: I FFCTS OF SURFACE TENSION AND VISCOSITY ON TAYLOR INSTABILITY*

BY

RICHARD BELLMAN AND RALPH H. PENNINGTON

Rand Corporation, Santa Monica, Cal.

Abstract. The model used is that of two fluids of infinite depth, with the interface initially in the form of a sine wave with amplitude small compared to wave length. The fluids are considered incompressible, and only the linear terms in the equations of hydrodynamics are used. The first four sections discuss the effects of surface tension and viscosity. The fifth gives a few numerical results to illustrate the main points of the preceding sections.

Introduction. If two different fluids having a common plane boundary are accelerated in a direction perpendicular to the boundary, any small irregularities in the boundary will tend to change in shape. If the acceleration is directed from the more dense to the less dense medium, the irregularities will tend to smooth out (in the absence of external forces). Thus the plane configuration of the interface is a stable one. This can be illustrated by the usual example of a glass of water sitting at rest. If one considers the force of gravity to be replaced by an acceleration which produces the same effect, the water and the air are undergoing an upward acceleration. Since the acceleration is from the more dense to the less dense medium, the air-water interface is stable.

Returning to the general case, if the acceleration is directed from the less dense to the more dense medium, irregularities of the interface will tend to grow. This is the effect known as Taylor instability. An example is the case of glass of water turned upside down. Here again the force of gravity may be considered to be replaced by an upward acceleration. The acceleration is from the air to the water, and the air-water interface is instable. The water, instead of maintaining a nearly plane lower surface as it falls, will tend to jet out into long spikes. It is the formation and rate of growth of these spikes which we wish to investigate, taking into account the effects of surface tension and viscosity.

1. Taylor's results. Let us begin by presenting an account of the work done by Taylor himself [4] (see also [2]). The model used is that of two fluids of infinite depth. The interface (neglecting perturbations) is the plane $y = 0$, the y axis being vertical. The initial perturbation will be of the form cos kx, with amplitude small compared to wave length. The problem is then two-dimensional, and the true equation of the interface at any time is $y = \eta(x, t)$, where the function $\eta(x, t)$ is to be determined from hydrodynamic considerations. The fluids will be considered to be incompressible, and only the linear terms in the equations of hydrodynamics will be used.

The linearized hydrodynamical equations in either fluid are

$$u_x + v_y = 0, \tag{1.1}$$

$$u_t + \frac{1}{\rho} p_x = 0, \tag{1.2}$$

$$v_t + \frac{1}{\rho} p_y + g + g_1 = 0. \tag{1.3}$$

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Here, as usual, u and v denote the components of velocity in the x and y directions respectively, p the pressure, ρ the density, g the acceleration of gravity, and g_1 the upward acceleration of the system. These equations have solutions of the form

$$u = -\phi_x , \qquad v = -\phi_y , \tag{1.4}$$

$$p = p_0 - (g + g_1)\rho_y + \phi_t , \tag{1.5}$$

where $\phi_{xx} + \phi_{yy} = 0$ and p_0 is the mean pressure at the interface in the unperturbed condition.

For the upper fluid we take

$$\phi_1 = Ae^{-Ky}f(t) \cos Kx, \tag{1.6}$$

$$p_1 = p_0 - (g + g_1)\rho_{1y} + \rho_1(\phi_1)_t , \tag{1.7}$$

and for the lower fluid,

$$\phi_2 = -Ae^{Ky}f(t) \cos Kx, \tag{1.8}$$

$$p_2 = p_0 - (g + g_1)\rho_{2y} + \rho_2(\phi_2)_t , \tag{1.9}$$

the lower fluid being the more dense, i.e., $\rho_2 > \rho_1$. The above relations satisfy the conditions that velocities are finite at $y = \infty$ and $y = -\infty$, and that $v_1 = v_2$ at the (approximate) interface.

The free boundary condition at $y = \eta(x, t)$ is

$$v - \eta_x u - \eta_t = 0 \tag{1.10}$$

which upon neglecting the non-linear term yields $\eta_t = v = KAf(t) \cos Kx$, or

$$\eta = KA\left(\int_{t_0}^{t} f(t)\, dt\right) \cos Kx. \tag{1.11}$$

The pressures at the interface must satisfy the condition $p_1 = p_2$. Substituting from (1.7) and (1.9) we obtain, after some slight manipulation,

$$-(g + g_1)(\rho_2 - \rho_1)Kf(t) - (\rho_2 + \rho_1)f''(t) = 0, \tag{1.12}$$

so that we may take $f(t) = \sinh nt$, since this choice makes the fluid velocity zero at $t = 0$. The value for n^2 is given by

