, λ is an unknown constant parameter, and a_0, b_0 etc. are constants, such that $a_0 b_0$, $a_1 b_1 \ge 0$.

In this paper we consider a problem which is identical with the Sturm-Liouville problem in all respects except for the appearance of the parameter λ in the boundary conditions. This introduces certain difficulties. Our primary object is to discuss a transformation which eliminates λ from the boundary conditions and which thus enables us to carry over several important results of the Sturm-Liouville theory.

The method is then extended to higher order differential equations and finally physical examples are presented which provide an interpretation of the mathematical technique.

2. Comparison with the Sturm-Liouville Problem. The Sturm-Liouville problem is known to have the following properties.

Non-trivial solutions (eigenfunctions) exist only for certain positive values of λ (the eigenvalues). The eigenvalues λ_1 , $\lambda_2 \cdots \lambda_i \cdots$, in order of magnitude, form a countably infinite set and the corresponding eigenfunctions y_1 , $y_2 \cdots y_i \cdots$ constitute a complete, orthogonal system. Since the functions y_i are determined up to an arbitrary constant multiplier only, they can be normalized. The orthogonality and normalization conditions are then expressed by

$$\int_{x_0}^{x_1} wy_i y_j \, dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}. \tag{3}$$

By completeness we mean that any function $f(x)$ which is piecewise continuous can be approximated as closely as desired in the mean square error sense by a sufficiently large number of terms of the finite series $\sum_{i=1}^{n} c_i y_i$, where the coefficients c_i are given by

$$c_i = \int_{x_0}^{x_1} wfy_i \, dx. \tag{4}$$

*Received April 30, 1952.

The problem we wish to study here consists of the differential equation (1) together with the following boundary conditions:

$$y'(x_0) + K_0\lambda y(x_0) = 0,$$

$$y'(x_1) - K_1\lambda y(x_1) = 0,$$

$$\text{where} \qquad K_0, K_1 \geq 0. \tag{5}$$

We first note that the usual orthogonality relation (3) no longer holds. In fact it can be shown following standard procedures that

$$\int_{x_0}^{x_1} wy_iy_j\,dx = -K_1p(x_1)y_i(x_1)y_j(x_1) - K_0p(x_0)y_i(x_0)y_j(x_0), \qquad i \neq j. \tag{6}$$

For the Sturm-Liouville problem we know that a function f which satisfies certain conditions* can be expanded in a series

$$f = \sum_{i=1}^{\infty} c_iy_i$$

where the expansion coefficients c_i are given by (4). In our problem this is no longer true.

Before discussing the transformation of the problem which will give, among other results, a physical interpretation of (6) and will also readily yield a formula for the expansion coefficients, we show how these coefficients can be derived formally by assuming the validity of the expansion

$$f(x) = \sum_{i=1}^{\infty} \gamma_iy_i\,, \qquad x_0 \leq x \leq x_1\,. \tag{7}$$

Multiply by wy_j and integrate over the domain, reversing the order of summation and integration:

$$\int_{x_0}^{x_1} wfy_j\,dx = \sum_{i=1}^{\infty} \gamma_i \int_{x_0}^{x_1} wy_jy_i\,dx.$$

Using (6) we obtain

$$\int_{x_0}^{x_1} wfy_j\,dx = \gamma_j \int_{x_0}^{x_1} wy_j^2\,dx - \sum_{i=1}^{\infty}{}' \gamma_i[K_1p(x_1)y_i(x_1)y_j(x_1) + K_0p(x_0)y_i(x_0)y_j(x_0)]$$

where $\sum'$ indicates a summation in which the term $i = j$ is omitted. To eliminate the coefficients γ_i for $i \neq j$ we make use of the series (7) evaluated at the end points.

$$f(x_0) = \gamma_jy_j(x_0) + \sum_{i=1}^{\infty}{}' \gamma_iy_i(x_0),$$

$$f(x_1) = \gamma_jy_j(x_1) + \sum_{i=1}^{\infty}{}' \gamma_iy_i(x_1).$$

*See for example Courant Hilbert, Methoden der Math. Physik, Vol. I, Ch. VI, Sect. 3.3.

