

AN ITERATIVE SOLUTION TO THE EFFECTS OF CONCENTRATED LOADS APPLIED TO LONG RECTANGULAR BEAMS*

BY

C. A. M. GRAY

The University of Sydney

Summary. The analysis of the bending of thin deep rectangular beams by concentrated loads has already been treated by L.N.G. Filon and Th. v. Kármán. These authors consider an infinitely long beam and express the load as a Fourier integral, obtaining integral solutions for the stresses and displacements. In their integral form, these results are rather difficult to interpret, although F. Seewald, using Kármán's analysis, has calculated numerical values for the case of a single concentrated load.

In this paper, a totally different approach to the problem is made. The loaded area is conformally transformed to a circle, and, using Muschelisvili's transformation of the boundary conditions, the problem is solved in the plane of the circle. The solution is then obtained in the form of a complex power series. In this manner, a direct solution, to any required degree of accuracy, is readily obtained.

1. Theory. Consider a long deep rectangular beam resting on two supports placed at the same level, and loaded by a weight placed midway between them. The weight of the beam will be neglected, and the beam will be assumed to be in a state of generalised plane stress, the plane of the mean stress being that of the length and depth. To determine the local effects of the concentrated loads on the stresses and deflexion it is convenient to move the supports to infinity and treat the beam as an infinitely long strip of depth $2h$ as is shown in Fig. 1. This procedure is that adopted by L.N.G. Filon, H. Lamb,

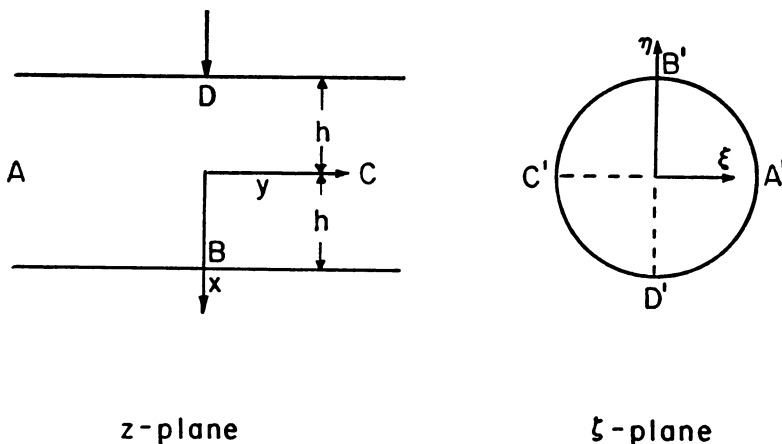


FIG. 1.

and Th. v. Kármán [1, 2] in their solutions of the problem.

Transform the strip ABCD to the unit circle, γ , in the ζ plane by the transformation

$$z = \frac{-2ih}{\pi} \log \frac{1 + \zeta}{1 - \zeta} = -4ih\pi^{-1}(\zeta + \zeta^3/3 + \zeta^5/5 + \dots). \quad (1)$$

*Received May 23, 1952.

Following Muschelivili [3], define two functions $\Omega(z)$ and $w'(z)$ of the complex variable z by,

$$\sigma_x + \sigma_y = R\Omega'(z), \quad \sigma_x - \sigma_y + 2i\tau_{xy} = -\frac{1}{2}\{z\bar{\Omega}''(\bar{z}) + \bar{w}''(\bar{z})\}, \tag{2}$$

where the dashes denote differentiation with regard to z and the bars denote the conjugate function. Then, the boundary conditions in the z plane can be expressed in the ζ plane by

$$\Omega[f(\sigma)] + f(\sigma)\bar{\Omega}'_z(\bar{\sigma}) + \bar{w}'[f(\bar{\sigma})] = F_1 + iF_2, \tag{3}$$

where,

$$z = f(\zeta) = \sum_{r=0}^{\alpha} u_r \zeta^r,$$

is the transformation relation, σ is the value of ζ on the unit circle,

$$\Omega'_z(\zeta) = \frac{d}{dz} \{ \Omega[f(\zeta)] \},$$

$F_1 + iF_2$ is the transform of $f_1 + if_2$, $f_1 + if_2 = 4i\int_0^s (X_s + iY_s) ds$, X_s, Y_s are the components of the boundary tractions, applied in the z plane, and s is the distance measured along the boundary in the z plane.