$$n^2 = \frac{-(g + g_1)(\rho_2 - \rho_1)K}{\rho_2 + \rho_1} \tag{1.13}$$

and the interface is given by

$$\eta = KAn^{-1} \cosh nt \cos Kx \tag{1.14}$$

If $(g + g_1)$ is negative, there is a positive value for n. The disturbance grows like $\cosh nt$, which means that the motion of the interface is instable. This instability exists for all positive K, i.e., for all wave lengths of the initial disturbance. Note that the smaller the wave length ($= 2\pi/K$), the more rapid the growth of the disturbance. This limits the use of the above result for arbitrary disturbances. Since the differential equations used were linear, one would expect to discuss an arbitrary disturbance by Fourier analysis. Let the surface at time $t = 0$ have the equation

$$y = f(x) = \sum_{K=0}^{\infty} a_K \cos Kx.$$

Then at time t, we have

$$y = \sum_{K=0}^{\infty} a_K \cosh nt \cos Kx,$$

where n is essentially $K^{1/2}$. For $t \neq 0$, the series will diverge unless the convergence of $\sum_{K=0}^{\infty} a_K \cos Kx$ is extremely rapid, since $\cos [h(Kt)^{1/2}]$ grows so rapidly.

2. Viscosity. The effects of viscosity on the arguments of Section 1 are clear intuitively. Viscosity is not to be expected to remove the instability, but only to reduce the rate of growth of the amplitude of the disturbance for any particular frequency. The amount of this reduction for small wave lengths is rather startling, however. In particular, as the frequency $\to \infty$, the rate of growth of amplitude $\to 0$.

The model to be used here is that of Section 1. The (linearized) equations governing the motion of an incompressible, viscous fluid are

$$u_x + v_y = 0, \tag{2.1}$$

$$u_t + \frac{1}{\rho} p_x = \frac{\mu}{\rho} (u_{xx} + u_{yy}), \tag{2.2}$$

$$v_t + \frac{1}{\rho} p_y + g + g_1 = \frac{\mu}{\rho} (v_{xx} + v_{yy}), \tag{2.3}$$

where μ is the coefficient of viscosity.

These equations are satisfied by

$$u = -\phi_x - \psi_y\,, \qquad v = -\phi_y + \psi_x\,, \tag{2.4}$$

$$p = p_0 - (g + g_1)\rho_y + \rho(\phi)_t\,, \tag{2.5}$$

provided that (cf. [1, 2])

$$\phi_{xx} + \phi_{yy} = 0, \qquad \frac{\mu}{\rho} (\psi_{xx} + \psi_{yy}) = \psi_t\,. \tag{2.6}$$

For the upper fluid we take

$$\phi_1 = Ae^{-Ky+nt} \cos Kx, \tag{2.7}$$

$$\psi_1 = Be^{-m_1y+nt} \sin Kx, \tag{2.8}$$

$$p_1 = p_0 - (g + g_1)\rho_{1y} + \rho_1(\phi_1)_t\,, \tag{2.9}$$

where

$$m_1^2 = K^2 + \frac{\rho_1 n}{\mu_1}\,.$$

For the lower fluid,

$$\phi_2 = Ce^{Ky+nt} \cos Kx \tag{2.10}$$

$$\psi_2 = De^{m_2y+nt} \sin Kx \tag{2.11}$$

$$p_2 = p_0 - (g + g_1)\rho_{2y} + \rho_2(\phi_2)_t \tag{2.12}$$

where

$$m_2^2 = K^2 + \frac{\rho_2 n}{\mu_2}\,.$$

In order that the velocity components remain finite for $y \to \pm\infty$, it is necessary that the real parts of m_1 and m_2 be positive.

Let the interface be given by $y = \eta(x, t)$. Then $\eta_t = v_1$ or $\eta_t = K(A + B)e^{nt} \cos Kx$, from which we obtain

$$\eta = K(A + B)n^{-1}e^{nt} \cos Kx \tag{2.13}$$

The boundary conditions at the interface are

$$u_1 = u_2 , \qquad v_1 = v_2 \tag{2.14}$$

$$-p_1 + 2\mu_1 \frac{\partial v_1}{\partial y} = -p_2 + 2\mu_2 \frac{\partial v_2}{\partial y} \tag{2.15}$$

$$\mu_1\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial y}\right) = \mu_2\left(\frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial y}\right) \tag{2.16}$$

