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LINEAR APPROXIMATIONS IN A CLASS OF NON-LINEAR VECTOR DIFFERENTIAL EQUATIONS*

BY

J. J. GILVARRY

The RAND Corporation, Santa Monica, California

1. Introduction. The class of vector differential equations considered in this paper is represented by

$$d^2\mathbf{r}/dt^2 - \mathbf{f}(\mathbf{r}) = \mathbf{b}(t), \quad (1)$$

in which \mathbf{r} is a radius vector in a Cartesian frame, $\mathbf{f}(\mathbf{r})$ is a point function derivable as the gradient of a scalar, and $\mathbf{b}(t)$ is an arbitrary function of the independent variable t . In its physical application, Eq. (1) determines the position of a particle subject to the acceleration $\mathbf{b} + \mathbf{f}$ at time t . To allow the wider interpretation of Eq. (1) as a curve in three-space (for fixed initial conditions), t will not be identified with time (except in the applications of Sec. 3). The basic assumption will be made that $\mathbf{f}(\mathbf{r})$ (which, in general, is a non-linear function of components of \mathbf{r}) is a slowly varying function as compared to arc length on any integral curve of Eq. (1). The scalar corresponding to $\mathbf{f}(\mathbf{r})$ will be assumed differentiable to fourth order at least, and the function $\mathbf{b}(t)$ will be assumed differentiable at the initial point of any integral curve of Eq. (1) to third order at least (but otherwise merely Riemann-integrable).

In a previous paper (referred to hereafter as *I*) by the author and others [1], a special case of Eq. (1) was treated, in which $\mathbf{f}(\mathbf{r})$ and $\mathbf{b}(t)$ were the gravitational and non-gravitational accelerations respectively of a particle. A quasi-linear approximation for two-dimensional motion was described, which takes into account terms in \mathbf{f} of higher order than does a linearized approximation, and which depends on the assumption that the radius of curvature of the corresponding trajectory be slowly varying. In this paper, the quasi-linear approximation in question is generalized in two directions: (a) to three dimensions (by assuming that the radius of torsion of an integral curve is likewise slowly varying); (b) to include an arbitrary $\mathbf{f}(\mathbf{r})$ derivable as the gradient of a non-harmonic scalar. The approximation exploits the fact that coordinate distances (canonical coordinates) along the tangent, principal normal, and binormal of an analytic integral curve are proportional (within dominant terms) to the first, second, and third powers of the arc length, respectively, by virtue of the Frenet-Serret formulas. Hence, non-linear terms in \mathbf{f} through third order in the arc length can be expressed as linear functions of the canonical coordinates and thus of any Cartesian coordinates. In a sense, the approximation considered is the analogue in the non-linear case of Liouville's approxi-

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mation [2, 3] (referred to as the Wentzel-Kramers-Brillouin-Jeffries approximation [4, 5, 6] in quantum mechanics). A prefatory discussion of the linearized approximation to Eq. (1) is given first.

Some remarks on the practical use of approximations of the type considered are in order. The general function of such approximations is to determine corrections to a zero- or first-order solution for a limited range of the independent variable. Approximations of this type cannot compete, of course, with modern numerical methods of solution for an extended range of the independent variable. For moderate ranges of the independent variable, however, linear approximations can be applied piece-wise by identifying initial conditions on one interval with terminal conditions on a preceding interval. Such a continuation solution has been used in *I* for the computation of an actual trajectory.

2. Linearized Approximation. The scalar potential φ , in terms of which $\mathbf{f} = \nabla\varphi$, will be expressed as a function of three coordinates x_i ($i = 1, 2, 3$) forming a Cartesian frame (right-handed in the order given) with origin at the initial point O of an integral curve in question. The Taylor expansion of φ in these coordinates at O will have the form

$$\varphi = \varphi^{(1)} + \varphi^{(2)} + \varphi^{(3)} + \varphi^{(4)} + \cdots, \quad (2)$$

where $\varphi = 0$ at O and each term is a rational integral homogeneous function of the x_i (the superscript notation indicates the degree). Through quadratic terms, φ will be written

$$\varphi = \sum A_i x_i + \sum B_{ij} x_i x_j, \quad (i, j = 1, 2, 3), \quad (3)$$

where $B_{ij} = B_{ji}$.