We now have

$$\int_{x_0}^{x_1} wfy_i\,dx = \gamma_i \int_{x_0}^{x_1} wy_i^2\,dx - K_0p(x_0)y_i(x_0)[f(x_0) - \gamma_i y_i(x_0)]$$
$$- K_1p(x_1)y_i(x_1)[f(x_1) - \gamma_i y_i(x_1)]. \quad (8)$$

If we require the y_i to be normalized so that

$$\int_{x_0}^{x_1} wy_i^2\,dx + K_0p(x_0)y_i^2(x_0) + K_1p(x_1)y_i^2(x_1) = 1, \quad (9)$$

then (8) gives

$$\gamma_i = \int_{x_0}^{x_1} wfy_i\,dx + K_0p(x_0)y_i(x_0)f(x_0) + K_1p(x_1)y_i(x_1)f(x_1). \quad (10)$$

The above is essentially a generalization of the method employed by some authors in the special case when the y_i are the trigonometric functions.*

3. Transformation to Quasi-Sturm-Liouville Form. We now show how the problem can be transformed into, essentially, the standard Sturm-Liouville form. We shall here use heuristic arguments only, although it is possible to put the reasoning on a more rigorous basis by means of suitable limit processes.

We define two functions $\delta(x_1 - \eta)$ and $\delta(\eta - x_0)$ in the following manner:

$$\delta(x_1 - \eta) = 0 \qquad \text{for} \qquad \eta \neq x_1\,,$$

and

$$\int_{x_0}^{x_1} v(\eta)\ \delta(x_1 - \eta)\,d\eta = v(x_1).$$

Similarly:

$$\delta(\eta - x_0) = 0 \qquad \text{for} \qquad \eta \neq x_0\,,$$

and

$$\int_{x_0}^{x_1} v(\eta)\ \delta(\eta - x_0)\,d\eta = v(x_0).$$

It is also convenient to define:

$$h(\xi) \equiv \int_{x_0}^{\xi} y(\eta)\ \delta(x_1 - \eta)\,d\eta = \begin{cases} 0; & x_0 \leq \xi < x_1 \\ y(x_1); & \xi = x_1 \end{cases},$$
$$g(\xi) \equiv \int_{\xi}^{x_1} y(\eta)\ \delta(\eta - x_0)\,d\eta = \begin{cases} 0; & x_0 < \xi \leq x_1 \\ y(x_0); & \xi = x_0 \end{cases}. \quad (11)$$

We now introduce a new dependent variable u by the relation:

$$u(x) = y(x) - \lambda K_1 \int_{x_0}^{x}\int_{x_0}^{\xi} y(\eta)\ \delta(x_1 - \eta)\,d\eta\,d\xi - \lambda K_0 \int_{x}^{x_1}\int_{\xi}^{x_1} y(\eta)\ \delta(\eta - x_0)\,d\eta\,d\xi. \quad (12)$$

*Timoshenko; Vibration Problems in Engineering, p. 319. K. Ludwig; Einwirkungen von Bodenschwingungen auf Pfeiler, Z. a. M. M. 14, 361-363 (1934).

From (11) we see that

$$\int_{x_0}^{x} h(\xi)\, d\xi = 0, \qquad x_0 \leq x \leq x_1 \,,$$

$$\int_{x}^{x_1} g(\xi)\, d\xi = 0, \qquad x_0 \leq x \leq x_1 \,. \tag{13}$$

Hence (12) becomes $u(x) = y(x)$, $x_0 \leq x \leq x_1$.

We now find the differential equation and boundary conditions satisfied by the function $u(x)$.

From equations (12) and (11) we have:

$$u'(x) = y'(x) - \lambda K_1 \int_{x_0}^{x} y(\eta)\, \delta(x_1 - \eta)\, d\eta + \lambda K_0 \int_{x}^{x_1} y(\eta)\, \delta(\eta - x_0)\, d\eta, \tag{14}$$

or

$$u'(x) = y'(x) - \lambda K_1 h(x) + \lambda K_0 g(x),$$

and therefore

$$u'(x) = y'(x), \qquad x_0 < x < x_1 \,,$$

$$u'(x_1) = y'(x_1) - \lambda K_1 y(x_1), \tag{15}$$

$$u'(x_0) = y'(x_0) + \lambda K_0 y(x_0).$$

Comparing with (5), the boundary conditions on y, we see that our boundary conditions on u are

$$u'(x_0) = u'(x_1) = 0. \tag{16}$$

From (14) and (13) we obtain:

$$u''(x) = y''(x) - \lambda K_1\, \delta(x_1 - x) y(x) - \lambda K_0\, \delta(x - x_0) y(x)$$

$$= y''(x) - \lambda K_1\, \delta(x_1 - x) u(x) - \lambda K_0\, \delta(x - x_0) u(x).$$