The last two equations state the equality of the components of the stress-tensor. Substitution in Eqs. (2.14) to (2.16) gives four conditions on the constants A, B, C, and D. From (2.14) we obtain

$$KA + m_1B - DC + m_2D = 0 \tag{2.17}$$
$$A + B + C - D = 0$$

From (2.15) we have, after some simplification,

$$\left[\frac{-(g + g_1)(\rho_2 - \rho_1)K}{n} - \rho_1 n - 2\mu_1K^2\right]A + \left[\frac{-(g + g_1)(\rho_2 - \rho_1)K}{n} - 2\mu_1Km_1\right]B$$
$$+ [\rho_2 n + 2\mu_2K^2]C - 2\mu_2Km_2D = 0 \tag{2.18}$$

Henceforth, we will let $-(g + g_1)(\rho_2 - \rho_1)K = \beta$. Similarly, using (2.16) and some algebra, we obtain

$$2\mu_1K^2A + \mu_1(K^2 + m_1^2)B + 2\mu_2K^2C - \mu_2(K^2 + m_2^2)D = 0 \tag{2.19}$$

These four equations are linear and homogeneous in A, B, C, and D. They have non-trivial solutions if and only if the determinant of the coefficient vanishes,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ K & m_1 & -K & m_2 \\ 2\mu_1K^2 & \mu_1(K^2 + m_1^2) & 2\mu_2K^2 & -\mu_2(K^2 + m_2^2) \\ \frac{\beta}{n} - \rho_1 n - 2\mu_1K^2 & \frac{\beta}{n} - 2\mu_1Km_1 & \rho_2 n + 2\mu_2K^2 & -2\mu_2Km_2 \end{bmatrix} = 0 \tag{2.20}$$

This equation reduces to

$$[-\beta + (\rho_1 + \rho_2)n^2][(\mu_1K + \mu_2m_2) + (\mu_2K + \mu_1m_1)]$$
$$+ 4nK[\mu_1K + \mu_2m_2][\mu_2K + \mu_1m_1] = 0 \tag{2.21}$$

Equation (2.21), using the values of m_1 and m_2 given above, yields an equation of tenth degree in n. Since the roots cannot be directly determined, it will be more profitable to avoid rationalization and see what information can be obtained by other means.

In Section 1 we found that n was positive, which implied that instability occurred, when $(g + g_1)$ was negative. The value of n which determined chiefly how fast the amplitude of the disturbance grew was

$$n = +\left(\frac{-(g + g_1)(\rho_2 - \rho_1)K}{\rho_1 + \rho_2}\right)^{1/2} \tag{2.22}$$

In the present case, then we expect that when $\beta = -(g + g_1)(\rho_2 - \rho_1)K$ is positive, there will be at least one root of equation (2.21) with positive real part, and that for this root,

$$Re(n) \leq \left(\frac{\beta}{(\rho_1 + \rho_2)}\right)^{1/2} = \left(\frac{-(g + g_1)(\rho_2 - \rho_1)K}{\rho_2 + \rho_1}\right)^{1/2} \tag{2.23}$$

We shall determine whether equation (2.21) has a root with positive real part by considering n a complex variable and applying the principle of the argument to the right half plane. So long as n remains in the right half plane, the quantities $m_1 = (K^2 + \rho_1 n/\mu_1)^{1/2}$ and $m_2 = (K^2 + \rho_2 n/\mu_2)^{1/2}$ remain on one branch of their domain of values, so we will run into no confusion if we write equation (2.21) as

$$\begin{aligned}[-\beta + (\rho_1 + \rho_2)n^2][\mu_1 K + (\mu_2^2 K^2 + \mu_2\rho_2 n)^{1/2} + \mu_2 K + (\mu_1^2 K^2 + \mu_1\rho_1 n)^{1/2}]& \\ + 4nK[\mu_1 K + (\mu_2^2 K^2 + \mu_2\rho_2 n)^{1/2}][\mu_2 K + (\mu_1^2 K^2 + \mu_1\rho_1 n)^{1/2}] = 0&\end{aligned} \tag{2.24}$$

Let us use the contour C consisting of the part of the imaginary axis between $(0, R)$ and $(0, -R)$ and the semi-circle in the right half plane with this segment as diameter.