Since $\Omega'_z(\zeta)$ is analytic within the unit circle, we can express it by the Taylor's Series,

$$\Omega'_z(\zeta) = \sum_{r=0}^{\alpha} b_r \zeta^r. \tag{4}$$

Let us multiply both sides of (3) by $(1/2\pi i) d\sigma/(\sigma - \zeta)$ and integrate around γ . Using Cauchy's Theorem and, neglecting the constant, we have,

$$\Omega[f(\zeta)] = (1/2\pi i) \int_{\gamma} (F_1 + iF_2) d\sigma/(\sigma - \zeta) - (1/2\pi i) \int_{\gamma} f(\sigma)\bar{\Omega}'_z(\bar{\sigma}) d\sigma/(\sigma - \zeta) \tag{5}$$

Similarly from the relation conjugate to (3) we obtain,

$$w'[f(\zeta)] = (1/2\pi i) \int_{\gamma} (F_1 - iF_2) d\sigma/(\sigma - \zeta) - (1/2\pi i) \int_{\gamma} \bar{f}(\bar{\sigma})\Omega'_z(\sigma) d\sigma/(\sigma - \zeta). \tag{6}$$

Substituting (4) into (5), carrying out the integration and noting that

$$\frac{d}{d\zeta} \frac{1}{2\pi i} \int_{\gamma} (F_1 + iF_2) \frac{d\sigma}{\sigma - \zeta}$$

can be expressed as a power series, $\sum_{r=0} p_r \zeta^r$, we obtain the following infinite set of equations

$$\sum_0^{n-\nu} (1 + \nu)u_{\nu+1}b_{n-\nu} + (1 + n) \sum_{\nu=0}^{\infty} u_{1+n+\nu}\bar{b}_{\nu} = p_n. \tag{7}$$

Solving equation (7) gives Ω'_z . To complete the solution, we require $w'(z)$ which is established in exactly the same way, using (6). The formula for $w'(z)$ is then

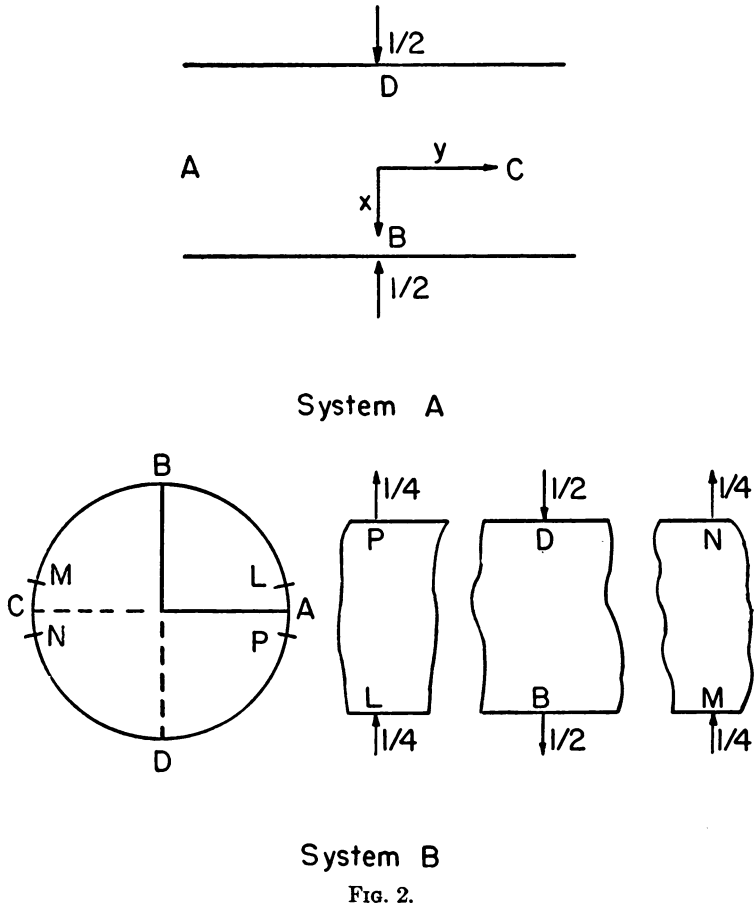
$$w'(z) = (1/2\pi i) \int_{\gamma} (F_1 - iF_2) d\sigma/(\sigma - \zeta) - \sum_{r=0}^{\infty} c_r \zeta^r, \tag{8}$$

