MODIFIED STURM-LIOUVILLE SYSTEMS*

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1. Heat conduction problem. Consider the steady-state temperatures $U(x,z)$ of a thin slab of material in the shape of a square with edges at $x = 0$, $x = 1$, $z = 0$, and $z = 1$. Let the edge at $x = 0$ be kept at temperature zero and each point of the edge at $z = 0$ kept at temperature $F(x)$. Suppose the edge at $x = 1$ is in perfect thermal contact with a thin strip of anisotropic material the coefficient of conductivity of which is much greater in the x direction than in the z direction so that the temperatures in the strip can be regarded as a function of z alone. Suppose that an analogous situation holds for the edge at $z = 1$ so that the temperatures in that strip may be regarded as a function of x alone. If at the faces heat is transferred according to a linear law into an external medium at temperature zero and the coefficient of heat transfer $r(x)$ is a continuous function, the boundary value problem may be written**

$$\begin{gathered} U_{zz} + U_{xx} - cr(x)U = 0, \\ U(0, z) = 0, \\ U_x(1, z) - k_1 U_{zz}(1, z) = 0, \\ U(x, 0) = F(x), \\ U(x, 1) - k_2 U_{xx}(x, 1) = 0. \end{gathered}$$

In this statement of the problem $1/c$ is the conductivity of the slab, $k_1 = cq_1$, $k_2 = cq_2$, where q_1 and q_2 are the coefficients of conductivity of the strips along $x = 1$ and $z = 1$ in the z and x directions respectively.

The assumption made above of a narrow (relative to the size of the slab) strip of material whose temperature varies in only one direction may be closely realized in practice. For example, consider a material such as concrete which has imbedded in it pipes containing moving water. Also, consider a laminated material such as a brick wall.

In the solution of the problem we are led to the classical method of separation of variables, since, because of the boundary conditions at $x = 1$ and $z = 1$ and the non-constant coefficient in the differential equation, there is little hope for success in the employment of a finite Fourier transform or the Laplace transform. In deriving solutions of the homogeneous conditions of the form $U(x,z) = y(x)w(z)$ we obtain the following eigenvalue problem in $y(x)$:

$$\begin{gathered} y''(x) + [\lambda - cr(x)]y(x) = 0, \\ y(0) = 0, \\ ck_1 r(1)y(1) - y'(1) - k_1 y''(1) = 0. \end{gathered} \tag{2}$$

*Received July 21, 1952. This paper is a condensation of parts of the author's doctoral dissertation "Modified Sturm-Liouville problems and associated integral transforms" written at the University of Michigan, 1951, under Professor R. V. Churchill whose advice and suggestions are gratefully acknowledged.

**Throughout this paper function values at an end point of an interval shall mean the limit the function assumes as the point is approached from the interior of the interval.

In order to justify the classical method of satisfying the non-homogeneous boundary condition $U(x,0) = F(x)$ of problem (1) by combining solution of the type $y(x)w(z)$, an expansion theorem for the eigenvalue problem (2) is required. In this paper we shall deal with properties of and expansion theorems for problems of which type (2) is typical.

2. The modified Sturm-Liouville problem. The eigenvalue problem (2) is a special case of the problem consisting of the Sturm-Liouville equation

$$y''(x) + [\lambda + p(x)]y(x) = 0 \tag{3}$$

in which $p(x)$ is a real, continuous function for $0 \leqq x \leqq 1$, with the boundary conditions

$$\begin{aligned} U_0[y] &\equiv a_1 y(0) + a_2 y'(0) + a_3 y''(0) = 0, \\ U_1[y] &\equiv b_1 y(1) + b_2 y'(1) + b_3 y''(1) = 0 \end{aligned} \tag{4}$$

in which the a's and the b's are real constants. In view of the differential equation (3), the boundary conditions may be written in the alternative form

$$\begin{aligned} U_{\lambda 0}[y] &\equiv \{a_1 - a_3[\lambda + p(0)]\}y(0) + a_2 y'(0) = 0, \\ U_{\lambda 1}[y] &\equiv \{b_1 - b_3[\lambda + p(1)]\}y(1) + b_2 y'(1) = 0. \end{aligned} \tag{5}$$

In this form the boundary conditions are of the form of the usual Sturm-Liouville problem except for the appearance of the parameter λ in the coefficients.