If we denote the left-hand side of equation (2.24) by $f(n)$, the principle of the argument states that

$$\begin{aligned}\int_C \frac{f'(n)}{f(n)}\,dn &= i[\text{change in argument of } f(n) \text{ around } C] \\ &= 2\pi i[\text{number of zeros of } f(n) \text{ within } C]\end{aligned} \tag{2.25}$$

so that

$$\lim_{R\to\infty} \int_C \frac{f'(n)}{f(n)}\,dn = 2\pi i[\text{number of zeros of } f(n) \text{ in right-half plane}] \tag{2.26}$$

Now

$$\lim_{R\to\infty} \int_C \frac{f'(n)}{f(n)}\,dn = \lim_{R\to\infty} \int_{-\pi/2}^{\pi/2} \frac{f'(Re^{i\phi})}{f(Re^{i\phi})}\, i\,Re^{i\phi}\,d\phi + \lim_{R\to\infty}\int_{iR}^{-iR} \frac{f'(n)}{f(n)}\,dn \tag{2.27}$$

provided that the limits on the right-hand side exist. Since they do, we shall evaluate them separately. To evaluate the first term, note that the highest power of n appearing is $n^{5/2}$. In the limit this is the only term which will matter, so

$$\begin{aligned}\lim_{R\to\infty} \int_{-\pi/2}^{\pi/2} \frac{f'(Re^{i\phi})}{(Re^{i\phi})}\, i\,Re^{i\phi}\,d\phi &= \lim_{R\to\infty} \int_{-\pi/2}^{\pi/2} \frac{5/2n^{3/2}}{n^{5/2}}\, i\,Re^{i\phi}\,d\phi \\ &= \lim_{R\to\infty} \int_{-\pi/2}^{\pi/2} \frac{5/2(Re^{i\phi})^{3/2}}{(Re^{i\phi})^{5/2}}\, i\,Re^{i\phi}\,d\phi = \int_{-\pi/2}^{\pi/2} 5/2\, i\,d\phi = 5\pi i/2.\end{aligned} \tag{2.28}$$

Hence, in the limit the change of argument of $f(n)$ when n traverses the semicircle is $5\pi/2$. The second term in equation (2.27) is i times the change of argument of $f(n)$ as n traverses the imaginary axis from $+i\infty$ to $-i\infty$. This change of argument can be seen directly.

We next ask for an upper limit on the value of the real part of this root. We note first of all that the root itself is real. For positive real n, $f(n)$ is real and continuous in n for all K. For $n = 0$, $f(n) = -2\beta(\mu_1 + \mu_2)K < 0$. For $n = (\beta/(\rho_1 + \rho_2))^{1/2}$, $f(n) = 4nK[\mu_1 K + (\mu_2^2 K^2 + \mu_2\rho_2 n)^{1/2}][\mu_2 K + (\mu_1^2 K^2 + \mu_1\rho_1 n)^{1/2}] > 0$. Hence there is a positive real root between $n = 0$ and $n = (\beta/(\rho_1 + \rho_2))^{1/2}$. We already have, then, an upper limit on the root. At the root, $n < (\beta/(\rho_1 + \rho_2))^{1/2}$. This is just as expected, for the positive value of n when viscosity was neglected, see (1.13), was $(\beta/(\rho_1 + \rho_2))^{1/2}$.

We can find an upper bound on this root which shows more about the nature of the root. To do this, we rewrite (2.24) as

$$[(g_1 + g)(\rho_2 - \rho_1)K + (\rho_1 + \rho_2)n^2]\left[\frac{1}{\mu_1 K + (\mu_2^2 K^2 + \mu_2\rho_2 n)^{1/2}} + \frac{1}{\mu_2 K + (\mu_1^2 K^2 + \mu_1\rho_1 n)^{1/2}}\right] + 4nK = 0 \tag{2.29}$$

keeping in mind that $g + g_1$ is negative. Consider also the comparison equation

$$[(g_1 + g)(\rho_2 - \rho_1)K + (\rho_1 + \rho_2)z^2]\left[\frac{1}{\mu_1 K + \mu_2 K} + \frac{1}{\mu_2 K + \mu_1 K}\right] + 4zK = 0 \tag{2.30}$$

For any K, the positive root z of (2.30) must be greater than the positive root n of (2.29). The second factor of the first term has been increased, and at a root this must be counterbalanced by an increase in the second term or a decrease in the first factor of the first term, both of which require an increase in the root. For an upper bound on n, then, we have only to give the value of z. We rewrite (2.30) as

$$(\rho_1 + \rho_2)z^2 + 2zK^2(\mu_1 + \mu_2) + (g + g_1)(\rho_2 - \rho_1)K = 0 \tag{2.31}$$

The positive root is

$$z = \frac{-(\mu_1 + \mu_2)K^2 + ((\mu_1 + \mu_2)^2 K^4 - (g + g_1)(\rho_2 - \rho_1)(\rho_2 + \rho_1)K)^{1/2}}{\rho_2 + \rho_1} \tag{2.32}$$