where the c 's are given by the relations,

$$c_n = \sum_{r=0}^{\infty} \bar{u}_r b_{n+r}. \tag{9}$$

A full and complete account of the derivation of (7) and (8) is given in Reference 4.

2. Solution. Returning to the original problem, consider the given loading as a combination of the following two systems, where in system B, the points L, M, N, P are supposed to be very remote.



System A

System B

FIG. 2.

Case A. Load function:

$$\text{along } BCD, F_1 + iF_2 = -2i,$$

$$\text{along } DAB, F_1 + iF_2 = 0.$$

Then,

$$(1/2\pi i) \int_{\gamma} (F_1 + iF_2) d\sigma/(\sigma - \zeta) = -\pi^{-1} \log (\sigma_D - \zeta)/(\sigma_B - \zeta)$$

$$\frac{d}{d\zeta} (1/2\pi i) \int_{\gamma} (F_1 + iF_2) d\sigma/(\sigma - \zeta) = \frac{2i}{\pi} (1 + \zeta^2)^{-1}$$

$$(1/2\pi i) \int_{\gamma} (F_1 - iF_2) d\sigma/(\sigma - \zeta) = \pi^{-1} \log (e^{i3\pi/2} - \zeta)/(e^{i\pi/2} - \zeta).$$

Equations (7) become

$$\begin{aligned}
 2b_0 + .33333b_2 + .2b_4 + .142857b_6 + .11111b_8 + \dots &= -(2h)^{-1}, \\
 2b_0 + 1.6b_2 + .42857b_4 + .33333b_6 + .27273b_8 + \dots &= (2h)^{-1}, \\
 2b_0 + 1.7108b_2 + 1.5556b_4 + .45455b_6 + .38462b_8 + \dots &= -(2h)^{-1}, \\
 2b_0 + 1.778b_2 + 1.6364b_4 + 1.5385b_6 + .46666b_8 + \dots &= (2h)^{-1}, \text{ etc.}
 \end{aligned}$$

Using an iterative process to solve these equations we obtain for the 6th and 7th iterations the following values:

Iteration	$b_0 \times h$	$b_2 \times h$	$b_4 \times h$	$b_6 \times h$	$b_8 \times h$	$b_{10} \times h$
6	-.32113	.91461	-1.0132	.98457	-.98724	.98556
7	-.34898	.88018	-1.0381	.97263	-1.0143	1.0143

From these we obtain

$$\begin{aligned}
 \Omega'_z &= -.329/h + .895\zeta^2/h - 1.025\zeta^4/h + .979\zeta^6/h + \dots, \\
 &= -\frac{1}{2h} \frac{1 - \zeta^2}{1 + \zeta^2} + .171/h - .105\zeta^2/h - .025\zeta^4/h + \dots, \tag{a}
 \end{aligned}$$

$$\Omega''_z = \frac{d\Omega'}{d\zeta} \frac{d\zeta}{dz} = \frac{i\pi}{4h^2} \zeta(1 - \zeta^2) \left[\frac{2}{(1 + \zeta^2)^2} - .21 - .10\zeta^2 - .126\zeta^4 + \dots \right]. \tag{b}$$