A matrix notation $\{a\}$ will be used interchangeably with \mathbf{a} for a vector, where $\{a\}$ stands for the column vector $\{a_1, a_2, a_3\}$ of components. The linearized approximation $\mathbf{f}^{(1)}$ to \mathbf{f} consists in retaining only terms in φ through second order. From Eq. (3), it is

$$\{f^{(1)}\} = \mathbf{A} + \mathbf{P}\{r\}, \quad (4)$$

where $\{r\} = \{x_i\}$, $\mathbf{A} = \{A_i\}$, and the coefficient matrix \mathbf{P} is

$$\mathbf{P} = 2\mathbf{B} \quad (5)$$

in terms of the matrix $\mathbf{B} \equiv [B_{ij}]$. With the notation $\mathbf{r}' = dx/dt$, the corresponding linearized approximation to Eq. (1) is

$$\{r''\} - \mathbf{P}\{r\} = \{b\} + \mathbf{A}, \quad (6)$$

which assumes that \mathbf{f} is a slowly-varying function of position with respect to arc length on an integral curve.

Since \mathbf{P} is a symmetric matrix, its eigenvalues are real, and Eq. (6) can be solved directly by reduction to normal coordinates. The solution for the normal coordinate x'_i corresponding to the negative eigenvalue $-\omega_i^2$ of \mathbf{P} is

$$x'_i = v'_i \omega_i^{-1} \sin \omega_i t + \omega_i^{-1} \int_0^t [b'_i(\tau) + A'_i] \sin \omega_i(t - \tau) d\tau, \quad (7)$$

where v'_i is the initial value of dx'_i/dt and $\mathbf{b}' + \mathbf{A}'$ is the transform in the normal coordinates of $\mathbf{b} + \mathbf{A}$. For a positive eigenvalue of \mathbf{P} , the circular functions in Eq. (7) must

be replaced by the corresponding hyperbolic functions. In the classical theory of small oscillations of a particle, eigenvalues leading to such hyperbolic solutions are usually excluded. Such exclusion is not necessary (at least on purely mathematical grounds) in the application to particle motion of the linear approximations discussed here, since the application is to the initial phase of a forced motion. For this reason, the approximations do not necessarily carry implications on the boundedness in the large of the corresponding motion. Note that, from Earnshaw's theorem [7], all the eigenvalues of \mathbf{P} can be of one sign only if φ is non-harmonic.

Ordinarily, the coefficients B_{ij} must be determined by direct expansion of φ . The Appendix describes a method of determining these coefficients which relates them to geometric parameters of the equipotential passing through the origin O . For the purpose of formulating this result, take the x_3 -axis in the direction of $-\nabla\varphi$, and take the x_1 -, x_2 - axes in the directions of the lines of curvature of the equipotential surface $\varphi = 0$ in the tangent plane of this equipotential at O . Let r_1 and r_2 be the principal radii of normal curvature of this surface at O in the directions of x_1 and x_2 respectively¹. Let $r_{1,2}$ be the derivative evaluated at O of the principal radius of normal curvature in the x_1 -direction with respect to arc length on the line of curvature in the x_2 -direction; and let $r_{2,1}$ be defined correspondingly. The matrix \mathbf{B} is then

$$\mathbf{B} = \frac{f_0}{2} \begin{bmatrix} r_1^{-1} & 0 & -r_{2,1}r_2^{-1} \\ 0 & r_2^{-1} & -r_{1,2}r_1^{-1} \\ -r_{2,1}r_2^{-1} & -r_{1,2}r_1^{-1} & -(r_1^{-1} + r_2^{-1} + \sigma_0/f_0) \end{bmatrix}, \quad (8)$$

where σ_0 is a mass density defined by Poisson's equation $\nabla^2\varphi = -\sigma$ at O , and $f_0 = |\nabla\varphi|$ at O . The vector \mathbf{A} is $\{0, 0, -f_0\}$ in the particular frame introduced here. Equation (8) is convenient when the form of the equipotential is a datum given directly (as it is for \mathbf{f} a gravitational acceleration). Use will be made of it in an application in Sec. 3.

3. Quasi-Linear Approximation. For a non-vanishing initial $d\mathbf{r}/dt$, the corresponding integral curve is analytic at the initial point O , since $\nabla\varphi$ and \mathbf{b} are assumed differentiable at O (to third order, at least). Hence, the radius vector \mathbf{r} to a point on the curve can be expanded in the Taylor series [8]

$$\mathbf{r} = \alpha s + \frac{\beta}{2\rho} s^2 - \left\{ \frac{\alpha}{\rho^2} + \frac{\rho'\beta}{\rho^2} + \frac{\gamma}{\rho\tau} \right\} \frac{s^3}{6} + \dots, \quad (9)$$

where s is the arc length on the curve; α , β , γ are the unit tangent, unit principal normal, and unit binormal at O respectively; and ρ , τ , ρ' are the (non-vanishing) radius of curvature, (non-vanishing) radius of torsion, and derivative of ρ with respect to s respectively on the integral curve at O . Equation (9) is equivalent to the standard Frenet-Serret formulas. It is understood that α , β , γ , ρ , τ , ρ' are constants evaluated at O , but specifying subscripts will be omitted for convenience.

