AN ASYMMETRICAL FINITE DIFFERENCE NETWORK*

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Introduction. Finite difference techniques have been used extensively in recent years in the solution of two-dimensional second order boundary value problems that have proved to be intractable by other methods. The differential equation is replaced by a system of linear algebraic equations, the solution of which gives the values of the wanted function at a finite number of points lying at the intersections of a gridwork. The use of regular polygons, either squares or equilateral triangles, in the formation of these gridworks has the desirable property that the equations associated with each node (intersection) point have a particularly simple, symmetrical form that is identical for all interior points. There are, however, two troublesome problems which arise in connection with the use of regular polygons. The first of these arises when the region has curved boundaries. In such cases some node points near the boundary will be connected to the boundary by gridwork elements of irregular lengths, necessitating the use of special equations for these points. The second problem concerns the change of mesh size at points within the boundary. It is frequently uneconomical from the point of view of the labor of computation to use the same mesh size at all points. In the neighborhood of a sharp corner or near other types of singularities, the mesh size must be reduced if an accuracy is to be obtained that is comparable with the accuracy of the solution in parts of the region where the behaviour of the wanted function is more uniform. Both of these problems have received attention from writers on relaxation methods and it is with these problems that the present paper is principally concerned. A method will be described by means of which the coefficients of the system of algebraic equations can be computed for an arbitrary distribution of node points. The positions of these node points can then be chosen to fit the boundary conditions and other special requirements of each problem.

In the construction of a finite difference gridwork to be used in the solution of physical problems, it is helpful to associate physical properties with the elements of the gridwork. Southwell and his co-workers have regarded the gridwork as a network of tensioned strings [1] while others have regarded the gridwork as a network of electrical elements [2]. In the case of a second order boundary value problem, this clear physical picture of the gridwork is lost if the differential equation is replaced by difference equations involving difference operators higher than the second order. In order to preserve the physical picture and to simplify the calculations, the higher order difference terms are usually regarded as corrections which are added in the final stage of calculation, if the relaxation method is to be used [3].

Another important reason for eliminating higher order difference operators arises in connection with analog computing devices, in which the physical picture of the network is realized. Electrical circuits have been used extensively in the solution of many kinds of boundary value problems [4,5,6]. In the construction of these circuits one consideration enters that is not present when the finite-difference equations are solved by purely numerical methods, namely that the circuit must be physically realizable. If the

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circuit is to be constructed of resistors only, it must contain no "negative" resistors, and the resistors must have the same resistance looked at from either end. In terms of the matrix of coefficients of the finite difference equations, the necessary and sufficient conditions that a network of resistors satisfying the equations is physically realizable are that the matrix must be symmetrical; that the nondiagonal terms in any row must all have a sign opposite to that of the diagonal term; and that the absolute value of their sum must be less than the absolute value of the diagonal term. These assertions can be easily verified by an examination of the equations of a network of resistors. Such restrictions are not imposed on purely numerical solutions of the difference equations. Since the author of this paper is primarily interested in the solution of boundary value problems by means of electrical analogy, these restrictions have been imposed on the methods to be presented. These restrictions will have the effect of eliminating difference operators of higher than the second order in the equations for the network.

The problem of changing cell size within a given rectangular gridwork has been solved by Southwell (Ref. 1, pp. 98–100) by a method which leads to network elements which are "physically unrealizable" according to the rules laid down above. A network for the solution of Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1}$$

is shown in Fig. 1. The finite difference equivalent of Laplace's equation for non-ex-

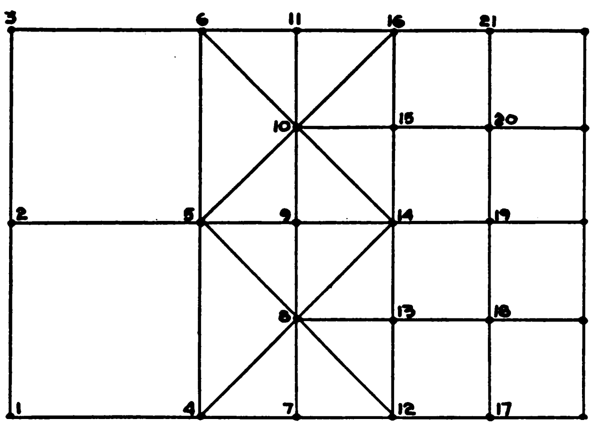

FIG. 1. Network for changing cell-size in a network of squares by Southwell's method.

ceptional points in a square gridwork is, neglecting fourth order and higher difference operators,

$$\sum_{n=1}^{4} (\phi_n - \phi_0) = 0, \tag{2}$$

where ϕ_0 is the value of the function at the point in question, and ϕ_n is the value of the function at one of the nearest neighbors of the point in question.

This equation is made to apply to exceptional points by placing upon it the following interpretation: the value of the wanted function at any point is equal to the average of its values at the vertices of a square, the center of which is at the point in question. For example in the network of Fig. 1:

$$\begin{aligned} \phi_{14} + \phi_6 + \phi_2 + \phi_4 - 4\phi_5 &= 0, \\ \phi_{14} + \phi_{16} + \phi_6 + \phi_5 - 4\phi_{10} &= 0, \\ \phi_{14} + \phi_{10} + \phi_5 + \phi_8 - 4\phi_9 &= 0. \end{aligned} \tag{3}$$

The network resulting from these equations is physically unrealizable because the matrix of coefficients is not symmetrical. For instance, the coefficient of ϕ_{10} in the third equation is $+1$ but the coefficient of ϕ_9 in the second equation is zero.