Using these values in equations (8) and (9) we calculate $w'(z)$ and $w''(z)$ as follows:

$$\begin{aligned}
 w'(z) = \frac{1}{\pi} \log \left(\frac{-i - \zeta}{i - \zeta} \right) - \frac{i\zeta}{2} \frac{1 - \zeta^2}{1 + \zeta^2} + \frac{2i}{\pi} [-.55014\zeta + .771667\zeta^3 \\
 + .043\zeta^5 + \dots], \tag{c}
 \end{aligned}$$

$$w''(z) = \frac{1 - \zeta^2}{h} \left[\frac{1.2854}{1 + \zeta^2} - \frac{\pi}{2} \left[\frac{\zeta}{1 + \zeta^2} \right]^2 - .11763 + .1075\zeta^2 + .107\zeta^4 + \dots \right] \tag{d}$$

Case B. Load function:

For	BM	MCN	ND	DP	PAL	LB
$F_1 + iF_2$	0	-i	-2i	0	-i	-2i

Then,

$$\frac{1}{2\pi i} \int_{\gamma} (F_1 + iF_2) \frac{d\sigma}{\sigma - \zeta} = \frac{1}{2\pi i} \left[-i \log \left(\frac{\sigma_N - \zeta}{\sigma_M - \zeta} \right) \left(\frac{\sigma_D - \zeta}{\sigma_N - \zeta} \right) \left(\frac{\sigma_L - \zeta}{\sigma_P - \zeta} \right) \left(\frac{\sigma_P - \zeta}{\sigma_L - \zeta} \right)^2 \right]$$

Now when L and P move to A $\sigma_L = \sigma_P = 1$, and when M and N move to C $\sigma_M = \sigma_N = -1$. Thus,

$$\frac{1}{2\pi i} \int_{\gamma} (F_1 + iF_2) \frac{d\sigma}{\sigma - \zeta} = \frac{-1}{\pi} \log \frac{-1 - \zeta^2}{1 - \zeta^2}.$$

The constants b are purely imaginary and equations (7) become

$$\begin{aligned} b_1 - 2b_1/3 - 2b_3/5 - 2b_5/7 - 2b_7/9 + \dots &= -1/h, \\ b_1 + b_3 - 4b_1/5 - 4b_3/7 - 4b_5/9 - 4b_7/11 + \dots &= 0, \\ b_1 + b_3 + b_5 - 6b_1/7 - 6b_3/9 - 6b_5/11 - 6b_7/13 + \dots &= -1/h, \text{ etc.} \end{aligned}$$

Assume

$$\Omega'_z = i\pi b_1 z/4h + f_3 \zeta^3 + f_5 \zeta^5 + \dots,$$

i.e., assume that the b_n are of the form

$$\begin{aligned} b_1 &= b_1, \\ b_3 &= b_1/3 + f_3, \\ b_5 &= b_1/5 + f_5, \text{ etc.} \end{aligned}$$

Then since

$$\begin{aligned} 1 - \frac{2}{3} - \frac{2}{5.3} - \frac{2}{7.5} - \frac{2}{7.9} + \dots &= 0, \\ 1 + \frac{1}{3} - \frac{4}{1.5} - \frac{4}{3.7} - \frac{4}{5.9} + \dots &= 0, \\ 1 + \frac{1}{3} + \frac{1}{5} - \frac{6}{1.7} - \frac{6}{3.9} - \frac{6}{5.11} + \dots &= 0, \text{ etc.,} \end{aligned}$$

b_1 can be eliminated from the equations and we are left with the following system for the values of f :

$$\begin{aligned} -.2f_3 - .142857f_5 - .111111f_7 - .090909f_9 + \dots &= -1/2h, \\ .107143f_3 - .111111f_5 - .090909f_7 - .076923f_9 + \dots &= 0, \\ .055556f_3 + .075758f_5 - .076923f_7 - .066667f_9 + \dots &= -1/6h, \text{ etc.} \end{aligned}$$

