THE LOCATION OF THE ROOTS OF POLYNOMIAL EQUATIONS BY THE REPEATED EVALUATION OF LINEAR FORMS*

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1. Introduction. The author was recently engaged in problems connected with the solution of polynomial equations with the aid of an electrolytic tank analog[1]. In connection with this work he evolved a simple and apparently novel computational system of locating the complex roots of polynomial equations. This system is particularly suitable for "punched card" and "digital electronic" computing machines, because it is essentially the evaluation of linear forms, repeated systematically. The present paper describes the principles of the system.

2. The connection between a polynomial and a two-dimensional field. Let the polynomial the zeros of which are to be located[2] be

$$G(Z) = \sum_{q=0}^{q=n} L_q Z^q = \sum_{q=0}^{q=n} (l_q^{(r)} + i l_q^{(i)}) Z^q \tag{1}$$

Let u be a scale factor and Z' an auxiliary variable

$$Z = uZ' \tag{2}$$

As Lucas[3] pointed out, a rational function $H(Z')$ which has $m > n$ first order poles at arbitrarily selected points $Z_s (s = 1, 2, \cdots, m)$ and whose zeros coincide with those of $G(uZ')$ can be derived by dividing $kG(uZ')$ by

$$F(Z') = \prod_{s=1}^{s=m} (Z' - Z'_s), \tag{3}$$

where k is a constant. $H(Z')$ can be expressed as the sum of partial fractions

$$H(Z') = k \frac{G(uZ')}{F(Z')} = \sum_{s=1}^{s=m} \frac{A_s}{Z' - Z'_s}, \tag{4}$$

where

$$A_s = a_s^{(r)} + i a_s^{(i)} = k \frac{G(uZ'_s)}{B_s(Z'_s)} \tag{5}$$

and

$$B_s(Z'_s) = \prod_{q=1 \text{ to } (s-1)}^{q=(s+1) \text{ to } m} (Z'_s - Z'_q). \tag{6}$$

As is the residue of $H(Z')$ at Z'_s. By eliminating in (1) to (4) the expressions Z', $G(Z)$, $F(Z')$, and $H(Z')$, we obtain two power series in Z. The comparison of the first coefficients

*Received Nov. 7, 1952.

[1]L. Tasny-Tschiassny and A. G. Doe, *The solution of polynomial equations with the aid of the electrolytic tank*, Aust. J. Sci. Research **4**, 231-257 (1951).

[2]Capital letters stand for complex numbers, small letters for real ones.

[3]F. Lucas, Résolution des équations par l'électricité, C. R. Ac. Sci. Paris **106**, 645, 1072 (1888).

shows that

$$\left.\begin{aligned} \sum_{s=1}^{s=m} A_s &= 0, &&\text{if} \quad m \geq n+2, \\ \sum_{s=1}^{s=m} A_s &= ku^{n-1}L_n\,, &&\text{if} \quad m = n+1. \end{aligned}\right\} \tag{7}$$

The two-dimensional vector corresponding to the conjugate of $H(Z')$ is proportional to the field strength at Z' of a two-dimensional field produced by sources of the intensity $a_s^{(r)}$ and vortexes of the intensity $a_s^{(i)}$ positioned at the poles Z'_s. If $\sum_{s=1}^{s=m} A_s = 0$, the field is self-contained, if $\sum_{s=1}^{s=m} A_s \neq 0$, a source and a vortex of intensities given by $(-\sum_{s=1}^{s=m} A_s)$ is to be imagined at $Z' = \infty$. Since the zeros of $H(Z')$ and $G(uZ')$ coincide and the zeros of $H(Z')$ are saddle points of the potential, these saddle points determine the roots of the equation $G(uZ') = 0$. This relation has been utilized for electrolytic tank analogs[4,5].

The second statement of (4) can be employed for a purely computational exploration of the field conditions. By a systematic cut and try method one can approach the points Z'_r for which $H(Z'_r) = 0$, with any desired accuracy. In general, this method will be very cumbersome, the main reason being that the accuracy required in the computation of the terms $A_s/(Z' - Z'_s)$ is considerably greater than the accuracy obtained in the result $H(Z')$.

In the special arrangements discussed in the present paper the described computational exploration becomes very simple, because it is essentially the evaluation of linear forms

$$f = \sum a_q b_q \tag{8}$$

in which the set of quantities a_q depends on the special numerical problem in hand, and the set of quantities b_q is taken from tables compiled once and for all. Digital electronic and punched card machines are particularly suitable for this type of work, but it appears that a satisfactory efficiency can be obtained with ordinary commercial multiplying machines. When evaluating linear forms on these machines, no need arises to make a record of intermediate products, because when a certain number has appeared in the result register, a further multiplication adds or subtracts the additional product to this number. If the numbers a_q are recorded on a strip of paper in a way that in a certain position all numbers a_q can be made adjacent to the corresponding numbers b_q of the table, the corresponding multiplicands and multipliers can be directly read off without risk of errors. The tables for b_q can be arranged in a way that the same a_q-strip can be used for different b_q-sets.