Substitution into the differential equation (1) yields

$$(pu')' - qu + \lambda[wu + K_0 p\, \delta(x - x_0) u(x) + K_1 p\, \delta(x_1 - x) u(x) - K_0 p' g(x) + K_1 p' h(x)] = 0.$$

Now g and h vanish everywhere except at the end points, and at the end points they are irrelevant in view of the presence of the terms containing $\delta(x - x_0)$ and $\delta(x_1 - x)$, respectively. Hence the differential equation for u can be written:

$$(pu')' - qu + \lambda \rho u = 0 \tag{17}$$

where $\rho = w + K_0 p(x_0)\, \delta(x - x_0) + K_1 p(x_1)\, \delta(x_1 - x)$.

The problem defined by (16) and (17) is in the standard Sturm-Liouville form except for the singularities of the function ρ at the ends. We shall formally regard it as a Sturm-Liouville problem and assume without attempting a rigorous justification that many results of the Sturm-Liouville theory, such as those concerning the nature of the eigenvalues, the completeness of the eigenfunctions, etc. are still applicable.

The orthogonality (3) yields

$$\int_{x_0}^{x_1} \rho u_i u_j \, dx = \int_{x_0}^{x_1} \rho y_i y_j \, dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

Substituting for $\rho(x)$, the integral becomes

$$\int_{x_0}^{x_1} w y_i y_j \, dx + K_0 p(x_0) \int_{x_0}^{x_1} y_i y_j \; \delta(x - x_0) \, dx + K_1 p(x_1) \int_{x_0}^{x_1} y_i y_j \; \delta(x_1 - x) \, dx,$$

and this gives

$$\int_{x_0}^{x_1} w y_i y_j \, dx + K_0 p(x_0) y_i(x_0) y_j(x_0) + K_1 p(x_1) y_i(x_1) y_j(x_1) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}. \tag{18}$$

This is immediately seen to be identical with (6) and the previously introduced normalization (9).

The coefficients γ_i in the approximation

$$f \sim \sum_{i=1}^{n} \gamma_i u_i = \sum_{i=1}^{n} \gamma_i y_i$$

are immediately found by using the appropriate functions in formula (4);

$$\begin{aligned} \gamma_n &= \int_{x_0}^{x_1} \rho f u_n \, dx = \int_{x_0}^{x_1} \rho f y_n \, dx \\ &= \int_{x_0}^{x_1} w f y_n \, dx + K_0 p(x_0) f(x_0) y_n(x_0) + K_1 p(x_1) f(x_1) y_n(x_1), \end{aligned}$$

and this is identical with (10).

Completely analogous results can of course be derived if λ only appears in one of the boundary conditions, the other condition being as in (2). In that case only one of the two double integrals in (12) will have to be used in the definition of the new variable $u(x)$.

4. Extension to Higher Order Self-Adjoint Differential Equations. An analogous method applies to the equation

$$(p_2 y'')'' - (p_1 y')' + pu - \lambda w u = 0, \qquad x_0 \leq x \leq x_1 \,. \tag{91}$$

This is to be solved subject to four homogeneous boundary conditions, one or two of which involve the parameter λ in the following manner:

$$\begin{aligned} y'''(x_0) - K_0 \lambda y(x_0) &= 0, \\ y'''(x_1) + K_1 \lambda y(x_1) &= 0, \\ K_0 \, , K_1 &\geq 0. \end{aligned} \tag{20}$$

As an illustration consider the case in which, in addition to (20), the conditions $y''(x_0) = y''(x_1) = 0$ are to be satisfied.