The most interesting thing about this root is the fact that it has a maximum for some K. Thus the introduction of viscosity has eliminated the tendency for disturbances of small wave-length to increase without bound. We would like to know the value of z at this maximum, since this value will be an upper bound on n for all K. Differentiating (2.32) with respect to K and setting $dz/dK = 0$, we obtain

$$K = \frac{-(g + g_1)(\rho_2 - \rho_1)}{4z(\mu_1 + \mu_2)} \tag{2.33}$$

Substituting this value in (2.32), the result is

$$z = \frac{[-(g + g_1)(\rho_2 - \rho_1)]^{2/3}}{2(\rho_1 + \rho_2)^{1/3}(\mu_1 + \mu_2)^{1/3}} \tag{2.34}$$

This occurs for

$$K = \frac{[-(g + g_1)(\rho_2 - \rho_1)]^{1/3}(\rho_1 + \rho_2)^{1/3}}{2(\mu_1 + \mu_2)^{2/3}} \tag{2.35}$$

so that for all K we have

$$n < \frac{[-(g + g_1)(\rho_2 - \rho_1)]^{2/3}}{2(\rho_1 + \rho_2)^{1/3}(\mu_1 + \mu_2)^{1/3}} \tag{2.36}$$

One can, of course, make better approximations for n. For the general case this process does not seem to offer much, since the general state of affairs is now established.

A quite complicated, but straightforward, calculation shows that n has only one maximum.

Note that in this present Section one cannot satisfy the condition that the velocities be zero when $t = 0$. Apparently because of the linearization performed, one obtains no motion at all if one attempts to satisfy this condition.

3. Surface tension. We now introduce the effects of surface tension into the arguments of Section 1, cf. [2]. It is to be expected that the presence of surface tension will remove the instability for sufficiently small wave lengths. This is indeed the case, as will be shown.

To introduce surface tension into the arguments of Section 1, we have merely to replace the condition $p_1 = p_2$ by

$$p_2 - p_1 + T_1\eta_{xx} = 0 \tag{3.1}$$

Substituting as in Section 1, we obtain

$$-(g + g_1)(\rho_2 - \rho_1)\eta + \rho_2(\phi_2)_t - \rho_1(\phi_1)_t + T_1\eta_{xx} = 0 \tag{3.2}$$

or

$$-(g + g_1)(\rho_2 - \rho_1)AKn^{-1} \sinh nt \cos Kx - (\rho_2 + \rho_1)An \sinh nt \cos Kx \\ - T_1AK^3n^{-1} \sinh nt \cos Kx = 0, \tag{3.3}$$

so that

$$n^2 = \frac{-(g + g_1)(\rho_2 - \rho_1)}{\rho_1 + \rho_2} K - \frac{T_1}{\rho_1 + \rho_2} K^3 \tag{3.4}$$

The condition given for Taylor instability was that $g + g_1$ be negative. But we see from equation (3.4) that the amplitude of the initial disturbance grows only when

$$\frac{-(g + g_1)(\rho_2 - \rho_1)}{\rho_2 + \rho_1} K - \frac{T_1K^3}{\rho_1 + \rho_2} > 0 \tag{3.5}$$

or

$$K < \left[\frac{-(g + g_1)(\rho_2 - \rho_1)}{T_1}\right]^{1/2} \tag{3.6}$$

or

$$\lambda > 2\pi\left(\frac{T_1}{-(g + g_1)(\rho_2 - \rho_1)}\right)^{1/2} \tag{3.7}$$

where $\lambda = 2\pi/K$ is the wave-length of the initial disturbance. Thus for wave-lengths smaller than those satisfying condition (3.7), there is no instability.*

Another fact of importance is expressed by equation (3.4). Since the right-hand side has an absolute maximum, there is a "most dangerous frequency," i.e., a frequency for which the amplitude of the disturbance grows most rapidly.

The most dangerous frequency is that frequency for which n, or n^2, is a maximum. At this frequency, then,

$$\frac{d}{dK}\left[-\frac{(g+g_1)(\rho_2-\rho_1)}{\rho_2+\rho_1}K-\frac{T_1}{\rho_2+\rho_1}K^3\right]=0 \tag{3.8}$$

from which

$$K=\frac{[-(g_1+g)(\rho_2-\rho_1)]^{1/2}}{(3T_1)^{1/2}}. \tag{3.9}$$

Substituting this value in equation (4), we have

$$n^2=\frac{2}{3(3T_1)^{1/2}}\frac{[-(g+g_1)(\rho_2-\rho_1)]^{3/2}}{\rho_2+\rho_1} \tag{3.10}$$

It is remarkable to note the small effect which the numerical value of the surface tension has on the rate of growth of amplitude. Although it is the quantity which places a limit on the rate of growth of amplitude, it is felt numerically only in the one-fourth power, as equation (3.10) shows.