For two eigenfunctions y_i and y_j corresponding to distinct eigenvalues λ_i and λ_j we obtain by a familiar procedure the Green's formula

$$(\lambda_i - \lambda_j) \int_0^1 y_i y_j \, dx = [y_i y_j' - y_j y_i']_0^1 .$$

Supposing that $a_2 \neq 0$ and $b_2 \neq 0$ we substitute from (5) into the right member and obtain the orthogonality relation

$$\int_0^1 y_i y_j \, dx - \frac{a_3}{a_2} y_i(0) y_j(0) + \frac{b_3}{b_2} y_i(1) y_j(1) = 0. \tag{6}$$

It is easily verified that if $a_2 = 0$ or $b_2 = 0$ the term of the orthogonality relation involving that quantity does not appear.

Expansion theorems for various cases of the problem (3)(4), or the equivalent problem (3)(5), have been established by Langer [7], Gaskell [4], and Churchill [1]. Boundary value problems giving rise to the eigenvalue problem are described in the first two of these papers. Among these three papers, only Churchill's deals with the problem* involving the differential equation with non-constant coefficients. In this paper we shall establish an expansion theorem for the problem having non-constant coefficients in the differential equation. The author believes that the theoretical development leading up to the expansion theorem as well as the theorem itself helps fill the gap in theory between eigenvalue problems which involve the parameter in the coefficients of the boundary conditions and those which do not.

*Another paper by Langer [8] presents, in another way, derivations of some properties of eigenvalue problems more general than ours.

3. Definitions and properties of normal and positive-definite eigenvalue problems. We make the following definitions:

Definition 1. A function $u(x)$ shall be called a "V-function" if it is real, not identically zero, of class C^2, and satisfies the boundary conditions (4).

Definition 2. The inner product corresponding to the eigenvalue problem (3), (4) for two bounded integrable functions $f(x)$ and $g(x)$ is defined as follows:

$$[f, g] = \int_0^1 fg\, dx - \frac{a_3}{a_2} f(0)g(0) + \frac{b_3}{b_2} f(1)g(1).$$

We note here that the inner product is defined for any two V-functions. Also, the orthogonality relation (6) is simply $[y_i, y_j] = 0$ and the Fourier coefficient with respect to the function $f(x)$ and corresponding to the eigenfunction $y_i(x)$ is written formally as

$$\frac{[f, y_i]}{[y_i, y_i]}.$$

The inner product has the important properties of linearity and commutativity with respect to its functional arguments.

Definition 3. The eigenvalue problem (3), (4) is "normal" if for every V-function $u(x)$ it is true that

$$[u, u] > 0.$$

It is easy to verify that a necessary and sufficient condition that the problem (3), (4) is normal is that $a_2a_3 \leqq 0$ and $b_2b_3 \geqq 0$.

Theorem 1. If the eigenvalue problem (3), (4) is normal, it has only real and simple eigenvalues and real eigenfunctions.

Proof. Suppose λ is a complex eigenvalue, y the corresponding eigenfunction, and the imaginary part of λ is not zero. Then $\bar{y}$, the conjugate of y, is an eigenfunction and, according to the orthogonality relation (6), we have $[y,\bar{y}] = 0$. But this is a contradiction because the eigenvalue problem is normal ($a_2a_3 \leqq 0$, $b_2b_3 \geqq 0$) and the integral term of the inner product is greater than zero while the last two terms are greater than or equal to zero. Therefore, if the problem (3), (4) is normal it has only real eigenvalues.

Suppose next that $u(x)$ and $v(x)$ are two eigenfunctions both corresponding to the same real eigenvalue. Since both of these functions are solutions of (3), the Wronskian of the two functions is a constant. If this constant is evaluated by means of either boundary condition in (5), it is found to be equal to zero. The functions u and v are, therefore, linearly dependent. That is, any eigenvalue of (3), (4) is simple in the sense that there cannot be two linearly independent eigenfunctions corresponding to it. It follows immediately that any complex eigenfunction $u(x) + iv(x)$ is equal to $(k + i)v(x)$ for some k; any eigenfunction is real up to a constant factor.