The coefficients of the terms of order s^4 and higher in Eq. (9) contain products of powers and derivatives with respect to s of $1/\rho$ and $1/\tau$. If all derivatives of ρ and τ with respect to s are sufficiently small, one can take s sufficiently small so that terms

¹The principal radii r_1 and r_2 are taken positive or negative according as the corresponding centers of curvature lie on the positive or negative half of the x_3 -axis.

of the form s^4/ρ^3 , $s^4/\rho^2\tau$, \dots (which do not involve derivatives) are negligible in Eq. (9). In this case, \mathbf{r} is given by the terms of the series (9) which are indicated explicitly. The permissible range of s for this approximation is given by $s \ll \rho$, τ and is thus smaller as ρ and τ are smaller. Correspondingly, the terms of order s^2 and s^3 in Eq. (9) are then smaller for ρ and τ smaller. Since these terms form the basis of the quasi-linear approximation to be described, the range of s over which this approximation yields a significant correction to the linearized approximation is greater for ρ and τ larger.

On these assumptions, Eq. (9) determines any x_i as a linear function of s , s^2 and s^3 , which is given by

$$\{r\} = \mathbf{\Gamma}'\mathbf{S}\{s^i\}, \quad (10)$$

where $\{s^i\}$ designates the column vector $\{s, s^2, s^3\}$, $\mathbf{\Gamma}'$ is the transpose of the matrix $\mathbf{\Gamma}$ of direction cosines of α , β , γ ,

$$\mathbf{\Gamma} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}, \quad (11)$$

and the matrix \mathbf{S} is

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & -(6\rho^2)^{-1} \\ 0 & (2\rho)^{-1} & -\rho'(6\rho^2)^{-1} \\ 0 & 0 & -(6\rho\tau)^{-1} \end{bmatrix}. \quad (12)$$

Note that the components of the column vector $\mathbf{S}\{s^i\}$ in Eq. (10) are simply canonical coordinates of the curve. Equation (10) can be inverted and written

$$\{s^i\} = \mathbf{S}^{-1}\mathbf{\Gamma}\{r\}, \quad (13)$$

where

$$\mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & -\tau\rho^{-1} \\ 0 & 2\rho & -2\rho'\tau \\ 0 & 0 & -6\rho\tau \end{bmatrix}, \quad (14)$$

provided \mathbf{S} is non-singular, which is true if the determinant

$$|\mathbf{S}| = -(12\rho^2\tau)^{-1} \quad (15)$$

be non-vanishing. Thus the following results apply to an integral curve of Eq. (1) which is curved ($1/\rho \neq 0$) and twisted ($1/\tau \neq 0$) at the origin; the special case of an integral curve which is plane at the origin will be treated later. Under these restrictions, Eq. (13) expresses s , s^2 and s^3 linearly in terms of coordinates x_i of the curve.

If quintic and higher terms in φ are neglected, \mathbf{f} can be written

$$\{f\} = \mathbf{A} + \mathbf{P}\{r\} + \{\nabla\varphi^{(3)}\} + \{\nabla\varphi^{(4)}\}. \quad (16)$$

In general, the cubic and quartic terms of φ , $\varphi^{(3)}$ and $\varphi^{(4)}$, are polynomials having respectively 10 and 15 coefficients (these two sets of coefficients can depend on only 7 and 9 independent constants respectively if φ is harmonic) [9]. These terms will be written

$$\varphi^{(3)} = \sum C_{ijk} x_i x_j x_k, \quad (i, j, k = 1, 2, 3), \tag{17a}$$

$$\varphi^{(4)} = \sum D_{ijkl} x_i x_j x_k x_l, \quad (i, j, k, l = 1, 2, 3), \tag{17b}$$