3. The computation of the residues A_s for symmetrically arranged poles Z'_s. A convenient number is chosen for the scale factor u in a way that at least some of the points Z'_r for which $G(uZ'_r) = 0$, are within the circle of unit radius with the origin as centre (in the following called the "unit circle"). Let the poles Z'_s be the complex roots of the equation

$$Z'^m = 1,$$

[4]A. R. Boothroyd, E. C. Cherry, and R. Makar, *An electrolytic tank for the measurement of steady-state response, transient response, and allied properties of networks*, Proc. Instn. Elec. Engrs. **96**, 163 (1949).

[5]A. Bloch, *Solution of algebraic equations by means of an electrolytic tank*, VIIth Internat. Congr. Appl. Mech., London, Paper No. IV-28.

i.e.,

$$Z'_s = e^{i(2\pi/m)s}, \qquad (s = 1, 2, \cdots, m). \tag{9}$$

The quantity $B_s(Z'_s)$ [equation (6)] may be expressed as

$$B_s(Z'_s) = \lim_{Z' \to Z'_s} \left[\frac{Z'^m - 1}{Z' - Z'_s}\right] \tag{10}$$

which after substitution from (9) and evaluation of the indeterminate form leads to

$$B_s(Z'_s) = me^{-2\pi si/m} \tag{11}$$

From (1), (2), and (5) we obtain for $k = m$

$$A_s = \sum_{q=0}^{q=n} (L_q u^q) e^{2\pi i s(q+1)/m}, \qquad (s = 1, 2, \cdots, m). \tag{12}$$

The real and imaginary parts of A_s are linear forms of the type of equation [8], because the trigonometrical functions appearing in them can be tabulated once and for all, if m is fixed. Equations [7] can be used as a check for the absence of errors in the computations.

In the language of electrical power engineering the quantities A_s are m-times the "symmetrical coordinates" of the quantities $(L_q u^q)$[6,7]. Previous authors[8] have expressed the different roots of a polynomial equation in terms of their symmetrical coordinates, but these investigations have no bearing on the present problem.

4. The computation of H(Z′) for symmetrically arranged poles Z'_s. Let the unit circle be divided into g equal sectors. g is either equal to m or a multiple of it.

$$g = k'm \tag{13}$$

Let the radius of the unit circle be divided into $1/p_0$ equal parts ($1/p_0$ is an integer). In Fig. 1 this subdivision is shown for $m = 8$, $g = 16$, and $1/p_0 = 4$. The subdividing radii and circles determine "grid" points by their mutual intersection. Let equation [4] be rewritten as