Define the new variable $u(x)$ by

$$\begin{aligned} u(x) = y(x) &+ \lambda K_1 \int_{x_0}^{x} \int_{x_0}^{r} \int_{x_0}^{s} \int_{x_0}^{\xi} y(\eta) \; \delta(x_1 - \eta) \, d\eta \, d\xi \, ds \, dr \\ &+ \lambda K_0 \int_{x}^{x_1} \int_{r}^{x_1} \int_{s}^{x_1} \int_{\xi}^{x_1} y(\eta) \; \delta(\eta - x_0) \, d\eta \, d\xi \, ds \, dr. \end{aligned} \tag{21}$$

Proceeding as before we obtain

$$\begin{aligned} u(x) &= y(x), \\ u'(x) &= y'(x), \\ u''(x) &= y''(x), \qquad x_0 \leq x \leq x_1 . \end{aligned} \tag{22}$$

Also

$$u'''(x) = y'''(x) + \lambda K_1 h(x) - \lambda K_0 g(x)$$

where g and h are defined as before by (11). Hence:

$$\begin{aligned} u'''(x) &= y'''(x), \qquad x_0 < x < x_1 , \\ u'''(x_1) &= y'''(x_1) + \lambda K_1 y(x_1), \\ u'''(x_0) &= y'''(x_0) - \lambda K_0 y(x_0). \end{aligned} \tag{23}$$

Comparing (22) and (23) with (20) we see that the boundary conditions on u are:

$$u''(x_0) = u'''(x_0) = u''(x_1) = u'''(x_1) = 0. \tag{24}$$

Finally we have:

$$u^{\mathrm{IV}} = y^{\mathrm{IV}} + \lambda K_1 \, \delta(x_1 - x)y(x) + \lambda K_0 \, \delta(x - x_0)y(x).$$

Substituting in the differential equation (19), we obtain

$$(p_2 u'')'' - (p_1 u')' + pu - \lambda \rho u = 0, \tag{25}$$

where $\rho = w + K_0 p_2(x_0) \, \delta(x - x_0) + K_1 p_2(x_1) \, \delta(x_1 - x)$.

Hence we have again succeeded in converting the original problem into a standard form, the differential equation having been changed only by the replacement of the original weight function w by a new weight function ρ identical with w everywhere except for singularities at the boundaries. We are again in a position to apply the results of the standard theory.

The orthogonality relations and the formulas for the coefficients γ_i in the approximation

$$f \sim \sum_{i=1}^{n} \gamma_i u_i = \sum_{i=1}^{n} \gamma_i y_i$$

are found precisely as before.

This technique can be extended in a self-evident manner to self-adjoint differential equations of order $2m$, $m = 1, 2, \cdots$ if one or two of the boundary conditions are of the form

$$y^{(2m-1)}(x_i) + K_i \lambda y(x_i) = 0, \qquad i = 0, 1$$

and the K_i have the proper sign to make $\rho(x)$ positive.

5. Physical Interpretation. We now consider three illustrative examples of physical systems, the mathematical analysis of which leads to the type of eigenvalue problem under discussion.

(a) Longitudinal vibrations of a bar with a mass at the end.

The differential equation is

$$w \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(AE \frac{\partial v}{\partial x}\right) \tag{26}$$

where v is the longitudinal displacement of a cross section of the bar, w is the mass per unit length, A is the cross section, and E is Young's modulus.

If the bar is built in at $x = 0$, the boundary condition there is $v(0,t) = 0$. If a mass M is attached to the end $x = l$, the force in the bar at that end must be equal to the rate of change of momentum of the mass M.

Hence the boundary condition at $x = l$ is:

$$AE \frac{\partial v}{\partial x} = -M \frac{\partial^2 v}{\partial t^2} \qquad \text{at} \qquad x = l, \qquad t > 0. \tag{27}$$

To solve this problem we look for normal modes in the form

$$v(x, t) = \left.\begin{matrix} \sin \omega t \\ \\ \cos \omega t \end{matrix}\right\} \cdot y(x)$$

and hence obtain the following differential equation and boundary conditions for y:

$$(AEy')' + \lambda w y = 0,$$

$$y = 0 \qquad \text{at} \qquad x = 0,$$

$$AEy' - \lambda M y = 0 \qquad \text{at} \qquad x = l$$

$$\text{where} \qquad \lambda = \omega^2. \tag{28}$$

This is precisely of the form (1) and (5) with a standard boundary condition at $x = 0$.