where the indices run over the ranges given to yield no redundancy of symmetric terms (differing only in subscript order in the coefficients C_{ijk} , D_{ijkl}). Thus, the components of $\Delta \varphi^{(3)}$ and $\Delta \varphi^{(4)}$ are homogeneous polynomials of degree two and three respectively in x_i . But, from Eq. (10), any power x_i^q ($1 \leq q \leq 3$) can be expressed as a polynomial in s of lowest term s^q . If the expressions derived in this way for x_i^q are substituted in the expressions for $\nabla \varphi^{(3)}$ and $\nabla \varphi^{(4)}$ corresponding to Eqs. (17a) and (17b) respectively, and powers of s above the third are dropped, one obtains

$$\{f\} = \mathbf{A} + \mathbf{P}\{r\} + (\mathbf{E} + \mathbf{F})\{s^i\}, \tag{18}$$

where the square matrices \mathbf{E} and \mathbf{F} are derived from the terms $\nabla \varphi^{(3)}$ and $\nabla \varphi^{(4)}$ respectively of \mathbf{f} in Eq. (16). Since $\nabla \varphi^{(3)}$ is quadratic in x_i , it can contribute no term to \mathbf{f} which is linear in s ; hence, $\mathbf{E} \equiv [E_{ij}]$ can be partitioned by columns thus

$$\mathbf{E} = [0, E_{i2}, E_{i3}]. \tag{19}$$

Similarly, $\nabla \varphi^{(4)}$ can contribute no terms to \mathbf{f} which are linear or quadratic in s ; hence, $\mathbf{F} \equiv [F_{ij}]$ can be partitioned

$$\mathbf{F} = [0, 0, F_{i3}]. \tag{20}$$

The non-vanishing elements of \mathbf{E} and \mathbf{F} are given by

$$E_{i2} = 3C_{i11}\alpha_i^2 + 2\alpha_i \sum_p C_{i1p}\alpha_p + \sum_{p,q} C_{i pq}\alpha_p\alpha_q, \tag{21a}$$

$$E_{i3} = \rho^{-1} \left[3C_{i11}\alpha_i\beta_i + \sum_p C_{i1p}(\alpha_i\beta_p + \beta_i\alpha_p) + \frac{1}{2} \sum_{p,q} C_{i pq}(\alpha_p\beta_q + \beta_p\alpha_q) \right], \tag{21b}$$

$$F_{i3} = 4D_{i111}\alpha_i^3 + 3\alpha_i^2 \sum_p D_{i11p}\alpha_p + 2\alpha_i \sum_{p,q} D_{i1pq}\alpha_p\alpha_q + \sum_{p,q,r} D_{i pqr}\alpha_p\alpha_q\alpha_r, \tag{21c}$$

where the indices p, q, r run over the range 1, 2, 3 exclusive of i to yield no redundancy of terms differing only in order of subscripts in the coefficients C_{ijk} , D_{ijkl} . Equation (18) contains the non-linear terms of $\nabla \varphi$ through order s^3 expressed as a linear function of s^2 and s^3 .

Finally, by means of the linear relationship (13) between $\{s^i\}$ and $\{r\}$, Eq. (18) yields the quasi-linear approximation $\mathbf{f}^{(2)}$ to \mathbf{f} ,

$$\{f^{(2)}\} = \mathbf{A} + (\mathbf{P} + \mathbf{Q})\{r\}, \tag{22}$$

where

$$\mathbf{Q} = (\mathbf{E} + \mathbf{F})\mathbf{S}^{-1}\mathbf{\Gamma} \tag{23}$$

is a constant square matrix whose elements depend only on coefficients of φ and the initial conditions of the motion (through the parameters $\alpha, \beta, \gamma, \rho, \tau, \rho'$). The parameters α, β, γ entering \mathbf{Q} through Γ can be evaluated directly from standard formulas [8] as

$$\alpha = \frac{\mathbf{v}_0}{v_0}, \quad \beta = \frac{\mathbf{v}_0 \times (\mathbf{b}_0 + \mathbf{A})}{|\mathbf{v}_0 \times (\mathbf{b}_0 + \mathbf{A})|} \times \alpha, \quad \gamma = \alpha \times \beta, \quad (24)$$

where \mathbf{v}_0 (of magnitude $v_0 \neq 0$) is the initial value of $d\mathbf{r}/dt$, and $\mathbf{b}_0 \neq -\mathbf{A}$ is the initial value of \mathbf{b} . The values of ρ, τ and ρ' in \mathbf{S}^{-1} are given by

$$1/\rho = v_0^{-3} |\mathbf{v}_0 \times (\mathbf{b}_0 + \mathbf{A})|, \quad (25a)$$