$$H(Z') = \sum_{v=1}^{v=g} \frac{A'_v}{Z' - Z'_v}, \tag{14}$$

where

$$Z'_v = e^{2\pi i v/g} \qquad (v = 1, 2, \cdots, g) \tag{15}$$

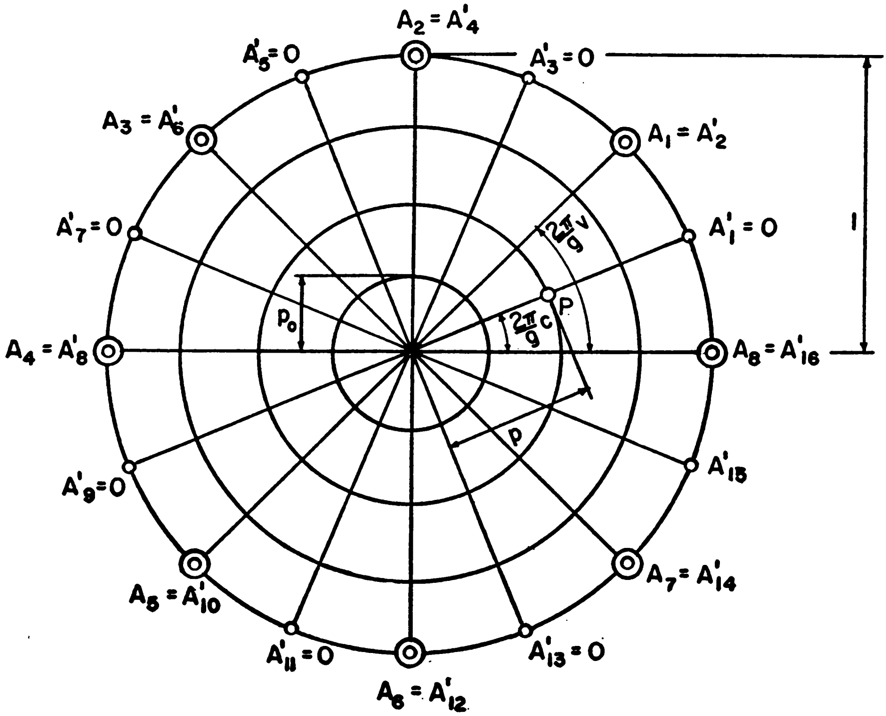
and

$$\left.\begin{aligned} A'_v &= a'^{(r)}_v + i a'^{(i)}_v = A_{(v/k')}, && \text{if } v/k' \text{ is integral,} \\ A'_v &= 0 && \text{if } v/k' \text{ is not integral.} \end{aligned}\right\} \tag{16}$$

[6] C. L. Fortescue, *Method of symmetrical coordinates applied to the solution of polyphase networks*, Trans. A.I.E.E. **37**, 1315-1327 (1918).

[7] C. F. Wagner and R. D. Evans, *Symmetrical components*, First Ed., McGraw-Hill, New York and London, Ch. XVI, 328-344 (1933).

[8] C. L. Fortescue and G. Calabrese, *L'applicazione delle coordinate simmetriche alla risoluzione delle equazioni algebriche*, Atti del Congresso Internazionale dei Matematici. Bologna, p. 159 (1928).



UNIT CIRCLE AND GRID POINTS

Let

$$Z' = pe^{i2\pi c/g} \tag{17}$$

be a grid point, so that c is an integer. By substituting (15) and (17) in (14) and introducing

$$\left.\begin{aligned} h &= v - c, && \text{if} \quad v > c \\ h &= g + v - c, && \text{if} \quad v \leq c \end{aligned}\right\} \tag{18}$$

we obtain after a few transformations

$$H(Z') = e^{-i2\pi c/g}\overline{H}(Z'), \tag{19}$$

where

$$\overline{H}(Z') = \sum_{h=1}^{h=g} \frac{A'_{h+c}}{p - e^{i2\pi h/g}}. \tag{20}$$

Let the conjugate of $\overline{H}(Z')$ be written as

$$\overline{H}^*(Z') = -(\rho + i\tau). \tag{21}$$

Then $(-\rho)$ is the radial and $(-\tau)$ the tangential component of the intensity of the field at the grid point Z'. The use of [16] and a transformation of [20] results in

$$\rho = \sum_{h=1}^{h=g} a'^{(r)}_{h+c} r_h - \sum_{h=1}^{h=g} a'^{(i)}_{h+c} t_h\ , \tag{22}$$

$$\tau = \sum_{h=1}^{h=g} a'^{(r)}_{h+c} t_h + \sum_{h=1}^{h=g} a'^{(i)}_{h+c} r_h\ , \tag{23}$$

where

$$r_h = \frac{\cos(2\pi h/g) - p}{1 + p^2 - 2p\cos(2\pi h/g)} \tag{24}$$

and

$$t_h = \frac{\sin(2\pi h/g)}{1 + p^2 - 2p\cos(2\pi h/g)}. \tag{25}$$

Equations (22) and (23) are linear forms of the type of equation (8), because the values of r_h and t_h can be tabulated once and for all. Tables 1 and 2 are examples of such tables

TABLE 1

Values of r_h for $1/p_0 = 8$ and $g = 16$.

$2\pi h/g$	$p = 0$	$p = 0'125$	$p = 0'250$	$p = 0'375$	$p = 0'500$	$p = 0'625$	$p = 0'750$	$p = 0'875$	$p = 1'000$
0°	+1'00 000	+1'14 285	+1'33 333	+1'59 998	+2'00 000	+2'66 657	+4'00 000	+8'00 000	*
22°30'	+0'92 388	+1'01 814	+1'12 210	+1'22 597	+1'29 981	+1'26 768	+0'98 421	+0'32 843	−0'50 000
45°	+0'70 711	+0'69 394	+0'64 477	+0'54 417	+0'38 150	+0'16 204	−0'08 547	−0'31 786	−0'50 000
67°30'	+0'38 268	+0'28 010	+0'15 230	+0'00 900	−0'13 527	−0'26 562	−0'37 160	−0'44 923	−0'50 000
90°	0'00 000	−0'12 308	−0'23 529	−0'32 877	−0'40 000	−0'44 944	−0'48 000	−0'49 557	−0'50 000
112°30'	−0'38 268	−0'45 684	−0'50 460	−0'53 073	−0'54 064	−0'53 916	−0'53 015	−0'51 644	−0'50 000
135°	−0'70 711	−0'69 784	−0'67 590	−0'64 760	−0'61 680	−0'58 567	−0'55 547	−0'52 683	−0'50 000
157°30'	−0'92 388	−0'84 140	−0'76 904	−0'70 840	−0'65 500	−0'60 848	−0'56 774	−0'53 183	−0'50 000
180°	−1'00 000	−0'88 889	−0'80 000	−0'72 727	−0'66 667	−0'61 538	−0'57 143	−0'53 333	−0'50 000

