NOTE ON RECTANGULAR PLATES: DEFLECTION UNDER PYRAMIDAL LOAD*

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Using the Euler-Fourier method, a direct procedure leading to the evaluation of Fourier coefficients in the computation of the deflection of thin plates is developed.

Consider a thin rectangular plate of uniform thickness, placed horizontally on four supports and acted upon by an arbitrary distributed load (Fig. 1). To solve the well known differential equation of the rectangular plate,

$$\frac{EI}{1-\mu^2}\nabla^2\nabla^2 w = p(x, y), \tag{1}$$

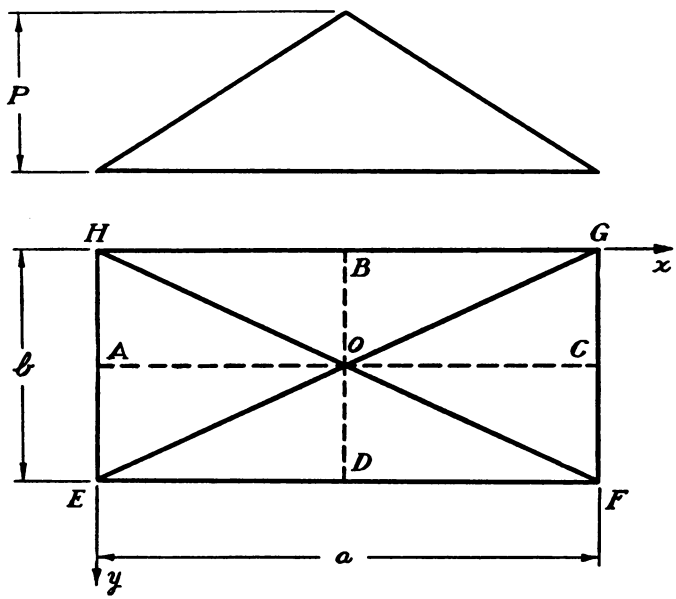

Fig. 1.

the deflection is represented in a form of a double infinite series whose every term satisfies the boundary conditions:

$$w = \sum_m \sum_n A_{mn} \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b}. \tag{2}$$

Substituting this expression in the plate equation (1), we obtain

$$\sum_m \sum_n A_{mn}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b} = \frac{1-\mu^2}{\pi^4 EI} p(x, y). \tag{3}$$

Multiplying both sides of equation (3) by sin $(m'\pi x/a)\cdot dx$ and integrating from 0 to a and then multiplying both sides of the equation by sin $(n'\pi y/b)\cdot dy$ and integrating

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from 0 to b we obtain

$$A_{mn} = \frac{4(1 - \mu^2)}{\pi^4 EIab}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2} \int_0^b \left[\int_0^a p(x, y) \sin m\pi \frac{x}{a}\, dx\right] \sin n\pi \frac{y}{b}\, dy. \tag{4}$$

The expression (4) has been developed for any arbitrary, continuously distributed load, but it may, with small modifications, be used for a concentrated load.

Application to pyramidal distribution of load. The load is symmetrical with respect to two central axes AC and BD, see Fig. 1. If loads of equal magnitude are applied at two points equidistant from the axis BD, at $P_1(x_1, y_1)$ and at $P_2(x_2, y_1)$ then it is obvious that due to symmetry $x_1 + x_2 = a$ and

$$\sin (m\pi x_1/a) = \sin m\pi(1 - x_2/a) = \sin (m\pi x_2/a) \tag{5}$$

and similarly $\sin (n\pi y_1/b) = \sin (n\pi y_2/b)$. Starting with the determination of A_{mn} due to the loads OEH and OFG, we see that when the contribution of the partial load OAH is known, then it is only necessary to multiply that value by four to have the expression for A_{mn} due to loads OEH and OFG. By the same reasoning, when we multiply by four the contribution made by load OHB to A_{mn}, the value of A_{mn} due to loads OHG and OEF is obtained. The maximum load on the plate is at the center, where its value is P. At any point (x,y) on OAH the load is

$$p = \frac{2Px}{a}, \tag{6}$$

i.e., a function of x alone. Substituting this value of p in the expression (4), we obtain

$$A_{mn} = \frac{8P(1 - \mu^2)}{\pi^4 EIab}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2} \int_0^{a/2} \left[\int_{y_0}^{b/2} \sin n\pi \frac{y}{b}\, dy\right] \frac{x}{a} \sin m\pi \frac{x}{a}\, dx. \tag{7}$$