$$1/\tau = \rho v_0^{-3} \mathbf{v}_0 \times (\mathbf{b}_0 + \mathbf{A}) \cdot \mathbf{d}_0 \quad (25b)$$

$$\rho' = \rho v_0^{-3} [3\mathbf{v}_0 \cdot (\mathbf{b}_0 + \mathbf{A}) - \rho\beta \cdot \mathbf{d}_0], \quad (25c)$$

where the vector \mathbf{d}_0 is defined by

$$\{d_0\} = \{(b)_0\} + \mathbf{P}\{v_0\}. \quad (26)$$

The corresponding quasi-linear approximation to the differential equation (1) is

$$\{r''\} - (\mathbf{P} + \mathbf{Q})\{r\} = \{b\} + \mathbf{A}. \quad (27)$$

The matrix \mathbf{Q} vanishes and hence Eq. (27) reduces to the linearized equation (6) as $\rho, \tau \rightarrow 0$; thus the magnitude of the quasi-linear correction as well as the range of s over which it is valid is greater for ρ and τ larger.

The quasi-linear equation (27) differs from the linearized equation (6) in an important respect. Unlike \mathbf{P} , the matrix $\mathbf{P} + \mathbf{Q}$ is not symmetric in general, and thus Eq. (27) implies some unilateral coupling of its component equations. Since $\mathbf{P} + \mathbf{Q}$ can have complex eigenvalues, reduction of the equation to normal coordinates is not generally possible. In view of this complication, analytic solution of the system (27) is best carried out by the Laplace transform method or other operational technique.

The quasi-linear approximation (22) for \mathbf{f} is equivalent to the expression of Eq. (18), which clearly contains all terms in \mathbf{f} of order s^3 or lower. Since $s = O(t)$ when $v_0 \neq 0$, it follows that the Taylor expansion of the quasi-linear solution agrees with the true Taylor expansion of \mathbf{r} through terms² of order t^5 , as compared to the linearized solution, which agrees through terms of order t^3 . For v_0 non-vanishing, an integral curve of the quasi-linear equation has contact of fifth order with the actual integral curve at the initial point. The approximation is of osculating type in general, since it amounts to evaluating \mathbf{f} on the osculating sphere of the integral curve at the initial point.

The constraint that an integral curve lie in an equipotential surface requires a condition on \mathbf{b} for a given \mathbf{f} ; the condition [10] is

$$b_n + f_n = (\mathbf{r}')^2/r_i, \quad (28)$$

where $b_n + f_n$ is the component of $\mathbf{b} + \mathbf{f}$ normal to the surface and r_i is the radius of normal curvature of the surface in the direction of the curve. When the potential φ is spherically symmetric, the quasi-linear equation (27) becomes exact for integral curves

²The partial sums through terms of order t^5 of both Taylor expansions exist, because of the differentiability assumptions on $\nabla\varphi$ and \mathbf{b} .

lying entirely in an equipotential. To show this result, select x_3 in the direction of $-\nabla \varphi$ (of magnitude f_0) at O , and let R_0 be the directed distance from O to the center of symmetry. From the symmetry of the field $\nabla \varphi$, one can write directly

$$\{r''\} - f_0 R_0^{-1} \{r\} = \{b\} + \{0, 0, -f_0\} \tag{29}$$

for an integral curve on the equipotential $\varphi = 0$. For such an integral curve which is twisted at O , the partial sums through terms of order t^5 of the Taylor expansions of \mathbf{r} are identical for the solution of Eq. (29) and the solution of the quasi-linear equation (27). This fact is sufficient to show that $f_0 R_0^{-1}$ is a triple eigenvalue of $\mathbf{P} + \mathbf{Q}$, and thus one has

$$\mathbf{P} + \mathbf{Q} = f_0 R_0^{-1} \mathbf{I} \tag{30}$$

in this case, where \mathbf{I} is a unit matrix. A corresponding theorem can be proved for the case where φ is cylindrically symmetric; the theorem for plane symmetry is trivial. In these special cases, the matrix $\mathbf{P} + \mathbf{Q}$ is symmetric and all its eigenvalues are of one sign (even if φ is harmonic).