*The value of this limit depends on the direction of approach.

TABLE 2

Values of t_h for $1/p_0 = 8$ and $g = 16$.

$2\pi h/g$	$p = 0$	$p = 0'125$	$p = 0'250$	$p = 0'375$	$p = 0'500$	$p = 0'625$	$p = 0'750$	$p = 0'875$	$p = 1'000$
0°	0	0	0	0	0	0	0	0	*
22°30'	0'38 268	0'48 771	0'63 722	0'85 475	1'17 347	1'62 311	2'16 607	2'57 126	2'51 383
45°	0'70 711	0'84 295	0'99 741	1'15 863	1'30 252	1'39 541	1'40 903	1'33 874	1'20 713
67°30'	0'92 388	1'00 427	1'06 053	1'08 232	1'06 522	1'01 273	0'93 466	0'84 301	0'74 831
90°	1'00 000	0'98 461	0'94 118	0'87 671	0'80 000	0'71 910	0'64 000	0'56 637	0'50 000
112°30'	0'92 388	0'83 136	0'73 685	0'64 714	0'56 587	0'49 433	0'43 242	0'37 937	0'33 409
135°	0'70 711	0'59 301	0'49 935	0'42 318	0'36 131	0'31 088	0'26 954	0'23 546	0'20 711
157°30'	0'38 268	0'30 698	0'25 070	0'20 871	0'17 604	0'15 034	0'12 980	0'11 314	0'09 946
180°	0	0	0	0	0	0	0	0	0

*The value of this limit depends on the direction of approach.

for $1/p_0 = 8$ and $g = 16$. Since $r_h = r_{g-h}$ and $t_h = -t_{g-h}$, it suffices to tabulate the values between $h = 0$ and $h = g/2$.

If greater accuracy in the location of the roots is desired than determined by the mesh of radii and concentric circles, interpolation methods may be employed. $H(Z')$ is analytical and $(dH(Z'))/dZ' \neq 0$ at a root point, unless $(dG(uZ'))/dZ' = 0$ which corresponds to a multiple root. Consequently, in general, complex linear interpolation for $\overline{H}^*(Z')$ is admissible, if the values of $\overline{H}^*(Z')$ at two grid points are known. Other possibilities are repeating the described procedure for different values of u, or for equations whose roots are shifted with respect to the roots of the original equation, or for equations whose roots are powers of the roots of the original equation, etc.

After this paper had been completed a numerical method became known[9] that uses Lucas' principle to find with great accuracy "the roots of polynomial equations with exact coefficients when one has a first approximation as a starting point." Since Salzer's approach is entirely different from that in this paper, a combination of the two methods in question is not possible without a special investigation into its practicability.

5. Polynomial equations with real coefficients. It will be shown in this Section that real values of the residues A_s for a symmetrical arrangement of the poles along the unit circle can be obtained with the aid of a well-known conformal transformation, if the coefficients of the polynomial (1) are real. For real values A_s equations (22) and (23) simplify considerably.

By the relation

$$(W + i)(Z' + i) = -2 \tag{26}$$

the upper half of the Z'-plane is conformally transformed into the interior of the unit circle of the W-plane. The point $Z' = \infty$ corresponds to $W_\infty = -i$. The field configuration given by equation [4] in the Z'-plane, correlates to the field configuration given by the equation

$$H'(W) = \sum_{s=1}^{s=m} \frac{A_s}{W - W_s} - \frac{1}{W + i} \sum_{s=1}^{s=m} A_s \tag{27}$$

in the W-plane, if the poles W_s in the W-plane correspond to the poles Z'_s in the Z-plane according to [26]. Saddle points of the potential correlate in the two field maps, hence we obtain the condition

$$H'(W_r) = 0 \tag{28}$$

if W_r and Z'_r correlate. Consequently, the roots Z'_r of the equation $G(uZ'_r) = 0$ can be located by a field exploration according to Section IV carried out in the W-plane.