The treatment that has been given to the problem of a curved boundary by writers on relaxation methods is largely intuitive (Ref. 1, pp. 67–78). From a consideration of the equilibrium of his tensioned network of strings Southwell concludes that, in the neighborhood of a boundary where the wanted function is known, the tension of a string connecting an interior point to the boundary should be inversely proportional to the length of the string. As applied to Eq. (2) this means that each term for which ϕ_n normally would fall outside the boundary should be weighted inversely as the length of the distance between ϕ_0 and the boundary.

The treatment of boundaries along which the normal derivative of the function is specified is less simple. In terms of the physical model, the transverse load at the edge is replaced by a statically equivalent set of forces applied to nodes just inside the boundary and to "ficticious" nodes just outside the boundary as in Fig. 2. The coefficients of the terms of Eq. (2) involving "ficticious" nodes have values between zero and one. In some instances the coefficient for an element between "ficticious" nodes (e.g. the element between nodes 2 and 3 of Fig. 2) is set equal to zero and in other instances it is set equal

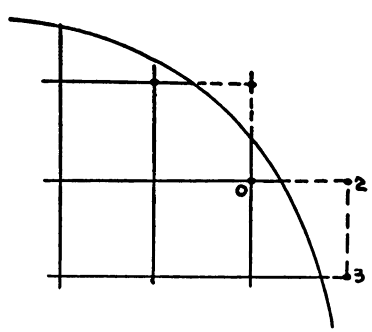

Fig. 2. Network of squares near a curved boundary.

to 1/2. In a recent article, which was largely concerned with the development of accurate network formulas, the conclusion was reached that "the writer has failed to find any completely satisfactory method of dealing accurately with boundary conditions [involving a derivative] when the direction of the normal cannot be identified with that of a mesh line as in the case of curved boundaries" [7].

The difficulty with a curved boundary is caused by the fact that for a network of regular polygons the location of node points is unalterable. By using irregular polygons it will be possible to put the node points on the boundary. In this paper the situation will be further clarified by giving a precise physical significance, in terms of the original field problem, to the terms of the resulting generalized difference equations.

Derivation of the asymmetrical network. The problem at hand is the solution of the following equation together with appropriate conditions on ϕ and its normal derivative at boundary points of a finite plane region.

$$\nabla \cdot (\sigma \nabla \phi) + \tau = 0. \tag{4}$$

The following physical interpretation may be made of the symbols appearing in this equation: ϕ is the electrical potential in a plane region of conducting material. σ is the conductivity of this material. τ is the density of currents inserted into the region from external sources. $-\sigma\nabla\phi$ is the vector density of currents flowing in the material. σ and τ may be scalar functions of position and τ may also be a linear function of ϕ.

This physical interpretation will aid in visualizing the constructions to be made.

The problem of forming an asymmetrical network whose equations will replace Eq. (4) can be stated in the following manner. Given a region in which Eq. (4) holds and a large number of points in the region chosen at random, in what way should the points be interconnected with "physically realizable" electrical resistors in order that the voltages at the nodes shall be as nearly as possible the correct solutions of the boundary value problem characterized by Eq. (4) and appropriate boundary conditions?

A unique answer cannot be given to this question at this time. A reasonable necessary condition that should be applied to the network is that for a homogeneous conductivity (σ = constant) and a uniform field ($\nabla\phi$ = constant), which is however arbitrarily oriented, the voltage at the nodes should give the exact solution of Eq. (4). It will be shown that more than one network connecting the given points can be constructed that satisfies this condition.

In the method of solution that has been chosen the first step is to connect the randomly chosen points by a network of triangles, as in Fig. 3. The network should be planar (no

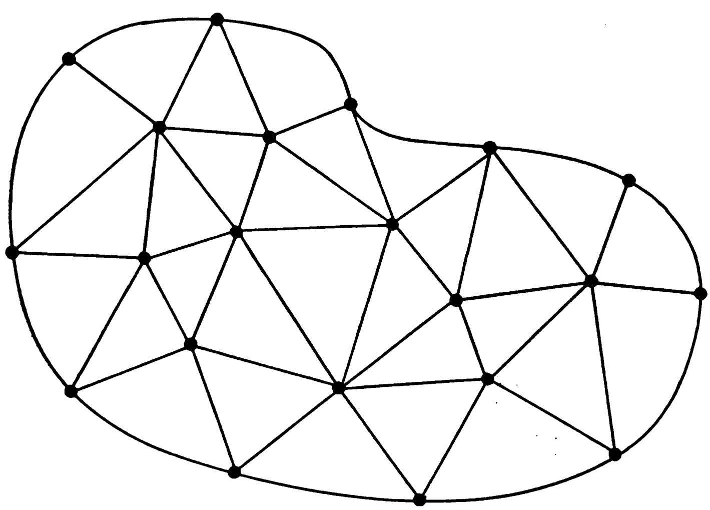

Fig. 3. Asymmetrical network of triangles.

cross-overs) and none of the interior angles of the triangles should be obtuse. It may be necessary to insert a few additional points in order to fulfill the last condition.