Let us now prove that the eigenvalues are simple in the sense that they are not double roots of the characteristic equation. Let $y(x,\lambda)$ be a solution of the differential equation (3) such that it satisfies the boundary conditions

$$U_{\lambda 0}[y(x, \lambda)] = 0, \; y(1, \lambda) = 1.$$

The function $y(x,\lambda)$ and its first and second derivatives are continuous functions of x and analytic functions of λ, for the system defining $y(x,\lambda)$ is an initial value problem with a complex parameter λ, and the coefficients of the problem are analytic functions of λ. Therefore we may expand $y(x,\lambda)$ in the following Taylor series about the eigenvalue λ_i :

$$y(x, \lambda) = y_i(x) + \sum_{n=1}^{\infty} A_n(x, \lambda_i)(\lambda - \lambda_i)^n, \tag{7}$$

where $y_i(x) = y(x,\lambda_i)$ is the eigenfunction corresponding to λ_i . Similar expressions can be written for $y'(x,\lambda)$ and $y''(x,\lambda)$.

In what follows we shall need to know that the coefficient of $(\lambda - \lambda_i)$ in the expansion of $y'(x,\lambda)$ is the same as $A_1'(x,\lambda_i)$. This is true if

$$\frac{\partial}{\partial\lambda}\frac{\partial}{\partial x} y(x, \lambda) = \frac{\partial}{\partial x}\frac{\partial}{\partial\lambda} y(x, \lambda) \tag{8}$$

which, in turn, is true if $(\partial/\partial\lambda)(\partial/\partial x)y(x,\lambda)$ is a continuous function of x and λ. Since $y'(x,\lambda)$ is analytic in λ,

$$\frac{\partial}{\partial\lambda} y'(x, \lambda) = \frac{1}{2\pi i}\int_C \frac{y'(x, \lambda)}{(z - \lambda)^2}\, dz,$$

where C is a finite closed path around λ. This expression shows that because $y'(x,\lambda)$ is continuous in x and λ, $(z - \lambda)^2$ is bounded from zero, and C is finite, $(\partial/\partial\lambda)y'(x,\lambda)$ is continuous in x and λ and (8) is valid. In a similar way it can be shown that $A_1''(x,\lambda_i)$ is the same as the coefficient of $(\lambda - \lambda_i)$ in the expansion of $y''(x,\lambda)$.

The problem which defines $y(x,\lambda)$ can be written as follows:

$$\begin{gathered} y'' + (\lambda_i + p)y = -(\lambda - \lambda_i)y, \\ \{a_1 - a_3[\lambda_i + p(0)]\}y(0, \lambda) + a_2 y'(0, \lambda) = a_3(\lambda - \lambda_i)y(0, \lambda), \\ y(1, \lambda) = 1. \end{gathered} \tag{9}$$

Into this problem we substitute the series (7) and the corresponding series for $y'(x,\lambda)$ and $y''(x,\lambda)$ and make use of the results of the preceding paragraph. The conditions in (9) hold for all λ and we may therefore equate coefficients of $(\lambda - \lambda_i)$ to obtain

$$\begin{gathered} A_1''(x, \lambda_i) + [\lambda_i + p(x)]A_1(x, \lambda_i) = -y_i(x), \\ a_2 A_1'(0, \lambda_i) = a_3 y_i(0), \\ A_1(0, \lambda_i) = 0. \end{gathered} \tag{10}$$

We multiply the differential equation in (10) by $y_i(x)$ and the differential equation $y_i'' + (\lambda_i + p)y_i = 0$ by $A_1(x,\lambda_i)$, subtract the two, integrate and obtain

$$-\int_0^1 [y_i(x)]^2\, dx = [A_1' y_i - A_1 y_i']_0^1 .$$

Making use of the fact that the eigenfunction y_i satisfies the boundary conditions $U_{\lambda_i 0} = U_{\lambda_i 1} = 0$ we suppose that $a_2 \neq 0$ and $b_2 \neq 0$ and evaluate the right member by means of the boundary conditions in (5) and the two boundary conditions in (10). Thus we obtain

$$[y_i , y_i] - \frac{b_3}{b_2}[y_i(1)]^2 = \frac{y_i(1)}{b_2} U_{\lambda_i 1}[A_1(x, \lambda_i)]. \tag{11}$$