Transformation of this problem to the form (16) and (17) gives:

$$(AEu')' + \lambda(w + M\ \delta(l - x))u = 0,$$

$$u = 0 \qquad \text{at} \qquad x = 0,$$

$$u' = 0 \qquad \text{at} \qquad x = l. \tag{29}$$

The physical meaning of the formulation (29) is that the mass M is treated as a part of the elastic system by making the mass per unit length infinite at $x = l$ in such a manner that the total mass added is M. The new boundary condition at $x = l$ is now applied at the outer end of the mass M and is the condition for a free end.

The new orthogonality condition, given by (18) is:

$$\int_0^l [(w + M\ \delta(l - x)]y_i y_j\, dx = \int_0^l w y_i y_j\, dx + M y_i(l) y_j(l) = 0, \qquad i \neq j. \tag{30}$$

It clearly shows the physical meaning of the transformation. The infinite mass density at $x = l$ changes the weight function to account for the extra mass M.

(b) Lateral vibrations of a beam with end load.*

*F. Gassmann, Über Querschwingungen eines Stabes mit Einzelmasse, Ingenieur—Archiv 2, 222-227, 1931.

The mathematical formulation of this problem is:

$$w \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 v}{\partial x^2}\right) = 0 \tag{31}$$

where v is the lateral displacement of a cross-section and EI is the flexural rigidity.

If the beam carries a mass M at $x = l$, the boundary conditions there are:

$$EI \frac{\partial^3 v}{\partial x^3} = M \frac{\partial^2 y}{\partial t^2},$$

and

$$\frac{\partial^2 v}{\partial x^2} = 0. \tag{32}$$

These conditions assume that the center of gravity of the mass M is very close to $x = l$ so that terms arising from this distance can be neglected.

The search for normal modes leads to:

$$(EIy'')'' - \lambda wy = 0, \tag{33}$$

and the boundary conditions at $x = l$ become:

$$\begin{aligned} EIy''' + \lambda My &= 0, \\ y'' &= 0. \end{aligned} \tag{34}$$

This is of the form discussed in section 4. If we carry out the transformation we again see that its physical interpretation is the inclusion of the mass M into the elastic system with the effect of obtaining a new weight function $w + M\,\delta(l - x)$.

(c) Heat Conduction in a rod with finite energy reservoir at end.

Consider one end of the rod to be in contact with a body in which heat diffusion is so rapid that it is permissible to consider the body to be at a uniform temperature equal to that at the point of contact. We also assume that the only energy flux into, or out of, the body occurs at the contact with the rod.

We have

$$cm \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(KA \frac{\partial v}{\partial x}\right) \tag{35}$$

where v is the temperature, K the termal conductivity, c the specific heat per unit mass, m the mass of the rod per unit length. If, at $x = l$, the rod is in contact with a body of mass M and specific heat γ, then the boundary condition there is:

$$-KA \frac{\partial v}{\partial x} = \gamma M \frac{\partial v}{\partial t}, \qquad x = l. \tag{36}$$

We try solutions:

$$v = e^{-\lambda t} y(x).$$

The equation for y is:

$$(KAy')' + \lambda(cm)y = 0. \tag{37}$$

The boundary condition at $x = l$ is:

$$KAy' - \lambda\gamma My = 0. \tag{38}$$

Comparing with (1) and (5) we see that

$$KA = p, \qquad cm = w, \qquad \text{and} \qquad \gamma M/KA = K_1 . \tag{39}$$

Our transformation yields:

$$(KAu')' + \lambda(cm + \gamma M\ \delta(l - x))u = 0,$$

and

$$u' = 0 \qquad \text{at} \qquad x = l. \tag{40}$$

The physical interpretation is the same as in the previous examples. The body at the end is thought of as a part of the system in which conduction takes place. This is done by making the heat capacity per unit length infinite at $x = l$ in such a manner that a total heat capacity γM is added to the system at that end.