Consider a portion of this network shown in Fig. 4. The perpendicular bisectors of the sides of the triangles divide the region into polygons surrounding each point. A network of resistors is now constructed connecting the vertices of the triangles. The voltage across each resistor shall be interpreted as the line integral of the gradient of the potential between the two points it connects. For example, for points A and B of Fig. 4:

$$V_B - V_A = \int_A^B \nabla\phi \cdot \mathbf{dl}. \tag{5}$$

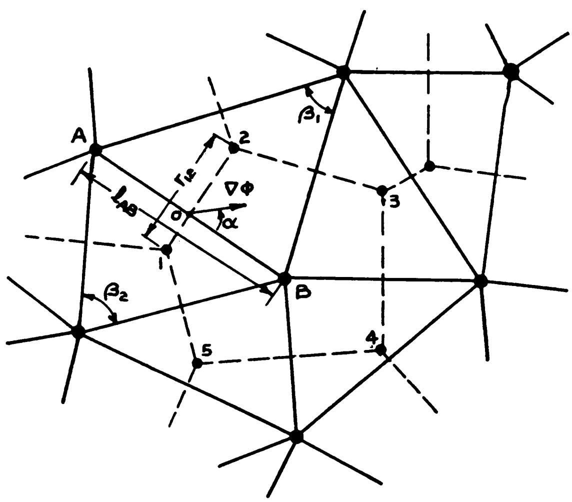

Fig. 4. Portion of the asymmetrical network of triangles.

The current in the resistor shall be interpreted as the total normal flux crossing the common boundary of the dotted polygons surrounding the two points. Since the current density is $-\sigma\nabla\phi$, we have:

$$I_{AB} = -\int_1^2 \sigma(\nabla\phi \cdot \mathbf{n})\, dr, \tag{6}$$

where $\mathbf{n}$ is a unit vector normal to dr. If $\nabla\phi$ and $\sigma\nabla\phi$ are now expanded in Taylor's series about the point 0, the midpoint of the segment $A-B$, and all terms except the first are neglected, $\nabla\phi \cong (\nabla\phi)_0$ and $\sigma\nabla\phi \cong \sigma_0(\nabla\phi)_0$. Since the segment AB is normal to the segment 1-2, the projection of $\nabla\phi$ on $\mathbf{dl}$ is the same as the projection of $\nabla\phi$ on $\mathbf{n}$. Hence

$$V_B - V_A \cong l_{AB} \mid \nabla\phi \mid_0 \cos\alpha, \tag{7}$$

$$I_{AB} \cong -\sigma_0 r_{12} \mid \nabla\phi \mid_0 \cos\alpha. \tag{8}$$

The value of the resistor connecting A and B is

$$R_{AB} = \frac{V_A - V_B}{I_{AB}} = \frac{l_{AB}}{\sigma_0 r_{12}}. \tag{9}$$

Hence R_{AB} depends only on the physical properties of the material and the manner in which the region is subdivided. If the segment 1-2 were not perpendicular to AB, the value of the resistor would depend on the orientation of the field. It can also be shown by a simple geometrical argument that

$$Y_{AB} = \frac{1}{R_{AB}} = \frac{\sigma_0}{2} (\text{ctn}\, \beta_1 + \text{ctn}\, \beta_2), \tag{10}$$

where β_1 and β_2 are the interior angles of the triangles subtended by the segment AB. If both β_1 and β_2 are acute angles R_{AB} will be physically realizable.

In addition to calculating the value of R_{AB} it is necessary to decide on an area element to be associated with the inhomogeneous term, τ, of Eq. (4). If Eq. (4) is integrated over the polygon 1-2-3-4-5 surrounding point B of Fig. 4,

$$\iint_B \nabla \cdot (\sigma \nabla \phi)\, dS + \iint_B \tau\, dS = 0. \tag{11}$$

By Gauss' integral theorem:

$$\iint_B \nabla \cdot (\sigma \nabla \phi)\, dS = \oint_B \sigma(\nabla \phi \cdot \mathbf{n})\, dr. \tag{12}$$

Hence

$$\oint_B \sigma(\nabla \phi \cdot \mathbf{n})\, dr + \iint_B \tau\, dS = 0. \tag{13}$$

If the surface integral in Eq. (13) is replaced by the value of τ measured at B multiplied by the area of the dotted polygon and the line integral is replaced by network currents from Eq. (6):

$$\sum_p I_{pB} + \tau_B A_B = 0, \tag{14}$$

where I_{pB} is the current flowing into node B from the p-th adjacent node. Eq. (14) is Kirchoff's law for the sum of the currents entering a node. It shows that the appropriate area for calculating the current to be inserted into node B is the area interior to the dotted polygon surrounding B.

By substituting Eq. (9) into Eq. (14) the generalized difference equation for node B is obtained:

$$\sum_p \sigma_0 \left(\frac{r_{Bp}}{l_{Bp}} \right) (V_p - V_B) + \tau_B A_B = 0, \tag{15}$$

where l_{Bp} is the distance between node B and node p and r_{Bp} is the length of the segment that is common to the polygons surrounding node B and node p.

The method that has been described will work for any network configuration in which the perpendicular bisectors of the branches meet at a point. Thus besides for triangles, the method will work for rectangles, regular hexagons and isosceles trapezoids. Later it will be shown that the perpendiculars to the sides need not bisect the sides, so long as they meet at common points.

From the method of derivation given here nothing can be inferred as to the accuracy of the solutions of Eq. (15), except that for a region of uniform conductivity with a constant, arbitrarily oriented potential gradient, the solutions will yield correct answers to the field problem. The magnitude of the errors will be investigated in a later section.