To bring these into a form suitable for the iteration process subtract each equation from the one above it. Thus,

$$\begin{aligned} -.307143f_3 - .031746f_5 - .020202f_7 - .013986f_9 + \dots &= -1/2h, \\ .051587f_3 - .186869f_5 - .013986f_7 - .010256f_9 + \dots &= 1/6h, \\ .021365f_3 - .027681f_5 - .135256f_7 - .007843f_9 + \dots &= -1/6h, \text{ etc.} \end{aligned}$$

From these we obtain

$f_3 \div i/h$	$1.581 \pm .004$	$f_9 \div i/h$	$-.72 \pm .01$
$f_5 \div i/h$	$-.581 \pm .001$	$f_{11} \div i/h$	$1.25 \pm .01$
$f_7 \div i/h$	$1.33 \pm .02$	$f_{13} \div i/h$	$-.77 \pm .02$

Thus

$$h\Omega'_z = ib_1z\pi/4 - i\zeta/(1 + \zeta^2) + i\zeta(1 + .581\zeta^2 + .419\zeta^4 + .33\zeta^6 + \dots) \tag{e}$$

and

$$h^2\Omega'_2 = ib_1h\pi/4 + \frac{\pi}{4}(1 - \zeta^2)^2/(1 + \zeta^2)^2 - \frac{\pi}{4}(1 - \zeta^2)[1 + 1.743\zeta^2 + 2.095\zeta^4 + \dots] \tag{f}$$

From formulae 8 and 9 and using the tabulated values of f , we have,

$$w'(z) = ib_1z^2\pi/8h + \pi^{-1} \log(1 + \zeta^2)/(1 - \zeta^2) + i - (1 + \zeta^2)^{-1} + (4/\pi)[c_0 + .972\zeta^2 + .749\zeta^4 + \dots], \tag{g}$$

and

$$hw''(z) = i\zeta/(1 + \zeta^2) + (i\pi/2)\zeta(1 - \zeta^2)/(1 + \zeta^2)^2 + ib_1z\pi/4 + i\zeta(1 - \zeta^2)(1.944 + 2.996\zeta^2 + \dots). \tag{h}$$

The stress distribution for which $\Omega'_z = ib_1z\pi/4h$, $w''(z) = ib_1z\pi/4h$ is that due to pure flexure, provided $b_1 = -iB_1/h$, where B_1 is real. From statics, we know that there is an infinite bending moment at the centre section due to the loads at infinity. But, from statics, at a section where the support loads are applied, the normal stress distribution must be equivalent to a moment of $h/2\pi$ as shown in Fig. 3. As this moment is applied

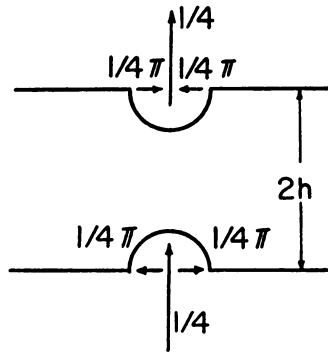


FIG. 3.

effectively at infinity, it must be included in the b_1 term. For as $\zeta \rightarrow \pm 1$, $R\Omega'_z \rightarrow B_1z/4h^2$. Hence, B_1 is made up of two parts, that due to an infinite moment which will be represented by b' and that due to a moment of amount $h/2\pi$. Thus

$$B_1 = b' + 3/\pi^2$$

and

$$h\Omega'_z = b'z\pi/4h + 3z/4\pi h - i\zeta/(1 + \zeta^2) + i\zeta(1 + .581\zeta^2 + .419\zeta^4 + \dots), \tag{i}$$

$$w'(z) = b'z^2\pi/8h^2 + 3z^2/8\pi h^2 + \pi^{-1} \log(1 + \zeta^2)/(1 - \zeta^2) + i - (1 + \zeta^2)^{-1} + (4/\pi)(c_0 + .972\zeta^2 + \dots) \tag{j}$$