The steps described in this Section so far can be comprised in a single procedure. Let the uniform distribution of the points W_s including $W_\infty = -i'$ be given by

$$W_s = -ie^{i(2\pi s/n+2)}, \qquad (s = 1, 2, \cdots, n + 2). \tag{29}$$

From (26) we obtain

$$Z'_s = -\frac{\sin[2\pi s/(n+2)]}{1 - \cos[2\pi s/(n+2)]}, \qquad (s = 1, 2, \cdots, n + 2) \tag{30}$$

[9]Herbert E. Salzer, *On calculating the zeros of polynomials by the method of Lucas,* J. Research Nat. Bur. Stands., **49**, 133-4 (1952).

The substitution of (1) in (5) gives

$$A_s = \sum_{q=0}^{q=n} (u^q L_q) \cdot {}^s h_q , \qquad (s = 1, 2, \cdots, r + 1), \tag{31}$$

where

$${}^s h_q = k \frac{Z_s'^q}{B_s(Z_s')}, \qquad \begin{Bmatrix} (s = 1, 2, \cdots, n + 1) \\ (q = 0, 1, 2, \cdots, n) \end{Bmatrix}. \tag{32}$$

A_s is a linear form of the type of equation [8], because the quantities ${}^s h_q$ can be computed and tabulated once and for all. Table 3 contains the values ${}^s h_q$ for $n = 6$. The residue at $W_\infty = W_{n+2} = -i$ is given by

TABLE 3
Values of ${}^s h_q$ *for* $n = 6$ $(k = 8)$.

	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$	$s = 6$	$s = 7$
$q = 0$	+0′02 512 6	−1	+4′97 487	−8	+4′97 487	−1	+0′02 512 6
$q = 1$	−0′06 066 0	+1	−2′06 066	0	+2′06 066	−1	+0′06 066 0
$q = 2$	+0′14 645	−1	+0′85 355	0	+0′85 355	−1	+0′14 645
$q = 3$	−0′35 355	+1	−0′35 355	0	+0′35 355	−1	+0′35 355
$q = 4$	+0′85 355	−1	+0′14 645	0	+0′14 645	−1	+0′85 355
$q = 5$	−2′06 066	+1	−0′06 066 0	0	+0′06 066 0	−1	+2′06 066
$q = 6$	+4′97 488	−1	+0′02 512 6	0	+0′02 512 6	−1	+4′97 488

$$A_{n+2} = -\sum_{s=1}^{s=n+1} A_s = -ku^{n-1}L_n . \tag{33}$$

The double statement in (33) can be used as a check for the absence of errors in the computations.

6. Methods to obtain real residues A_s for equations with complex coefficients. It may be sometimes desirable to work with real residues and to pay for this advantage with a larger number of poles. Two methods are suggested which achieve this and in which, at the same time, all roots or numbers closely associated with them are within the unit circle.

Method 1: The equation

$$\sum_{t=0}^{t=2n} k_t Z^t = \left[\sum_{q=0}^{q=n} L_q Z^q \right] \cdot \left[\sum_{h=0}^{h=n} L_h^* Z^h \right] = 0 \tag{34}$$

has real coefficients and its roots are the roots of the original equation and their conjugates. Equation (34) can be dealt with according to Section 5.

Method 2: If $G(Z)$ happens to be a polynomial of even degree in which any pair of coefficients equidistant from the two ends are complex conjugates, the residues A_s

computed from (12) turn out to be real. If the star denotes the conjugate, a subsidiary polynomial complying with this condition is

$$\sum_{t=0}^{t=2n} N_t Z'^t = \left[\sum_{q=0}^{q=n} (u^q L_q) Z'^q\right] \cdot \left[\sum_{h=0}^{h=n} (u^{n-h} L^*_{n-h}) Z'^q\right]. \tag{35}$$

The $(2n)$ zeros of the subsidiary polynomial are the n zeros of the original polynomial $G(uZ')$ and its n conjugate reciprocals.

If the values N_t are computed and substituted in (12) for $m = 2n + 2$, we obtain after some transformations

$$(-1)^s A_s = \left[\sum_{q=0}^{q=0} (u^q L_q) \cos \frac{2\pi sq}{2n+2}\right]^2 + \left[\sum_{h=0}^{h=n} (u^h L_h) \sin \frac{2\pi sq}{2n+2}\right]^2, \qquad (s = 1, 2, \cdots, 2n+2). \tag{36}$$

Equation (36) which is very closely related to linear forms of the type of equation (8) permits the direct computation of the values A_s from the coefficients of the original equation. A check for the absence of errors is the first statement of equations (7).