Applications of the asymmetrical network. The manner in which the asymmetrical network can be applied to the problem of a curved boundary is illustrated in Fig. 5. A certain number of points are placed on the boundary and lines joining boundary points are considered in the same manner as lines joining interior points, except that the conductivity of material outside the boundary is set equal to zero. If the outward normal gradient of the field, $\partial\phi/\partial n$, is specified at the boundary, an additional current equal to $\partial\phi/\partial n$ multiplied by the conductivity and the length of boundary associated with each

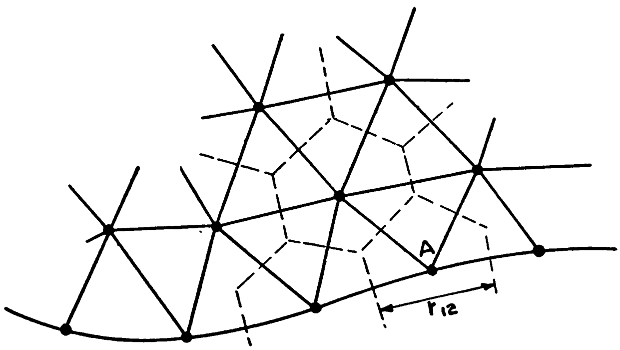

FIG. 5. Asymetrical network near a curved boundary.

node is fed into each boundary node. For point A of Fig. 5 for example this additional current is $\sigma_A[\partial\phi/\partial n]_A\, r_{12}$. This current is equal to the total flux crossing the boundary along the segment r_{12} . Hence for the asymmetrical network, boundary points are treated in almost the same manner as interior points.

The manner in which the principles of the asymmetrical network can be used to change cell size in a network of squares is illustrated in Figs. 6a and 6b. The numbers

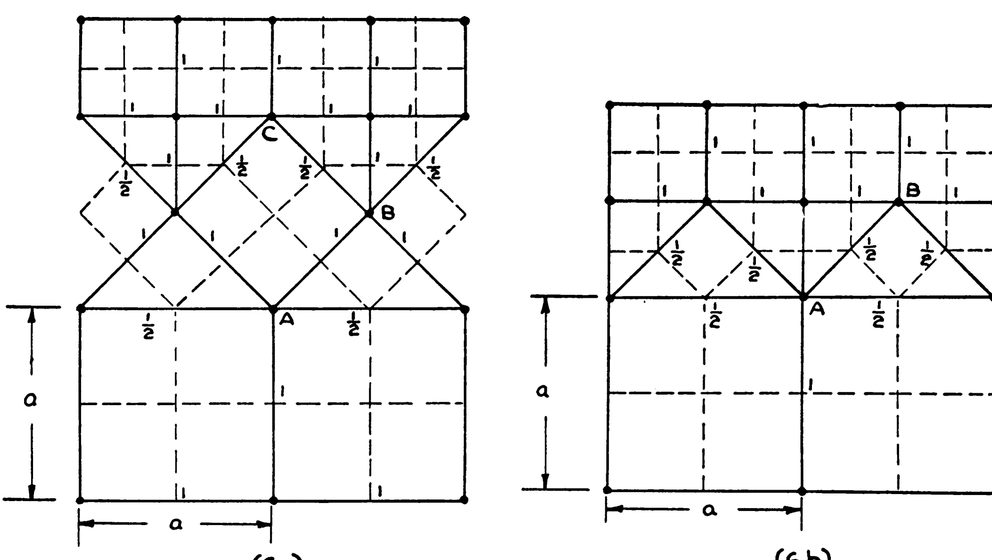

FIG. 6. Two ways in which to double cell size in a network of squares.

beside each branch of the network are the ratios r_{Bp}/l_{Bp} from Eq. (15). For example the branch connecting nodes B and C of Fig. 6a has a length $a/\sqrt{2}$ while the length of the common boundary is $a/2\sqrt{2}$ giving to r_{BC}/l_{BC} the value 1/2. It will be noted that the only values of this ratio occurring in these figures are 1/2 and 1. In contrast with the network of Fig. 1, the networks of Fig. 6 can be physically constructed and used for an analog computer solution. Another advantage that should not be underestimated is that the current in each element of the networks has a real physical significance. It represents the total normal flux crossing a known line segment.

An example of the application of the asymmetrical network to a complete problem is illustrated in Fig. 7. The problem concerns the calculation of the resonant frequencies

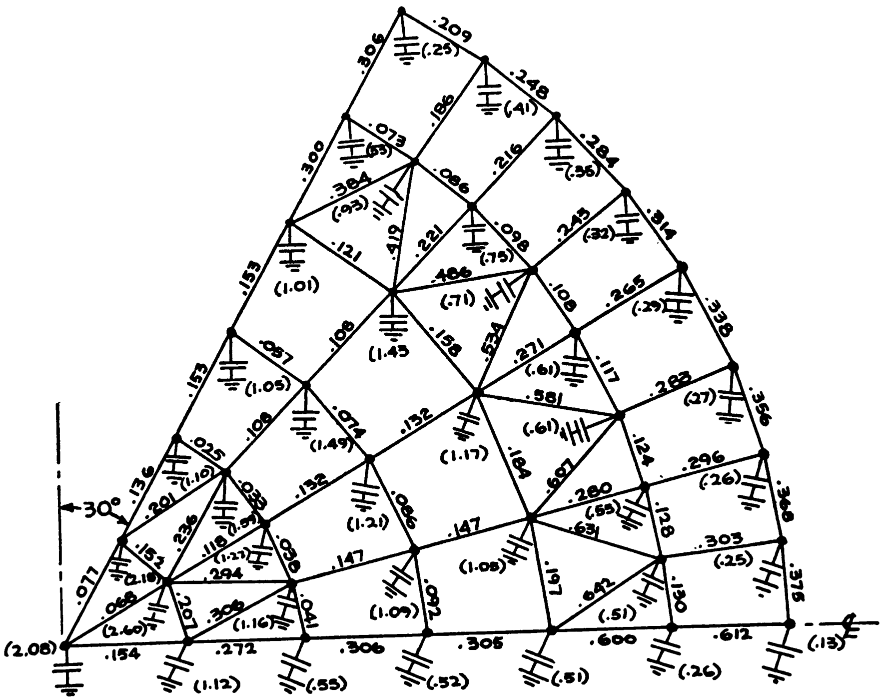

FIG. 7a. Network for the TEM modes of a conical line resonator.