In the second integral of the expression (7) the lower limit y_0 is equal to $y_0 = bx/a$ which is the equation of the diagonal HF. Because of symmetry of the load, both m and n are odd numbers, so that after integration the expression (7) takes the form

$$A_{mn} = \frac{8P(1 - \mu^2)}{\pi^5 EIn}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2} \int_0^{1/2} \frac{x}{a} \sin m\pi \frac{x}{a} \cos n\pi \frac{x}{a}\, d\left(\frac{x}{a}\right). \tag{8}$$

Setting $\pi x/a = u$, the integral of this expression becomes

$$\frac{1}{\pi^2}\int_0^{\pi/2} u \sin (mu) \cos (nu)\, du$$

$$= \frac{1}{2\pi^2}\int_0^{\pi/2} u \sin (m + n)u\, du + \frac{1}{2\pi^2}\int_0^{\pi/2} u \sin (m - n)u\, du. \tag{9}$$

Integrating by parts, we obtain the value of A_{mn} for the load OAH

$$A_{mn} = \frac{-2P(1 - \mu^2)}{\pi^6 EI}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2}\left[\frac{\cos [(m + n)\pi/2]}{n(m + n)} + \frac{\cos [(m - n)\pi/2]}{n(m - n)}\right]. \tag{10}$$

Interchanging a with b and m with n, the value of A_{mn} for the load OHB is obtained:

$$A_{mn} = \frac{-2P(1 - \mu^2)}{\pi^6 EI}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2}\left[\frac{\cos [(m + n)\pi/2]}{m(m + n)} + \frac{\cos [(n - m)\pi/2]}{m(n - m)}\right]. \tag{11}$$

Adding to the expression (10) the contributions made by the loads OAE, OCG and OCF, and further, adding to the expression (11) the influence of loads OBG, ODE and ODF, the term A_{mn} for the whole plate is

$$A_{mn} = \frac{-16P(1 - \mu^2)}{\pi^6 EI mn} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2} \cos(m\pi/2) \cos(n\pi/2). \tag{12}$$

The expression (11) gives a trivial value of A_{mn}, because with both m and n odd, A_{mn} would vanish, which of course is impossible. It is seen that the equations (10) and (11) are true only for $m \neq n$, which, however, does not give any practical result. Moreover, we cannot set $m = n$, in the expression (10) and (11) because a value A_{mn} equal to infinity would result. It is clear that the integration performed is true only when m equals n; hence we must go back to the expression (9) and set there $m = n$. With this substitution expression (9) will yield the following value

$$\frac{1}{2\pi^2} \int_0^{\pi/2} u \sin 2mu \, du = \frac{1}{4m\pi}. \tag{13}$$

For the load OAH, the term A_{mn} from (8) becomes

$$A_{mn} = \frac{2P(1 - \mu^2)}{\pi^6 m^6 EI} \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{-2} \tag{14}$$

and for the whole plate

$$A_{mn} = \frac{16P(1 - \mu^2)}{\pi^6 m^6 EI} \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{-2}. \tag{15}$$

Finally the expression for the deflection is

$$w = \frac{16P(1 - \mu^2)}{\pi^6 EI} \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{-2} \left\{ \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + \frac{1}{3^6} \sin\frac{3\pi x}{a} \sin\frac{3\pi y}{b} + \cdots \right\}. \tag{16}$$

Thus, through this operation a double series for the deflection of the rectangular plate under pyramidal load is reduced to a result involving but a single series.

A RANDOM WALK RELATED TO THE CAPACITANCE OF THE CIRCULAR PLATE CONDENSER*

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Abstract. It is shown that the solution of Love's equation for the capacitance of the circular plate condenser can be expressed in terms of the mean duration of a certain one-dimensional random walk with absorbing barriers. The interpretation as a random walk makes it possible to confirm the fact that the actual capacitance of the condenser is always larger than the value given by the standard approximation for small separations, and yields an upper bound as well. In addition to its theoretical interest, the

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