Since the function $y(x,\lambda)$ satisfies the first boundary condition in (5) for all λ the characteristic equation of the eigenvalue problem is

$$U_{\lambda 1}[y(x, \lambda)] = 0.$$

The derivative with respect to λ of this left member when evaluated at $\lambda = \lambda_i$ is

$$U_{\lambda_i 1}[A_1(x, \lambda_i)] - b_3 y_i(1).$$

We see from (11) that if this were zero we would have $[y_i , y_i] = 0$ which is a contradiction in view of the fact that the eigenvalue problem is normal. The proof follows through with minor changes if $a_2 = 0$ or $b_2 = 0$. This completes the proof of Theorem 1.

We next define the Rayleigh's quotient $R(u)$ for any V-function $u(x)$:

$$R(u) = \frac{-[u'' + pu, u]}{[u, u]}.$$

By taking the inner product of the left member of the equation $y_i'' + (\lambda_i + p)y_i = 0$ with y_i and solving for λ_i we see that $R(y_i) = \lambda_i$; the Rayleigh's quotient for the eigenfunction $y_i(x)$ (a V-function) is the corresponding eigenvalue λ_i .

Definition 4. The problem (3), (4) shall be called "positive definite" if it is normal and if $[u'' + pu,u] < 0$ for every V-function $u(x)$.

It follows that a positive-definite eigenvalue problem has only positive eigenvalues. By performing an integration by parts we find that if $p(x) < 0$ for $0 \leqq x \leqq 1$ the conditions $a_2a_3 \leqq 0$, $b_2b_3 \geqq 0$, $a_1a_2 \leqq 0$, and $b_1b_2 \geqq 0$ guarantee that the problem is positive-definite. Other combinations of conditions which guarantee a positive-definite problem can be found.

It is easily verified that the eigenvalue problem (2) arising from the boundary value problem (1) is positive-definite because in that problem $r(x) > 0$ for $0 \leqq x \leqq 1$ since r is the thermal emissivity of the faces of the slab, and k_1 and k_2 are positive since q_1, q_2 and $1/c$ are positive since they are coefficients of conductivity of the materials involved. Hence everything that has been said and will be said concerning positive-definite problems holds for the problem (2).

4. The Green's function solution of the non-homogeneous problem. Consider the following non-homogeneous problem with the complex parameter λ:

$$\begin{aligned} y''(x, \lambda) + [\lambda + p(x)]y(x, \lambda) &= f(x), \\ U_0[y(x, \lambda)] &= 0, \\ U_1[y(x, \lambda)] &= 0, \end{aligned} \tag{12}$$

where $f(x)$ is a bounded integrable function. Except for the function $f(x)$ in the differential equation this is the eigenvalue problem (3)(4). We make the stipulation here that if in the boundary conditions $a_2 = 0(b_2 = 0)$ then $a_3 = 0(b_3 = 0)$. Making use of the differential equation we may rewrite the boundary conditions in the form

$$\begin{aligned} U_{\lambda 0}[y(x, \lambda)] &= -a_3 f(0), \\ U_{\lambda 1}[y(x, \lambda)] &= -b_3 f(1). \end{aligned} \tag{13}$$

It is well known (see, for example, [5] p. 257) that if λ is not an eigenvalue of (3)(4) the solution of (12) with the boundary conditions in the form (13) is

$$y(x, \lambda) = \int_0^1 f(\xi)G(x, \xi, \lambda)\, d\xi - a_3 f(0)G_0(x, \lambda) - b_3 f(1)G_1(x, \lambda). \tag{14}$$

In this equation $G(x,\xi,\lambda)$ is the Green's function of the system and $G_0(x,\lambda)$ is the solution of the corresponding homogeneous problem except that the right member of the first condition in (13) is replaced by 1. Similarly, $G_1(x,\lambda)$ is the solution of the corresponding homogeneous problem except that the second condition in (13) is replaced by 1.