4. Viscosity and surface tension. In this Section we combine the results of the two preceding to give an over-all picture including both surface tension and viscosity. We would expect that as in Section 3, there would be no instability for small wave lengths; and that for longer wave lengths, the rate of growth of amplitude of the disturbance will be less than that given in Section 3.

The procedure will be to take the arguments of Section 2, where viscosity is considered, and alter them to include the effects of surface tension. To do this we must replace equation (2.15) of Section 2 by

$$-p_2+2\mu_2\frac{\partial v_2}{\partial y}+p_1-2\mu_1\frac{\partial v_1}{\partial y}-T_1\frac{\partial^2\eta}{\partial x^2}=0 \tag{4.1}$$

Substitution in this equation as in Section 2 yields

$$\left[-\frac{a}{n}+\rho_1 n+2\mu_1K^2\right]A+\left[-\frac{a}{n}+2\mu_1Km_1\right]B$$
$$+[-2\mu_2K^2-\rho_2 n]C+2\mu_2Km_2D=0, \tag{4.2}$$

where $\alpha = p(g+g_1)(\rho_2-\rho_1)K-T_1K^3$.

*This explains the hanging of water droplets on the underside of a horizontal surface, such as a ceiling. Such a droplet is undergoing an upward acceleration of 980 cm/sec^2 and will tend to drip because of Taylor instability unless its effective wave-length is too small to satisfy (3.7). For water, the critical wave-length is about $\lambda = 2\pi(74/890)^{1/2} = 1.73$ cm. Droplets of larger diameter will tend to drip, while smaller ones will tend to hang. (Actually, of course, the true critical diameter will be different because of circular symmetry, etc., but the above at least contains the principle involved.)

The other three conditions on A, B, C, and D are the same as those in Section 2, namely:

$$\begin{aligned} A + B + C - D &= 0 \\ KA + m_1B - KC + m_2D &= 0 \\ 2\mu_1K^2A + \mu_1(K^2 + m_1^2)B + 2\mu_2K^2C - \mu_2(K^2 + m_2^2)D &= 0 \end{aligned} \tag{4.3}$$

Equations (4.3) are linear and homogeneous in A, B, C, and D. They have non-trivial solutions if and only if the determinant of the coefficients vanishes,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ K & m_1 & -K & m_2 \\ 2\mu_1K^2 & \mu_1(K^2 + m_1^2) & 2\mu_2K^2 & -\mu_1(K^2 + m_2^2) \\ -\dfrac{\alpha}{n} + \rho_1 n + 2\mu_1K^2 & -\dfrac{\alpha}{n} + 2\mu_1Km_1 & -2\mu_2K^2 - \rho_2 n & 2\mu_2Km_2 \end{bmatrix} = 0 \tag{4.4}$$

This equation reduces to

$$[-\alpha + (\rho_1 + \rho_2)n^2][(\mu_1K + \mu_2m_2) + (\mu_2K + \mu_1m_1)] + 4nK[\mu_1K + \mu_2m_2][\mu_2K + \mu_1m_1] = 0 \tag{4.5}$$

This is precisely equation (2.21) of Section 2 with $\beta = -(g + g_1)(\rho_2 - \rho_1)K$ replaced by $\alpha = -(g + g_1)(\rho_2 - \rho_1)K - T_1K^3$. In Section 3, where surface tension alone was considered, we found stability for $\alpha < 0$ and instability for $\alpha > 0$. We shall show that these conditions still hold.

For $\alpha > 0$, the result is immediate from Section 2. Paraphrasing the results of Section 2 for $\alpha > 0$ instead of $\beta > 0$, we may state: for $\alpha > 0$, equation (4.5), where $\mu_1 m = (\mu_1^2K^2 + \mu_1\rho_1 n)^{1/2}$ and $\mu_2m_2 = (\mu_2^2K^2 + \mu_2\rho_2 n)^{1/2}$, has just one root with positive real part. This root is real and less than $(\alpha/(\rho_1 + \rho_2))^{1/2}$. We will return to the problem of a better estimate of this root after proving stability for $\alpha < 0$.

To establish stability for $\alpha < 0$, we again apply the principle of the argument, as in Section 2. The result is established by a series of straightforward but laborious arguments which we shall omit.