Equations *a* to *f* now give the solution to the problem. To calculate the stresses at any point we use equation (1) with the values of Ω'_z , Ω''_z and w'' obtained by substituting the corresponding values of ζ in the relevant equation *a* to *j*. Fig. 4 gives the distribution

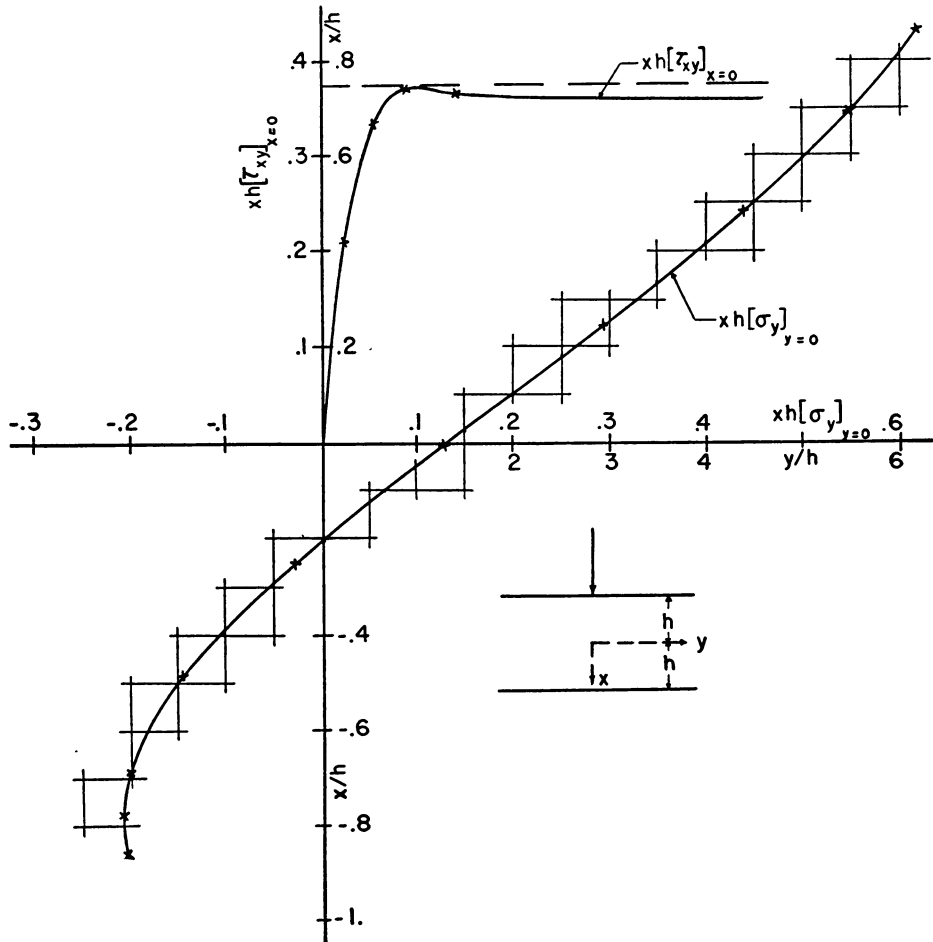


FIG. 4.

of normal stresses across *BD* and the shear stress distribution along the *y* axis.

The deflections *u* and *v* are readily obtained from formula (10)

$$8\mu(u + iv) = (3 - \nu)/(1 + \nu)\Omega(z) - z\bar{\Omega}'(\bar{z}) - \bar{w}'(\bar{z}), \tag{10}$$

where μ is the shear modulus; ν is Poisson's ratio, *u*, *v* are the components of displacement in the *x* and *y* direction respectively; and *D* is $u + iv$.

We are particularly concerned with the displacement and curvature of the *y* axis and for this case some simplification of 10 is possible.