Since the Green's function satisfies the homogeneous problem we have

$$U_{\lambda 0}[G(x, \xi, \lambda)] = \lim_{x\to 0} \{a_1 - a_3[\lambda + p(0)]G(x, \xi, \lambda) \qquad 0 < \xi \leqq 1$$
$$-a_2 G_x(x, \xi, \lambda)\} = 0$$

which gives

$$U_{\lambda 0}[\lim_{\xi\to 0} G(x, \xi, \lambda)] = U_{\lambda 0}[\lim_{\xi\to 0} G(x, \xi, \lambda)] - \lim_{\xi\to 0} U_{\lambda 0}[G(x, \xi, \lambda)]$$

the last term being zero. Because G is continuous and G_x has a unit jump across the line $x = \xi$ we see from this last expression that

$$U_{\lambda 0}[\lim_{\xi\to 0} G(x, \xi, \lambda)] = a_2 .$$

Because of the properties of the Green's function the function $\lim_{\xi\to 0} G(x,\xi,\lambda)$ satisfies the homogeneous differential equation corresponding to the one in (12) and the second boundary condition in (13). The function is also continuous and has a continuous derivative. Recalling the definition of $G_1(x,\lambda)$ we see that, except for a multiplicative constant, this function must be identical with the function $\lim_{\xi\to 0} G(x,\xi,\lambda)$. More precisely, if $a_2 \neq 0$ we have

$$G_0(x, \lambda) = \lim_{\xi\to 0} \frac{G(x, \xi, \lambda)}{a_2}.$$

In case $b_2 \neq 0$ a similar argument yields

$$G_1(x, \lambda) = -\lim_{\xi\to 1} \frac{G(x, \xi, \lambda)}{b_2}.$$

Therefore in view of (14) and our definition of the inner product the solution of (12) may be written

$$y(x, \lambda) = [f(\xi), G(x, \xi, \lambda)]. \tag{15}$$

According to Collatz [3], as long as λ_i is a simple eigenvalue the Green's function has a simple pole at $\lambda = \lambda_i$ and may be written

$$G(x, \xi, \lambda) = \frac{C_i y_i(x) y_i(\xi)}{(\lambda - \lambda_i)} + G_i^*(x, \xi, \lambda),$$

where G_i^* is analytic at $\lambda = \lambda_i$ and C_i is a constant. The quantity $C_i y_i(x) y_i(\xi)$ is, therefore, the residue of G at the simple pole $\lambda = \lambda_i$. It follows that

$$\lim_{\lambda\to\lambda_i} \{(\lambda - \lambda_i)[y_i(\xi), G(x, \xi, \lambda)]\} = C_i y_i(x)[y_i(\xi), y_i(\xi)]. \tag{16}$$

Using the fact that

$$y_i''(x) + [\lambda + p(x)]y_i(x) = (\lambda - \lambda_i)y_i(x)$$

and the result (15) which gives an implicit solution for $y_i(x)$, we see that the left member in (16) is simply $y_i(x)$. Therefore

$$C_i = \frac{1}{[y_i(\xi), y_i(\xi)]}.$$

Thus the residue of G at $\lambda = \lambda_i$ is completely determined and is found to be the term of the Fourier series of $f(x)$ corresponding to the eigenvalue λ_i. If $\lambda_i (i = 0, 1, \cdots, n)$ is the set of eigenvalues inside the circle $|\lambda| = R$ the meromorphic function G may be written

$$G(x, \xi, \lambda) = \sum_{i=0}^{n} \frac{y_i(x)y_i(\xi)}{(\lambda - \lambda_i)[y_i(\xi), y_i(\xi)]} + G_R^*(x, \xi, \lambda).$$

where G_R^* is an analytic function inside the circle $|\lambda| = R$. Also, in view of (15), the solution of the non-homogeneous problem (12) may be written

$$y(x, \lambda) = \sum_{i=0}^{n} \frac{y_i(x)[f(\xi), y_i(\xi)]}{(\lambda - \lambda_i)[y_i(\xi), y_i(\xi)]} + [f(\xi), G_R^*(x, \xi, \lambda)].$$

A note concerning an expansion theorem for V-functions. With the developments of this section and sections 2 and 3, the groundwork has been laid for the proofs of the existence of an infinite number of eigenvalues and an expansion theorem for V-functions for the problem (3)(4) which is positive definite. The proofs are analogous to those given by E. Kamke [6] for problems which do not involve the parameter in the boundary conditions. Because of the analogy with Kamke's work and because the expansion theorem we shall prove in the next section is more general, the proofs are omitted.