and field patterns of the so-called conical-line cavity resonator, a cross-section of which is illustrated in Fig. 7c. For TEM modes (modes in which electric field lines lie in planes passing through the vertical axis and magnetic field lines are concentric circles surrounding the axis) the equation governing the variation of the magnetic field in a plane passing through the vertical axis is

$$\nabla\cdot\left(\frac{1}{\rho}\nabla H_3\right)+\frac{\lambda^2}{\rho}H_3=0, \tag{16}$$

where ρ is the perpendicular distance to the vertical axis, H_3 is the covarient component of magnetic field intensity and λ^2 is an eigenvalue related to the frequency of oscillation. The physical component H_ϕ is equal to H_3/ρ. The boundary condition on H_3 is that $\partial H_3/\partial n = 0$ along the walls of the cavity. The cavity is assumed to have the shape of a

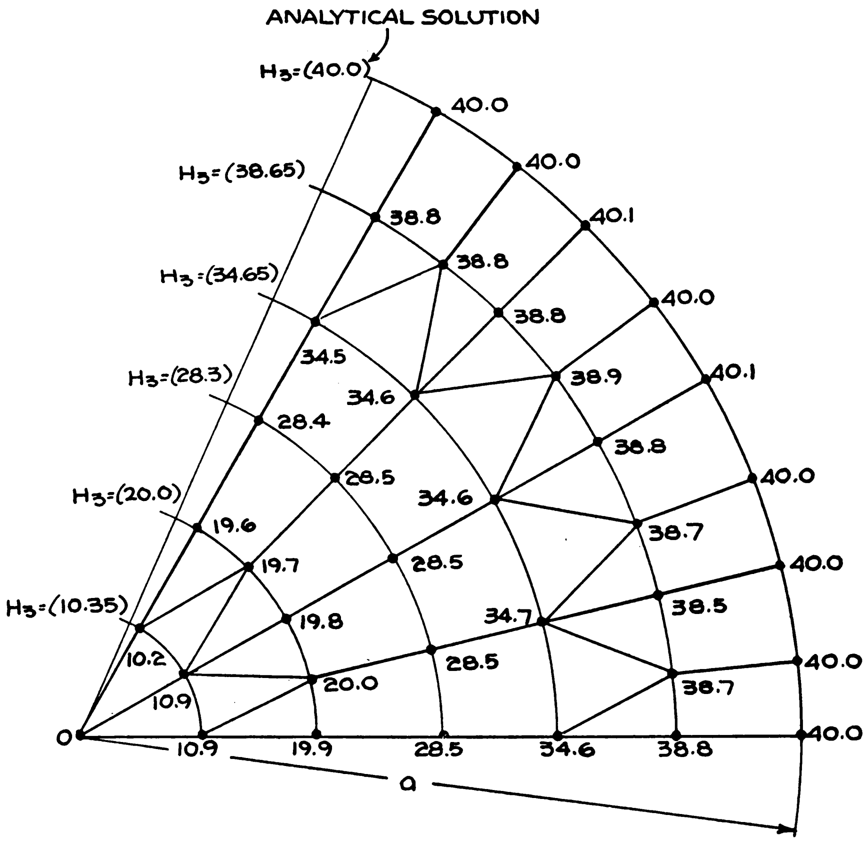

FIG. 7b. Computer solution of the lowest mode of a conical line resonator. Measured wavelength = 3.984a. Correct wavelength = 4a.

sphere where 30° conical dimples at the poles. By comparison with Eq. (4) we see that $\sigma = 1/\rho$ and $\tau = \lambda^2 H_3/\rho$.

Solutions for this problem were obtained by the electrical analogy method. An electrical circuit together with numerical values of the network elements are shown in Fig 7a. Since the coefficients of the second term in Eq. (16) is inherently positive and varies with the frequency of oscillation, a variable "negative" resistance is required for its realization. This difficulty is avoided by using inductors for the elements between nodes and capacitors for the negative elements. If the network is resonating with a frequency ω, the equation for the sum of currents at any node, B, is:

$$\sum_p \frac{1}{i\omega L_{Bp}} (V_p - V_B) - i\omega C_B V_B = 0. \tag{17}$$

Upon multiplying this equation by $i\omega$ and comparing the result with equation (15), using the values of σ and τ appropriate to this problem, it becomes apparent that:

$$L_{Bp} = \rho \frac{l_{Bp}}{r_{Bp}}, \qquad C_B = \rho A_B\,, \qquad \omega^2 = \lambda^2. \tag{18}$$