Differentiate both sides of (10) with regard to *y*, then,

$$8\mu \frac{\partial D}{\partial y} = i\Omega'_z(3 - \nu)/(1 + \nu) - i\bar{\Omega}'(\bar{z}) + iz\bar{\Omega}''(\bar{z}) + i\bar{w}''(\bar{z}).$$

Since it is only the loading condition of case *B* that will have any effect on the centre line we have for $x = 0$,

$$\zeta = \xi,$$

$$i\Omega'(z) = -i\bar{\Omega}'(\bar{z}),$$

$$i[z\bar{\Omega}''(\bar{z}) + \bar{w}''(\bar{z})] = \frac{-\pi}{4h^2} \cdot y \cdot \left(\frac{1-\xi^2}{1+\xi^2}\right)^2 + \frac{1}{h} \cdot \frac{\xi}{1+\xi^2} + \frac{\pi}{2h} \cdot \frac{(1-\xi^2)}{(1+\xi^2)^2} \xi \\ + .944\xi/h - .024\xi^2/h + \dots$$

and therefore

$$8\mu \left. \frac{\partial u}{\partial y} \right]_{x=0} = \frac{4}{1+\nu} \left[\frac{-\pi b'y}{4h^2} + \frac{1}{h} \cdot \frac{\xi}{1+\xi^2} - (.696\xi + .480\xi^3 + \dots) \right] - \frac{\pi}{4h^2} \cdot y \\ \cdot \left(\frac{1-\xi^2}{1+\xi^2}\right)^2 + \frac{1}{h} \cdot \frac{\xi}{1+\xi^2} + \frac{\pi}{2h} \cdot \xi \cdot \frac{1-\xi^2}{(1+\xi^2)^2} + (\xi/h)[.944 - .024\xi + \dots].$$

To compare this result with the usual engineering form let u_1 be the deflection of $x = 0$ and, considering the case $y > 0$ let $\xi = -t$ and E denote Young's Modulus. Then,

$$\frac{du_1}{dy} = \frac{\pi b'}{4h} \cdot \frac{y}{Eh} + \frac{3}{8} \cdot \left(\frac{y}{h}\right)^2 \frac{1}{Eh} - \frac{1}{Eh} \cdot \frac{t}{1+t^2} + \frac{1}{Eh} [.696t - .608t^2 \\ + .480t^3 - .405t^4 + .358t^5 + \dots] - \frac{1}{8\mu h} \left[\frac{\pi}{4} \cdot \frac{y}{h} \cdot \left(\frac{1-t^2}{1+t^2}\right)^2 + \frac{t}{1+t^2} \right. \\ \left. + \frac{\pi}{2} \cdot t \cdot \frac{1-t^2}{(1+t^2)^2} + .944t - .024t^2 + \dots \right]$$

Using Euler's summation process we have for the first series on the right-hand side

$$.696t - .608t^2 + .480t^3 + \dots = .696 \frac{t}{1+t} + .088 \left(\frac{t}{1+t}\right)^2 + \dots$$

Then if Δ is the correction to the usual engineering deflection,

$$\frac{d\Delta}{dy} = \frac{-1}{Eh} \cdot \frac{t}{1+t^2} + \frac{1}{Eh} \left(.696 \frac{t}{1+t} + .088 \left(\frac{t}{1+t}\right)^2 + \dots \right) \\ - \frac{1}{8\mu h} \left[\frac{\pi}{4} \cdot \frac{y}{h} \left(\frac{1-t^2}{1+t^2}\right)^2 + \frac{t}{1+t^2} + \frac{\pi}{2} \cdot t \cdot \frac{1-t^2}{(1+t^2)^2} + .944t - .024t^2 + \dots \right]$$