5. An expansion theorem by the Laplace transform. In this section we shall obtain an expansion theorem by the application of the Laplace transform to a problem in heat conduction. Let us consider the transient temperatures $U(x,t)$ of a thin slab in the shape of a rectangular parallelopiped which has its upper and lower edges insulated. At the edge $x = 1$ the slab is in perfect thermal contact with a well stirred liquid the container of which is exposed to an external medium at temperature zero. The edge $x = 0$ is kept at temperature zero. If the flat faces transfer heat into the surrounding medium according to a linear law and with thermal emissivity $p(x)$, and if the initial temperature of the slab is $F(x)$, the boundary value problem may be written as

$$U_t(x, t) = U_{xx}(x, t) - p(x)U(x, t),$$

$$U(0, t) = 0,$$

$$K_1U(1, t) + U_x(1, t) + K_2U_t(1, t) = 0,$$

$$U(x, 0) = F(x),$$

where the units are adjusted so that the thermal diffusivity equals 1. In this statement of the problem $K_1 = qA_2/KA_1$ and $K_2 = hM/KA_1$ and $K_2 = hM/KA_1$ where the constant q is the thermal emissivity at the end of the container, K is the coefficient of conductivity of the slab, h the specific heat of the liquid, M its mass, A_1 the area of the slab in contact with the liquid, and A_2 the area of the container in contact with the external medium. We note therefore that $K_1 > 0$, $K_2 > 0$; also, $p(x) > 0$ for $0 \leqq x \leqq 1$. Except for the variable coefficient of heat transfer of the slab which introduces the non-constant coefficient in the differential equation, this is essentially the problem considered by Langer [7] and Gaskell [4].

The application of the Laplace transform

$$L\{F(x)\} \equiv f(s) \equiv \int_0^\infty e^{-sx}F(x)\,dx$$

to the boundary value problem gives the following transformed problem:

$$\begin{aligned} u''(x, s) - [s + p(x)]u(x, s) &= -F(x), \\ u(0, s) &= 0, \\ (K_1 + K_2 s)u(1, s) + u'(1, s) &= K_2 F(1). \end{aligned} \tag{17}$$

We note that the homogeneous problem corresponding to this is a case of the eigenvalue problem (3)(4). From the results of Section 4 we see that the solution to this problem may be written

$$u(x, s) = -[F(\xi), G(x, \xi, s)], \tag{18}$$

where $G(x,\xi,s)$ is the Green's function for the system. From the definition of K_2 we noted that $K_2 > 0$ and hence the corresponding eigenvalue problem is normal; the problem has only real and simple eigenvalues s_n. According to the results of Section 4, $u(x,s)$ has simple poles at those eigenvalues and the residues of those poles are the terms of the Fourier series corresponding to $F(x)$.

The existence of an infinite set of eigenvalues and an expansion theorem for the eigenvalue problem can now be proved with the aid of a close examination of the order properties of various terms involved in the solution (18). In this proof we shall deal with the Laplace inversion integral

$$L_i^{-1}\{f(s)\} \equiv \lim_{\beta\to\infty} \frac{1}{2\pi i}\int_{\gamma-i\beta}^{\gamma+i\beta} e^{zt}f(z)\,dz$$

and make use of the following theorem (see [2], p. 159):

Theorem 5. Let $v(s)$ be a function of a complex variable s and of the order $O(s^{-k})$ in a half plane $\mathcal{R}(s) \geqq \gamma_0$ where $k > 1$. Then the inversion integral $L_i^{-1}\{v(s)\}$ converges along a line γ where $\gamma \geqq \gamma_0$ to a continuous function $V(t)$ which is independent of γ and such that $V(0) = 0$.