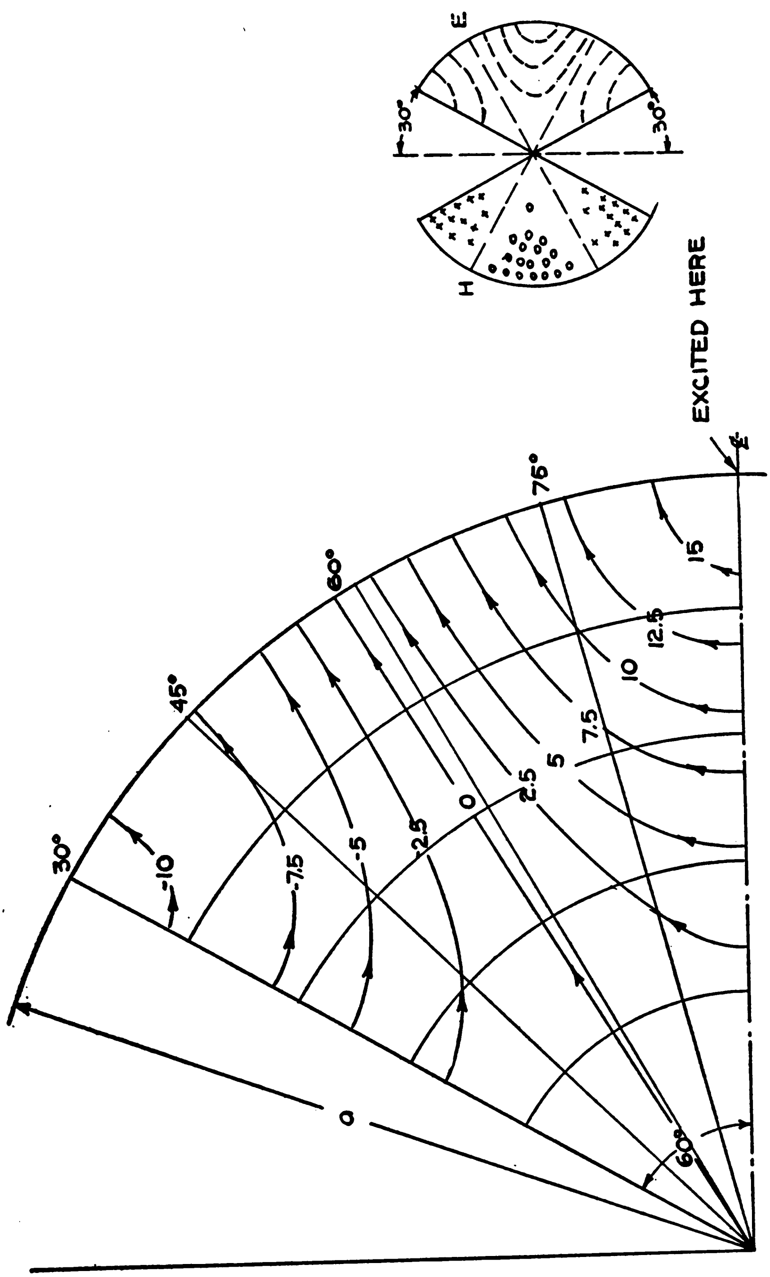

Fig. 7c. Higher TEM mode of a conical line resonator. Contours are lines of constant H_3, which are parallel to the electrical field. Measured wavelength = 1.457a.

The resonant frequencies and the corresponding eigenvalues were obtained experimentally.

The eigenfunction for the lowest mode of the conical-line resonator has the simple form

$$H_3 = \sin \frac{\pi r}{2a}, \tag{19}$$

where r is the distance from the center and a is the radius of the sphere. This solution is compared with values measured on the network in Fig. 7b. Lines of constant H_3, which are parallel to the electric field, are plotted in Fig. 7c for a higher TEM mode. In Fig. 7a it is seen that the principles of the asymmetrical network have been used to fit the location of nodes to the natural boundaries of the cavity resonator and also to effect changes in cell size so as to keep the cell area nearly constant in going from the center to the outer wall. The solutions were obtained on the California Institute of Technology Electric Analog Computer [8].

The use of Taylor's series expansions. A network for changing cell size within a network of squares, which has not been constructed according to the principles of the preceding section, is shown in Fig. 8. This network appeared without much explanation

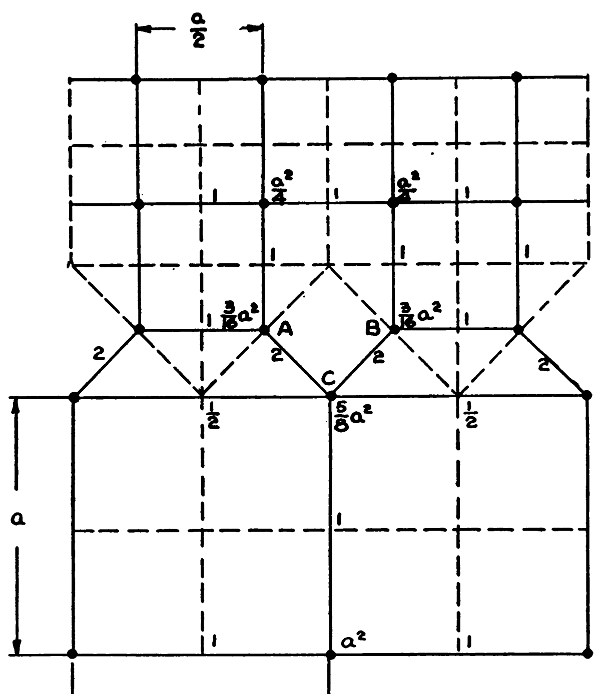

FIG. 8. Method of Reference (9) for doubling cell size in a network of squares.

in a paper by Spangenberg, Walters, and Schott [9]. The numbers beside each branch are coefficients corresponding to r_{B_ν}/l_{B_ν} in Eq. (15) and the numbers beside each node are the areas to be associated with each node. If this network had been constructed by the methods of the preceding section there would be, for example, a branch connecting nodes A and B. The values of the branch coefficients shown in the figure can be obtained if perpendiculars are drawn not through the midpoints of the branches but in the manner shown in Fig. 8. (These lines were not drawn in the paper quoted above.) This is permissible since, in the derivation leading to Eq. (15), the assumption that the perpendiculars to the branch segments were drawn through the *midpoints* of these segments was not essential. If the area of the dotted polygons surrounding each node are now computed, the results do not agree with the values given in the figure. For example the area of the polygon surrounding node C is $3/4\ a^2$ rather than $5/8\ a^2$. The question is thus raised as to which of these values is better. This and other questions can be investigated by means of an error analysis using Taylor's series expansions.