We turn now to the instable case, $\alpha > 0$. We found that in this case the equation

$$[-\alpha + (\rho_1 + \rho_2)n^2]\left[\frac{1}{\mu_1K + (\mu_2^2K^2 + \mu_2\rho_2 n)^{1/2}} + \frac{1}{\mu_2K + (\mu_1^2K^2 + \mu_1\rho_1 n)^{1/2}}\right] + 4nK = 0 \tag{4.6}$$

had one positive root. There are two immediate upper bounds for this root. The first, already given, is

$$n < \left(\frac{\alpha}{\rho_1 + \rho_2}\right)^{1/2} = \left(\frac{-(g + g_1)(\rho_2 - \rho_1)K - T_1K^3}{\rho_1 + \rho_2}\right)^{1/2} \tag{4.7}$$

and for all K,

$$n < \frac{2^{1/2}}{3^{3/4}} \frac{[-(g + g_1)(\rho_2 - \rho_1)]^{3/4}}{T_1^{1/4}(\rho_1 + \rho_2)^{1/2}} \tag{4.8}$$

Relations (4.7) and (4.8) state merely that the rate of growth when both viscosity and surface tension are considered is less than that when surface tension alone is considered.

The second upper bound on n comes from comparison of (4.6) with equation (2.30) of Section 2. Since

$$\alpha = -(g + g_1)(\rho_2 - \rho_1)K - T_1K^3 < (g + g_1)(\rho_2 - \rho_1)K, \tag{4.9}$$

the root of (4.6) must be less than that of equation (2.30) for given K. This may be seen in the following way. Suppose the value of α in (4.6) is increased. The first factor of the first term tends to become more negative. An increase in n will decrease both factors of the first term and increase the second term, to counter-balance the change in α. Thus the root of equation (2.30) is an upper bound on the root of (4.6). This is merely a statement of the physical fact that the rate of growth when both viscosity and surface tension are considered is less than when viscosity alone is considered.

From the study made of equation (2.30) we can give an upper bound for the root of (4.6), namely

$$n < \frac{-(\mu_1 + \mu_2)K^2 + ((\mu_1 + \mu_2)^2K^4 - (g + g_1)(\rho_2 - \rho_1)(\rho_2 + \rho_1)K)^{1/2}}{(\rho_2 + \rho_1)} \tag{4.10}$$

and for all K,

$$n < \frac{[-(g + g_1)(\rho_2 - \rho_1)]^{2/3}}{2(\rho_1 + \rho_2)^{1/3}(\mu_1 + \mu_2)^{1/3}} \tag{4.11}$$

The upper bounds on n given by (4.7)-(4.10) will not usually be of great practical value. For particular cases, numerical methods must be used.

A little can be said about the frequency for which (4.6) has a maximum root. The effect of viscosity is to shift the maximum towards smaller K, or greater wave lengths. Furthermore, n has a unique maximum as a function of K.

5. Numerical examples. In order to demonstrate the effects of surface tension and viscosity, we give some examples for ordinary fluids.

Example 1. If the two fluids involved are air and water, surface tension would be expected to play an important role in the development of Taylor instability. We use $\rho_{\text{air}} = 0$, $\rho_{\text{water}} = 1g/\text{cc}$, $T_1 = 74$ dynes/cm, $g + g_1 = -2 \times 10^4$ cm/sec^2 $\approx -20g$.

Figure 1 shows values on n vs. k when surface tension is considered and when it is neglected. The corresponding equations are $n^2 = 2 \cdot 10^4\, K - 74K^3$ and, $n^2 = 2 \times 10^4 K$.

For the surface tension case, n has a maximum of about 355 at $K = 9.5$ ($\lambda = 0.66$ cm) and drops to zero at $K = 16.4$($\lambda = 0.38$ cm). The deviation from the no-surface tension case is indistinguishable for $K < 3$($\lambda > 2.1$ cm).

Experiments have been made by Lewis [3] for accelerations on the order of that used above, at wave-lengths on the order of one centimeter. However, the published results are not in a form which allow comparison with those given above. It would appear that experimental verification of the effects of surface tension should not be difficult to obtain with apparatus like that used by Lewis.

Example 2. If the two fluids involved are air and glycerine, both surface tension and viscosity would be expected to play an important role in the development of Taylor instability. We use

$$\rho_{\text{air}} = 0, \qquad \rho_{\text{glycerine}} = 1.26 \text{ g/cc}, \qquad \mu_{\text{air}} = 0, \qquad \mu_{\text{glycerine}} = 14.9 \text{ poises},$$
$$T_1 = 63 \text{ dynes/cm}, \qquad g + g_1 = 2 \times 10^4 \text{ cm/sec}^2.$$