Let us now define functions $u_1(x,s)$ and $u_2(x,s)$ which are linearly independent solutions of the homogeneous differential equation corresponding to the one in (17) and are such that $u_1(0,s) = 0$ and $u_1'(0,s) = 1$, while $(K_1 + K_2 s)u_2(1,s) + u_2'(1,s) = 0$ and $u_2(1,s) = 1$. We write the initial value problem which defines $u_2(x,s)$ in the form

$$\begin{aligned} u_2''(x, s) - su_2(x, s) &= p(x)u_2(x, s), \\ u_2(1, s) &= 1, \\ u_2'(1, s) &= -(K_1 + K_2 s). \end{aligned}$$

By supposing that the right member of the differential equation is a known function, we find that the solution $u_2(x,s)$ may be written in the following implicit form:

$$u_2(x, s) = \frac{\sinh (xs^{1/2})}{\sinh (s^{1/2})} + \sinh [(1 - x)s^{1/2}]\left[\coth (s^{1/2}) + \frac{K_1 + K_2 s}{s^{1/2}}\right] - \frac{1}{s^{1/2}} \int_x^1 \sinh [(x - \xi)s^{1/2}]p(\xi)u_2(\xi, s)\, d\xi. \tag{19}$$

It can now be shown that the function

$$M(s) = \operatorname*{Max}_{0 \leq x \leq 1} \left| \frac{s^{1/2} \sinh (x^{1/2}) u_2(x, s) \exp [-(2 - x)s^{1/2}]}{K_1 + K_2 s} \right|$$

is bounded for R sufficiently large where $s = Re^{i\theta}$. Since $M(s)$ is bounded we can see from (19) that

$$\frac{2s^{1/2} \sinh (s^{1/2})u_2(0, s) \exp (-2s^{1/2})}{K_1 + K_2 s}$$

$$= [1 - \exp (-2s^{1/2})]\{1 - \exp (-2s^{1/2}) + [1 + \exp (-2s^{1/2})]O(s^{-1/2})\} + O(s^{-1/2}). \tag{20}$$

The zeroes of the function $u_2(0,s)$ are the eigenvalues as can be seen from the definition of $u_2(x,s)$. Since these eigenvalues are real we see from expression (20) that they cannot be large and positive. Therefore we set $s = -\rho^2$ where ρ is real and find that the characteristic equation $u_2(0,s) = 0$ takes the form

$$\sin \rho\left[\sin \rho + O\left(\frac{1}{\rho}\right) \cos \rho\right] + O\left(\frac{1}{\rho}\right) = 0$$

The applications of Rouche's theorem incomplex variable theory gives the fact that there exists an infinite number of $s_n (n = 0, 1, \cdots)$, that they are less than a fixed number γ_0, and that when n gets large $(-s_n)^{1/2}$ approaches $n\pi$. Since $u_2(0,s)$ is an analytic function it has only a finite number of zeroes in any finite region of the complex plane. Thus the existence of a denumerable set of eigenvalues is determined.

Treating the function $u_1(x,s)$ in a manner similar to that above for $u_2(x,s)$, the following result analogous to (20) can be obtained:

$$\frac{2s^{1/2}}{K_1 + K_2 s} [(K_1 + K_2 s)u_1(1, s) + u_1'(1, s)] = 1 - e^{(-2s^{1/2})} + O(s^{-1/2}) \tag{21}$$

In terms of the functions $u_1(x,s)$ and $u_2(x,s)$ the Green's function of the system (17) appearing in expression (18) may be written

$$G(x, \xi, s) = \frac{-u_2(\xi, s)u_1(x, s)}{u_2(0, s)}, \qquad 0 \leqq x < \xi,$$

$$= \frac{-u_1(\xi, s)u_2(x, s)}{u_2(0, s)}, \qquad \xi < x \leqq 1.$$

If we next make use of the properties of the Green's function and assume that $F(x)$ is continuous and has a sectionally continuous first derivative, we may integrate expression

(18) by parts to obtain

$$su(x, s) - F(x) = F(1)G_\xi(x, 1, s) - F(0)G_\xi(x, 0, s) - \int_0^1 G_\xi(x, \xi, s)F(\xi)\,d\xi$$
$$- \int_0^1 G_\xi(x, \xi, x)F(\xi)p(\xi)\,d\xi - \frac{K_2 sF(1)u_1(x, s)}{(K_1 + K_2 s)u_1(1, s) + u_1'(1, s)}.$$