The following investigation is limited to the case where the coefficients of Eq. (4) are constants, and for simplicity these coefficients will be set equal to unity. The results of the analysis are valid for any values of τ and σ. With

$$\nabla^2\phi + 1 = 0, \tag{20}$$

the corresponding generalized difference equation for the potential ϕ_0 of any node of the network is

$$\sum_p Y_{p0}(\phi_p - \phi_0) + A_0 = 0, \tag{21}$$

where ϕ_p is the potential at a neighboring node and A_0 is an element of area to be associated with ϕ_0. By a comparison of these equations we see that a measure of the error introduced by replacing Eq. (20) by Eq. (21) is

$$\epsilon_0 = A_0(\nabla^2\phi)_0 - \sum_p Y_{p0}(\phi_p - \phi_0). \tag{22}$$

If the point where ϕ_0 is defined is taken as the origin of a cartesian system of coordinates, the potential at any neighboring point can be obtained by a Taylor's Series expansion. The summation in Eq. (22) can be expressed in terms of such a series as

$$\sum_p Y_{p0}[\phi_p - \phi_0] = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_p Y_{p0}\left(x_p \frac{\partial}{\partial x} + y_p \frac{\partial}{\partial y}\right)_0^n \phi. \tag{23}$$

The coefficients of the terms of this series are arranged in tabular form below:

term	*coefficient*
$\left(\frac{\partial\phi}{\partial x}\right)_0$	$C_1 = \sum_p Y_{p0}x_p$
$\left(\frac{\partial\phi}{\partial y}\right)_0$	$C_2 = \sum_p Y_{p0}y_p$
$\left(\frac{\partial^2\phi}{\partial x^2}\right)_0$	$C_3 = \frac{1}{2}\sum_p Y_{p0}x_p^2$ (24)
$\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)_0$	$C_4 = \sum_p Y_{p0}x_py_p$
$\left(\frac{\partial^2\phi}{\partial y^2}\right)_0$	$C_5 = \frac{1}{2}\sum_p Y_{p0}y_p^2$
etc.	etc.

In this table all coefficients except C_3 and C_5 should vanish independently if the finite difference approximation is to be correct. The vanishing of the first two coefficients is equivalent to the statement, pertaining to an analogous problem in statics, that the center of gravity of loads, Y_{p0}, concentrated at the surrounding node points should be at the origin. It will be demonstrated that this will be the case if the Y_{p0}'s are calculated by a simple geometrical method to be described. In Fig. 9 a closed polygon is shown whose sides are each drawn perpendicular to line segments radiating from a common point. A concentrated load is placed at the end of each line segment equal to r_p/l_p where l_p is the length of the segment and r_p is the length of the side of the polygon perpendicular

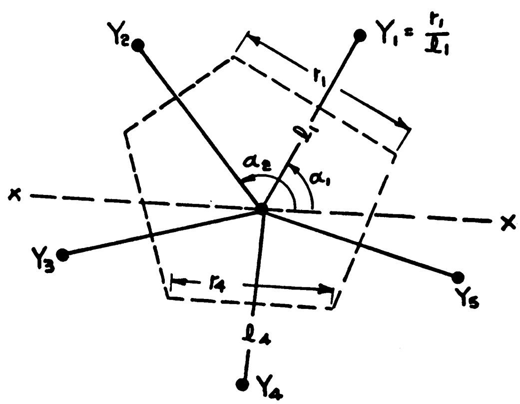

FIG. 9. Construction to prove that coefficients C_1 and C_2 vanish.

to l_p. The moment of the loads about any line, $X - X$, drawn through the common point making an angle α_p with each radiating line is

$$\text{Moment} = \sum_p \frac{r_p}{l_p} l_p \sin \alpha_p = \sum_p r_p \sin \alpha_p \tag{25}$$

The last expression is the sum of the projections of the sides of the polygon on the line $X - X$, which vanishes because the polygon is closed. Note that in this proof the polygon need not be drawn through the midpoints of the radiating lines.

In general the coefficients C_3 and C_5 of Eq. (24) will not be equal. If they are not, these terms should be combined to give a Laplacian and a "hyperbolic" operator, i.e.,

$$C_3\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + C_5\left(\frac{\partial^2\phi}{\partial y^2}\right)_0 = \frac{C_3 + C_5}{2}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)_0 + \frac{C_3 - C_5}{2}\left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right)_0. \tag{26}$$

The second term on the right can be put into the form, $(\partial^2\phi/\partial\xi\,\partial\eta)_0$, by a coordinate rotation through 45° while the first term is invariant in a coordinate rotation. From Eq. (22), the coefficient, $(C_3 + C_5)/2$, should be equal to A_0, and this provides a means for choosing the appropriate area for each cell.

$$A_0 = \frac{C_3 + C_5}{2} = \frac{1}{4}\sum_p Y_{p0}(x_p^2 + y_p^2). \tag{27}$$

If Y_{p0} is again considered as a load acting at x_p, y_p the appropriate area should be $1/4$ of the sum of the polar moments of inertia of the loads.