The preceding treatment is not valid for an integral curve which is plane ($1/\tau = 0$) at the initial point O , because the inversion of Eq. (10) to yield Eq. (13) is not possible. In this case, if ρ be finite and non-vanishing at the initial point, an analogous treatment can be carried out by noting that \mathbf{S} has a non-singular submatrix. In the corresponding development, of which the results will be stated, only terms in φ through the cubic $\varphi^{(3)}$ can be included. Let \mathbf{T}^{-1} and $\mathbf{\Delta}$ be the submatrices

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2\rho \end{bmatrix}, \quad \mathbf{\Delta} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}, \tag{31}$$

of \mathbf{S}^{-1} and $\mathbf{\Gamma}$ respectively. If one writes

$$\mathbf{R} = [0, E_{i2}] \mathbf{T}^{-1} \mathbf{\Delta}, \tag{32}$$

where the partitioned matrix $[0, E_{i2}]$ is of 3×2 order, the corresponding quasi-linear approximation to Eq. (1) is

$$\{r''\} - (\mathbf{P} + \mathbf{R}) \{r\} = \{b\} + \mathbf{A}, \tag{33}$$

which reduces to the linearized equation (6) for $\rho \rightarrow 0$. For v_0 non-vanishing, an integral curve of Eq. (33) has contact of fourth order with the actual integral curve at the initial point, and the corresponding Taylor expansion of \mathbf{r} agrees with the true Taylor expansion through terms of order t^4 .

4. Applications. The preceding results will be applied to the case where \mathbf{f} represents a gravitational acceleration. The value of the coefficient matrix \mathbf{P} of the linearized equation which corresponds to a spheroidal earth rotating with angular velocity Ω will be obtained from Eq. (8) on the basis of Clairaut's first-order theory [11] of the variation of gravity. This matrix for the case of a non-rotating spherical earth is given in *I*. A solution of the quasi-linear equation (33) for plane motion corresponding to this value of \mathbf{P} will be obtained.

The initial point O will be taken on the surface of the earth (Fig. 1), with the x_3 -axis directed along the outward normal to the surface, the x_1 -axis in the direction of a meri-

dian, and the x_2 -axis in the direction of a parallel of latitude (with right-hand sense). The potential φ at points external to the earth can be written

$$\varphi = \varphi_a + \varphi_c, \tag{34}$$

in which φ_a is due to Newtonian attraction, and

$$\varphi_c = \frac{1}{2} (\Omega R \sin \theta)^2 \tag{35}$$

is the potential due to centrifugal reaction at the point (R, θ) , where R is the radial distance from the earth's center and θ is the corresponding polar angle from the earth's axis. The potential φ_a can be represented as due to sources entirely within the earth and thus is harmonic at the earth's surface; hence, the mass density σ (which is fictitious in this case) is fixed by φ_c alone as

$$\sigma = -\Omega^2. \tag{36}$$

On the first-order theory [11], the potential (34) leads to a radius R of the earth for colatitude θ , and a surface value g of the gravitational acceleration given by

$$R = R_{e\theta}(1 - A \cos^2 \theta), \tag{37a}$$

$$g = g_{e\theta}(1 + B \cos^2 \theta), \tag{37b}$$

where $R_{e\theta}$ and $g_{e\theta}$ are equatorial values of R and g respectively, and A and B are parameters which depend on geometric constants of the earth, its mass, and its angular velocity. The lines of curvature of the (approximate) ellipsoid of revolution represented by Eq. (37a) are the meridians and parallels of latitude; the principal radii r_1 and r_2 of normal curvature at O are equal respectively to the (negative) radius of curvature of the meridian at this point and the (negative) distance from this point to the polar axis along the surface normal [8]. Thus, one can write from Eq. (8),

$$\mathbf{P} = -\omega_{e\theta}^2 \begin{bmatrix} 1 + P'_{11} & 0 & P'_{13} \\ 0 & 1 + P'_{22} & 0 \\ P'_{31} & 0 & -2 - P'_{33} \end{bmatrix}, \tag{38}$$

where

$$\omega_{e\theta}^2 = g_{e\theta}/R_{e\theta}, \tag{39}$$

and the assumption $A, B \ll 1$ yields

$$P'_{11} = 2A + (B - 3A) \cos^2 \theta, \tag{40a}$$

$$P'_{22} = (B - A) \cos^2 \theta, \tag{40b}$$

$$P'_{33} = 2\frac{2}{5}A + \frac{2}{5}B + 2(B - 2A) \cos^2 \theta, \tag{40c}$$

$$P'_{13} = P'_{31} = A \sin 2\theta. \tag{40d}$$

In determining Eq. (40c), use has been made of Clairaut's theorem [11],

$$\frac{\Omega^2}{\omega_{e\theta}^2} = \frac{2}{5}(A + B), \tag{41}$$

so that the elements of the coefficient matrix \mathbf{P} do not contain the earth's angular velocity Ω explicitly.