Because the quantities $u_2(0,s)$ and $(K_1 + K_2 s)u(1,s) + u_1'(1,s)$ appear in the denominators of the various factors and integrands above, the application of the results (20) and (21) yields

$$u(x, s) - \frac{F(x)}{s} = O(s^{-3/2}) \qquad \Re(s) \geqq \gamma_0,\ 0 < \epsilon \leqq x \leqq 1 - \epsilon$$

where ϵ is a fixed but arbitrary number. Thus we see that the conditions of Theorem 5 are satisfied. Let us designate the residue of $u(x,s)$ at $s = s_n$ by $R_n(x)$. If we suppose for the moment that the Laplace inversion integral of $u(x,s) - F(x)/s$ can be represented by the sum of the residues of the integrand we may write

$$L_t^{-1}\{u(x, s) - F(x)/s\} = \sum_{n=0}^{\infty} e^{s_n t}[R_n(x)] - F(x), \qquad t \geqq 0 \tag{22}$$

and therefore, according to Theorem 5,

$$\sum_{n=0}^{\infty} R_n(x) = F(x).$$

Because we have shown in Section 4 that the residues $R_n(x)$ are the terms of the Fourier series corresponding to $F(x)$, the proof of the expansion theorem rests on the proof that the equation (22) is valid, that the inversion integral can be represented by the sum of the residues of its integrand.

Since the numbers $(-s_n)^{1/2}$ approach $n\pi$ as n gets large, the parabolas P_n given by

$$r = \left(n + \frac{1}{2}\right)^2 \pi^2 \csc^2 \frac{\theta}{2} \qquad (n = 1, 2, \cdots)$$

pass between the poles of $u(x,s)$ when n is large. It can be shown that when s is a point on the parabola P_n and is such that $R(s) \leqq \gamma$

$$\left| u(x, s) - \frac{F(x)}{s} \right| \leq \frac{C}{(n + 1/2)^2} \tag{23}$$

for n sufficiently large where C is a positive constant; also, on the parabola P_n when θ is restricted to the range $-\pi < \theta_0 \leqq \theta \leqq \theta_0 < \pi$

$$\left| u(x, s) - \frac{F(x)}{s} \right| \leq \frac{D(\theta_0, \epsilon)}{(n + 1/2)^3}$$

where the positive number D, as indicated, depends on θ_0 and ϵ and gets large as θ_0 approaches π. In making these estimates we assume that $F(x)$, in addition to being continuous and having a sectionally continuous first derivative, has a sectionally continuous second integral around the closed path consisting of the parabola P_n and the line $\Re(s) = \gamma$ equals the sum of the residues of the poles enclosed. The order properties

(23) and (24) are sufficient to guarantee that the integral of $u(x,s) - F(x)/s$ over P_n goes to zero uniformly in x for $0 < \epsilon \leqq x \leqq 1 - \epsilon$ with increasing n. In other words, the inversion integral of $u(x,s) - F(x)/s$ may be represented by the sum of the residues of the integrand.

Thus we have proved that $F(x)$ may be expanded in the series of eigenfunctions of the problem

$$\begin{aligned} y''(x) - [\lambda + p(x)]y(x) &= 0, \\ y(0) &= 0, \\ (K_1 + K_2)y(1) + y'(1) &= 0, \end{aligned} \tag{25}$$

where K_1 and K_2 are constants with $K_2 > 0$, and $p(x)$ is any continuous function on $0 \leqq x \leqq 1$. The expansion is

$$F(x) = \sum_{i=0}^{\infty} A_i y_i(x), \tag{26}$$

where

$$A_i = \frac{[F(x),\, y_i(x)]}{[y_i(x),\, y_i(x)]}$$

in which the inner product is the one corresponding to problem (25). Our main results may be summed up with the following theorem:

Theorem 6. The eigenvalues λ_i of problem (25) are real and simple, and there exists a real number γ_0 such that $\lambda_i \leqq \gamma_0$ for all λ_i and the numbers $(-\lambda_i)^{1/2}$ approach $i\pi$ as i gets large. For any continuous function $F(x)$ such that $F'(x)$ and $F''(x)$ are sectionally continuous on $0 \leqq x \leqq 1$ the expansion (26) is valid on $0 < x < 1$ and the convergence is uniform on any closed subinterval of $0 < x < 1$.

If in (25) $K_2 = 0$ the problem reduces to a standard type Sturm-Liouville problem for which expansion theorems are well known (see, for example, [2], p. 259). Although (25) is a special case of problem (3)(4), only minor changes are necessary in the proof given in this section to extend it to the more general problem.

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