If the value of Y_{p0} is substituted into Eq. (27), then,

$$A_0 = \frac{1}{4}\sum_p r_p l_p \tag{28}$$

since

$$l_p^2 = x_p^2 + y_p^2 .$$

If the dotted polygon of Fig. 9 passes through the midpoints of the radiating line segments, the area of the polygon satisfies Eq. (28). In the general case it seems desirable to construct boundaries which clearly delineate the area to be associated with each node. A construction which satisfies this condition and Eq. (28) is shown in Fig. 10. The ap-

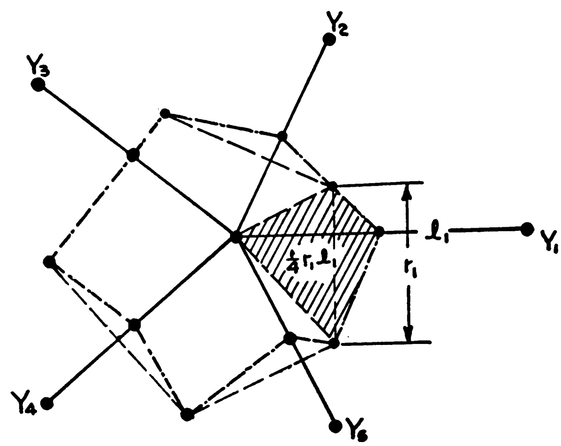

FIG. 10 Construction for obtaining the region appropriate to each nodal point.

propriate area is the area enclosed by joining the midpoints of the radiating line segments to the points of intersection of the perpendiculars to the segments. This procedure gives the same values of area coefficients for the network of Fig. 8 as those computed by by Spangenberg, Walters, and Schott [9].

Referring to Fig. 4, it can be seen that a network capable of solving the potential problem can be constructed in which the branches are the lines drawn perpendicular to the branches of the original network and in which the nodes are the vertices of the "dotted" polygons. For this new network the roles of r_p and l_p are interchanged, and this new network may be called the "geometrical dual" of the original one. Any network constructed according to the rules given in this paper has a geometrical dual. It will be observed that Fig. 8 is very nearly the geometrical dual of Fig. 6b.

The values of Y_{p0} and the area coefficient A_0 have been chosen so as to satisfy three conditions at every node. Since on the average there will be about five or six branches radiating from each node it seems that it should be possible to choose the values of the Y_{p0}'s to satisfy about three more conditions. However each Y_{p0} occurs in the equations for two nodes so that on the average for each node there are only about three independent coefficients. The non-vanishing terms of Eq. (23) will be simply regarded as errors which can, if desired, be incorporated as correction terms in the late stage of the calculation. ϵ_0 in Eq. (22) can be estimated from a trial solution and added to A_0 in Eq. (21). This follows the general procedure used in relaxation calculations [3].

The error series can be easily calculated from Eqs. (22) and (23) for each node of any given configuration. For a network of squares, with branch length a, the terms of lowest order in the error series are

$$\epsilon_0 = -\frac{a^4}{12}\left(\frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial y^4}\right)_0. \tag{29}$$

For a network of equilateral triangles with branch length a,

$$\epsilon_0 = -\frac{a^4}{16}(\nabla^4\phi)_0 . \tag{30}$$

For nodes without double symmetry in their branch patterns terms of lower order will occur.

The following table gives the errors at each type of asymmetrical node in Figs. 6a, 6b and 8. Terms of the fourth order and higher are not included.

	ϵ_0
Fig. 6a	
Node A	$+\frac{a^3}{8}\left(\frac{\partial^3}{\partial y^3} - \frac{\partial^3}{\partial x^2\,\partial y}\right)_A\phi$
Node B	$\left[\frac{a^2}{16}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) + \frac{a^3}{16}\left(\frac{\partial^3}{\partial x^2\,\partial y}\right)\right]_B\phi$
Node C	$\left[-\frac{a^2}{16}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) + \frac{a^3}{16}\left(\frac{\partial^3}{\partial x^2\,\partial y}\right)\right]_C\phi$
Fig. 6b	
Node A	$\left[\frac{a^2}{16}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) - \frac{a^3}{16}\left(\frac{\partial^3}{\partial x^2\,\partial y} - 2\,\frac{\partial^3}{\partial y^3}\right)\right]_A\phi$
Node B	$\left[-\frac{a^2}{16}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) + \frac{a^3}{16}\left(\frac{\partial^3}{\partial x^2\,\partial y}\right)\right]_B\phi$
Fig. 8	
Node A	$\left[\frac{a^2}{8}\frac{\partial^2}{\partial x\,\partial y} + \frac{a^3}{64}\left(\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial x^2\,\partial y} - \frac{\partial^3}{\partial x\,\partial y^2} - \frac{\partial^3}{\partial y^3}\right)\right]_A\phi$
Node B	$\left[-\frac{a^2}{8}\frac{\partial^2}{\partial x\,\partial y} + \frac{a^3}{64}\left(-\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial x^2\,\partial y} + \frac{\partial^3}{\partial x\,\partial y^2} - \frac{\partial^3}{\partial y^3}\right)\right]_B\phi$
Node C	$\left[-\frac{a^2}{32}\left(\frac{\partial^3}{\partial x^2\,\partial y} - 5\,\frac{\partial^3}{\partial y^3}\right)\right]_C\phi$

For each figure, the sum of the ϵ_0's in a vertical strip of width a is equal to $(a^3/8)(d^3\phi/dy^3)$. This indicates that the other terms contribute to a merely local distortion of the field pattern and that on a large network this distortion will be negligible. It is interesting that for a one dimensional network in which a two to one change in cell size is made, the leading term in the error series at the point where the change is made is just $(a^3/8)$ $(d^3\phi/dy^3)$.

Conclusion. A method has been described for constructing an asymmetrical finite difference network that can be used in the solution of second order boundary value problems. The coefficients of the difference equations that govern the network can be found by simple geometrical measurements. The asymmetrical network has the advantages that it provides a simple solution to the problem of fitting a gridwork to a curved boundary, and that it provides a means of changing cell size in such a way that the network is "realizable" by means of physical electrical elements. A clear interpretation has been given to the currents which flow along the branches of the network.

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