To determine the coefficient matrix \mathbf{R} of the quasi-linear equation for plane motion ($1/\tau = 0$) in a gravitational field, it is sufficiently accurate for the purpose at hand to take $\varphi = g_e R_e^2/R$, which corresponds to a non-rotating spherical earth of mean radius R_e and corresponding mean surface gravity g_e (as defined in *I*). The value of \mathbf{R} is then

$$\mathbf{R} = \omega_{ea}^2 [R'_{ij}], \tag{42}$$

where, for $j = 1, 2, 3$,

$$R'_{1j} = 6\lambda\alpha_1\alpha_3\beta_j\rho/R_{ea}, \tag{43a}$$

$$R'_{2j} = 6\lambda\alpha_2\alpha_3\beta_j\rho/R_{ea}, \tag{43b}$$

$$R'_{3j} = 3\lambda(1 - 3\alpha_3^2)\beta_j\rho/R_{ea}, \tag{43c}$$

in which the factor

$$\lambda = 1 + \frac{2}{3}A + \frac{1}{3}B \tag{44}$$

arises from expressing g_e and R_e in terms of g_{ea} and R_{ea} respectively by means of Eqs. (37) and results in *I*. Equations (42) and (43) generalize to an arbitrary plane of motion the corresponding result³ in *I*.

A solution of the quasi-linear equation (33) corresponding to \mathbf{P} of Eq. (38) and \mathbf{R} of Eq. (42) will be obtained for the special case of motion in a vertical plane at O . Take new axes x'_1, x'_3 at O (Fig. 1), where x'_1 is along the trace of this vertical plane in the x_1 ,

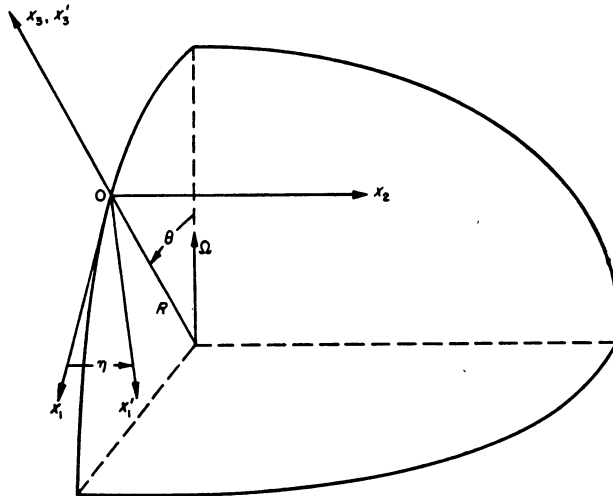


FIG. 1. Coordinate frames on spheroidal earth.

³The coefficient matrix \mathbf{A}_0 of Eq. (45) in *I* is in error as printed; the element in the first row and second column should have a negative sign prefixed. Otherwise, the results correspond with the identification $x_1, x_2, x_3 = x, y, z; \alpha_2 = \beta_2 = 0$ in Eqs. (42) and (43) above.

x_2 plane and $x'_3 = x_3$. Let η be the azimuth (positive sense from x_1 to x_2) of the x'_1 -axis relative to the x_1 -axis. The transform of Eq. (33) in the x'_1, x'_3 frame is

$$\begin{pmatrix} x'_1 \\ x'_3 \end{pmatrix} + \omega_{e_0}^2 \begin{bmatrix} 1 + p_{11} & p_{13} \\ p_{31} & -2 - p_{33} \end{bmatrix} \begin{pmatrix} x'_1 \\ x'_3 \end{pmatrix} = \begin{pmatrix} b'_1 \\ b'_3 - g \end{pmatrix}, \quad (45)$$

where

$$p_{11} = P'_{11} \cos^2 \eta + P'_{22} \sin^2 \eta - R'_{11}, \quad (46a)$$

$$p_{33} = P'_{33} + R'_{33}, \quad (46b)$$

$$p_{13} = P'_{13} \cos \eta - R'_{13}, \quad (46c)$$

$$p_{31} = P'_{31} \cos \eta - R'_{31}, \quad (46d)$$

in which the R'_{ij} are defined by Eqs. (43) in terms of direction cosines α', β' referred to the x'_1, x'_3 frame. Under the approximation $|p_{ij}| \ll 1$, the solution of the quasi-linear system (45) corresponding to the initial velocity $\{v'_1, v'_3\}$ is