Hence the correction to the usual engineering curvature is

$$K = \frac{d^2\Delta}{dy^2} = \frac{\pi}{4h^2} \cdot \frac{1-t^2}{E} \left(.696 - 1.216 \frac{t}{1+t} + .224 \left(\frac{t}{1+t}\right)^2 + \dots \right) \\ - \frac{\pi}{4h^2 E} \left\{ \frac{1-t^2}{1+t^2} \right\}^2 - \frac{\pi}{32\mu} \frac{1-t^2}{h^2} \cdot \left[.944 + 2 \frac{1-t^2}{(1+t^2)^2} - \frac{\pi y}{4h} \cdot \frac{8t(1-t^2)}{(1+t^2)^3} \right. \\ \left. + \frac{\pi}{2} \frac{1-6t^2+t^4}{(1+t^2)^3} \right]$$

and,

$$\Delta = \frac{-1}{\pi E} \log \frac{1+t^2}{1-t^2} + \frac{1}{E} \left[.370 \frac{y}{h} + \frac{1.656}{\pi} \cdot \frac{1}{1+t} - \frac{.088}{\pi} \frac{1}{1+t^2} + \dots \right]$$

$$- \frac{1}{8\mu} \frac{y}{h} \cdot \frac{t}{1+t^2} + \frac{1}{8\mu} \cdot \frac{1}{1+t^2} + \frac{.236}{\pi\mu} \log(1-t^2) - \frac{1}{8\mu} - \frac{1.568}{\pi E} .$$

Figure 5 gives a plot of K with y assuming $\nu = 0.3$ and shows that for $y > h$, K is

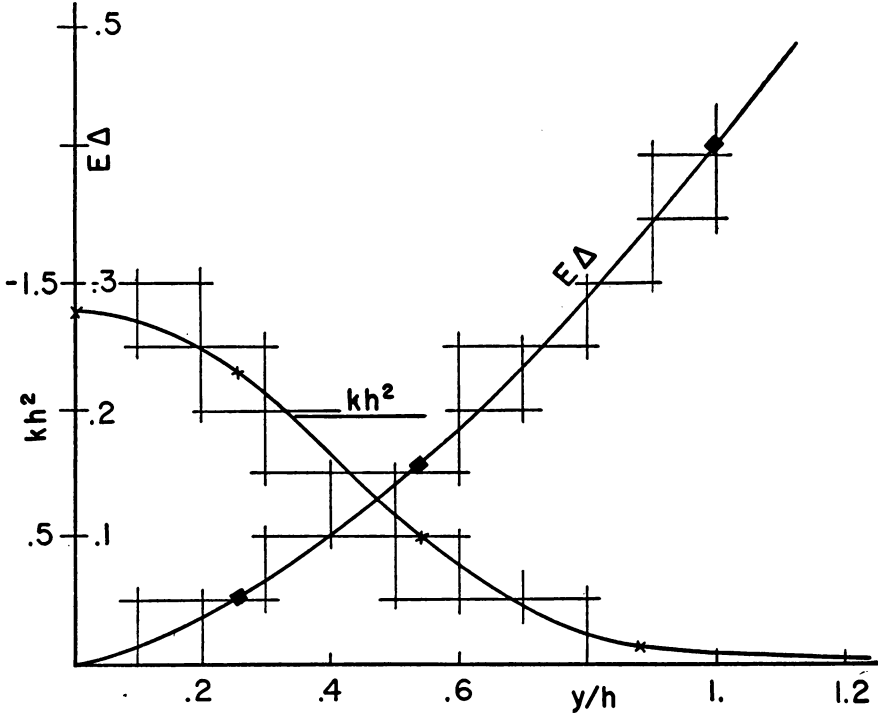


FIG. 5.

negligible. Hence the engineering approximation, that the curvature is proportional to the bending moment, is quite exact at a distance from the load greater than half the depth of the beam. Making the assumption that for $y > h$ the curvature of the centre line is proportional to the bending moment, we can then calculate the correction Δ which is also given in Fig. 5.

REFERENCES

1. L. N. G. Filon, Phil. Trans. (A) 201, 63 (1903).
2. Th. v. Kármán, see TIMOSHENKO, Theory of elasticity, p. 99-104.
3. N. Muschelisvili, Zeits. angew. Math. Mech. 13, 264 (1933).
4. C. A. M. Gray, Roy. Soc. N.S.W. Journal 85, 20 (1951).