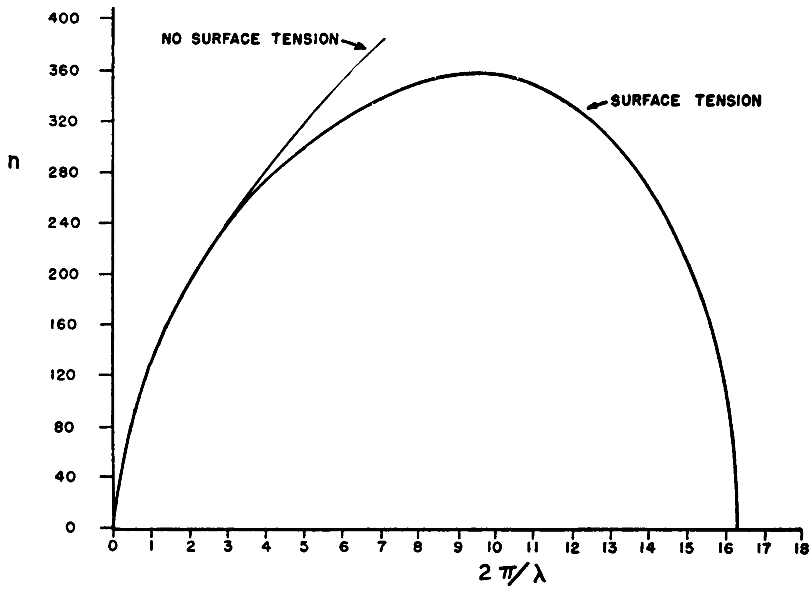

FIG. 1

Figure 2 shows values of n vs. k under four different conditions:

1. Neither surface tension nor viscosity acting.
2. Viscosity only acting.
3. Surface tension only acting.
4. Both viscosity and surface tension acting.

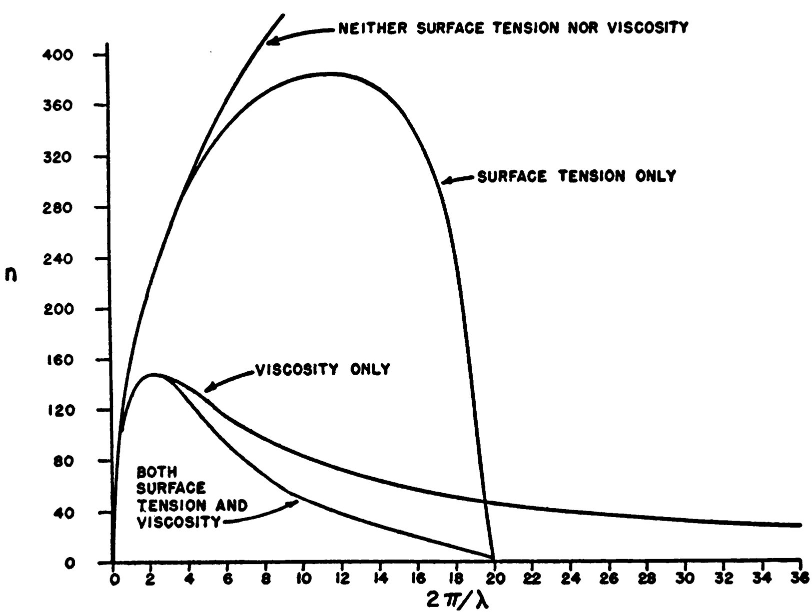

FIG. 2

In this way the relative importance of the two effects for various wave lengths are made apparent. The corresponding equations are:

1. $n^2 = 2 \cdot 10^4 K$

2. $$[-2 \cdot 10^4(1.26)K + 1.26n^2]\left[\frac{1}{((14.9)^2K^2 + (14.9)(1.26)n)^{1/2}} + \frac{1}{14.9K}\right] + 4nK = 0$$

3. $n^2 = 2 \cdot 10^4 K - \dfrac{63}{1.26} K^3$

4. $$[-2 \cdot 10^4(1.26)K + 63K^3 + 1.26n^2]\left[\frac{1}{((14.9)^2K^2 + (14.9)(1.26)n)^{1/2}} + \frac{1}{14.9K}\right] + 4nK = 0$$

It is seen that the viscosity is unimportant for $K < 1(\lambda > 6.28$ cm) and that the surface tension is unimportant for $K < 3(\lambda > 2.1$ cm). Experiments have been made by Lewis [3] for accelerations on the order of that used above, at wave-lengths on the order of one centimeter. It would seem that the viscosity effects would be apparent in these experiments. This would lead to an observed value of n much smaller than that preducted by the theory for non-viscous fluids. However, the experiments gave an observed value n greater than that predicted by the simple theory. Lewis explains this on the basis of viscous drag on the channel sides in the apparatus.

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