$$\begin{aligned} x'_1 = v'_1 \omega_1^{-1} \sin \omega_1 t + \omega_1^{-1} \int_0^t b'_1(\tau) \sin \omega_1(t - \tau) d\tau \\ + \frac{1}{3} p_{13} \left\{ v'_3 s(t) + \int_0^t [b'_3(\tau) - g] s(t - \tau) d\tau \right\}, \end{aligned} \quad (47a)$$

$$\begin{aligned} x'_3 = v'_3 \omega_3^{-1} \sinh \omega_3 t + \omega_3^{-1} \int_0^t [b'_3(\tau) - g] \sinh \omega_3(t - \tau) d\tau \\ + \frac{1}{3} p_{31} \left\{ v'_1 s(t) + \int_0^t b'_1(\tau) s(t - \tau) d\tau \right\}, \end{aligned} \quad (47b)$$

where

$$\omega_1 = \omega_{e_0} \left(1 + \frac{1}{2} p_{11} \right), \quad \omega_3 = 2^{1/2} \omega_{e_0} \left(1 + \frac{1}{4} p_{33} \right), \quad (48)$$

and the function $s(t)$ is defined by

$$s(t) = \omega_1^{-1} \sin \omega_1 t - \omega_3^{-1} \sinh \omega_3 t. \quad (49)$$

The corresponding linearized solution is obtained by letting $\rho \rightarrow 0$ in the parameters p_{ij} . Note that the restriction $|p_{ij}| \ll 1$ imposed to obtain the solution (47) requires $\rho \ll R_{e_0}$ in Eqs. (43) for the elements of the matrix \mathbf{R} .

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Appendix

In this appendix, Eq. (8) of the text for the matrix \mathbf{B} will be derived. Take coordinate axes as specified by the text in connection with this equation. It is known that, in these coordinates, a plane parallel to the tangent plane of a surface and at the signed distance

$p = x_3$ from it cuts the surface in a curve (the Dupin indicatrix) defined within terms of second order by [8]

$$p = x_1^2/2r_1 + x_2^2/2r_2 + \dots \quad (x_3 = p), \tag{1A}$$

where r_1 and r_2 are the principal radii of normal curvature of the surface at O in the directions of x_1 and x_2 , respectively. From Eq. (2), this curve is likewise given by

$$p = f_0^{-1}(B_{11}x_1^2 + 2B_{12}x_1x_2 + B_{22}x_2^2) + \dots, \quad (x_3 = p), \tag{2A}$$

if $f_0 = |\nabla \varphi|$ at O . Comparison of Eqs. (1A) and (2A) yields

$$B_{11} = \frac{1}{2} f_0 r_1^{-1}, \quad B_{12} = 0, \quad B_{22} = \frac{1}{2} f_0 r_2^{-1}. \tag{3A}$$

The coefficient B_{33} can be fixed by the condition that φ , in general, satisfies Poisson's equation $\nabla^2 \varphi = -\sigma$, where σ is a mass density. This condition yields

$$B_{33} = -\frac{1}{2}(f_0 K_m + \sigma_0), \tag{4A}$$

where K_m is the mean curvature $r_1^{-1} + r_2^{-1}$ of the equipotential and σ_0 is the mass density at O .

To evaluate B_{13} , consider a plane normal to the x_1 -axis at the signed distance $p = x_1$ from O . One can show that the equation of the curve of intersection of this plane with the equipotential $\varphi = 0$ is, within second-order terms [8],

$$x_3(1 + r_{2,1}p/r_2) = p^2/2r_1 + x_2^2/2r_2 + \dots, \quad (x_1 = p), \tag{5A}$$

where $r_{2,1}$ is the derivative evaluated at O of the principal radius of normal curvature in the x_2 -direction with respect to arc length on the line of curvature in the x_1 -direction. From Eqs. (2) and (3A), this curve is likewise defined by

$$x_3(1 - 2f_0^{-1}B_{13}p) = p^2/2r_1 + x_2^2/2r_2 + \dots, \quad (x_1 = p), \tag{6A}$$

which yields

$$B_{13} = -\frac{1}{2} f_0 r_{2,1} r_2^{-1} \tag{7A}$$

by comparison with Eq. (5A). Similarly, one has

$$B_{23} = -\frac{1}{2} f_0 r_{1,2} r_1^{-1}, \tag{8A}$$

where $r_{1,2}$ is the derivative evaluated at O of the principal radius of normal curvature in the x_1 -direction with respect to arc length on the line of curvature in the x_2 -direction. These results yield Eq. (8) of the text.

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