FREE VIBRATION OF A RECTANGULAR PLATE WITH DAMPING CONSIDERED*

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Introduction. This paper presents a method for calculating the natural frequencies of the normal modes of free vibration of a rectangular plate, fixed along each edge, with arbitrary shape ratio. Viscous damping of the plate material is considered.

Generally speaking, the objects of this paper are (a) to present a simple and practical solution for the stated problem, (b) to develop this solution in closed form so that a designer can evaluate the natural frequencies of a clamped plate of any shape ratio in a relatively short time, and (c) to make possible the determination of the damping coefficient by comparing experimental and theoretical values.

The theory developed is subjected to the following restrictions: (1) the plate is composed of a material which follows Hooke's law; (2) the deflection of the plate is small compared to its thickness; (3) the thickness of the plate is small compared to its lateral dimensions. The method which is used to solve this problem is that of Galerkin[1] [1] and belongs to the same general class as those of Rayleigh and Ritz. This method can be used (a) to determine an approximate solution of a differential equation with given boundary conditions admitting only the functions which satisfy the prescribed boundary condition exactly, and (b) to treat the problems which belong to non-conservative systems. This method was described by E. P. Grossman [2] and also by W. J. Duncan [3], [4].

The degree of accuracy expected can be increased by increasing the number of independent functions which are used in the solution.

W. J. Duncan has shown in his papers that when the functions are well chosen an excellent approximation can be obtained by use of a very small number of admissible functions. Since our system is non-conservative this method is most convenient for the solution of the problem.

Solution of the problem. If damping forces are proportional to velocity, the motion of the plate is governed by the following partial differential equation:

$$\nabla^4 w(x, y, t) + \frac{\rho h}{D} w_{tt}(x, y, t) = -\frac{k}{D} w_t(x, y, t), \tag{1}$$

where $w(x, y, t)$ is the transverse deflection of the plate, k is the damping coefficient, h is thickness of the plate, ρ is the mass density of the plate material and $D = Eh^3/12(1 - \nu^2)$ is the flexural rigidity of the plate. For a uniform plate vibrating harmonically with an amplitude $\phi(x, y)$ we can write

$$w(x, y, t) = \phi(x, y) \exp(-\alpha t) \cos \omega t, \tag{2}$$

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[1]Numbers in square brackets refer to the Bibliography at the end of the paper.

where ω is the angular frequency. Equations (1) and (2) lead to

$$\nabla^4\phi(x, y) \cos \omega t + \phi(x, y)\left\{\left[\frac{\rho h}{D}(\alpha^2 - \omega^2) - \alpha \frac{k}{D}\right] \cos \omega t + \left(2 \frac{\rho h}{D} \alpha\omega - \omega \frac{k}{D}\right) \sin \omega t\right\} = 0. \tag{3}$$

Since Eq. (3) must be satisfied for all values of t we obtain $(w/D)\ (2\rho h\alpha - k) = 0$. Therefore

$$\alpha = \frac{k}{2\rho h}. \tag{4}$$

Now, Eq. (3) becomes

$$\nabla^4\phi(x, y) - \left[\alpha \frac{k}{D} - \frac{\rho h}{D}(\alpha^2 - \omega^2)\right]\phi(x, y) = 0$$

or

$$\nabla^4\phi(x, y) - \lambda\phi(x, y) = 0, \tag{5}$$

where

$$\lambda = \frac{k^2}{4\rho h D}\left[1 + \left(\frac{2\rho h\omega}{k}\right)^2\right]. \tag{6}$$

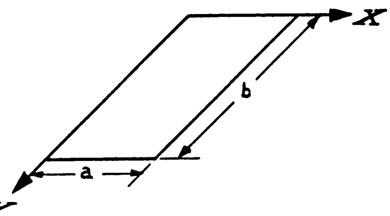

Fig. 1. Coordinate System for the Plate

Assume the solution of Eq. (5) to be given in the form

$$\phi(x, y) = \sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs} X_r Y_s \qquad (r, s = 1, 2, 3, \cdots), \tag{7}$$

in which the X_r and Y_s are admissible functions that satisfy only geometrical boundary conditions, but need not satisfy any "natural boundary conditions", and a_{rs} are the amplitude coefficients. Since the function $\phi(x, y)$ is an approximate solution of the problem, then the error caused in the solution is of the magnitude

$$\epsilon = \sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs}[\nabla^4(X_r Y_s) - \lambda X_r Y_s]. \tag{8}$$

The criterion that the approximation is best if ϵ tends to zero requires, as stated by Galerkin, that the integrals

$$J = \left| \int_0^a \int_0^b \epsilon X_p Y_q \, dx \, dy \right| \qquad (p, q = 1, 2, 3, \cdots) \tag{9}$$

be a minimum. The lateral dimensions of the plate are a and b. Equations (8) and (9) lead to

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs} \int_0^a \int_0^b [\nabla^4(X_r Y_s) - \lambda X_r Y_s] X_p Y_q \, dx \, dy = 0 \tag{10}$$

which can be written in the form

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs} \int_0^a \int_0^b [X_{r,xxxx} Y_s + 2X_{r,xx} Y_{s,yy} + X_r Y_{s,yyyy} - \lambda X_r Y_s] X_p Y_q \, dx \, dy = 0. \tag{11}$$

Equation (11) represents a system of linear homogeneous equations in the unknown coefficients a_{rs}. The natural frequencies λ_1, λ_2, $\cdots$ are determined from the condition that the determinant of the system must vanish.

Let the appropriate characteristic functions X_r, Y_s be given, following Young [5], in the form

$$\Psi_v(\xi) = \left(\cosh \mu_v \frac{\xi}{l} - \cos \mu_v \frac{\xi}{l}\right) - \beta_v\left(\sinh \mu_v \frac{\xi}{l} - \sin \mu_v \frac{\xi}{l}\right), \tag{12}$$

such that

$$\text{if} \quad v = r, \quad \Psi_v = X_r, \quad \xi = x, \quad l = a;$$

$$\text{if} \quad v = s, \quad \Psi_v = Y_s, \quad \xi = y, \quad l = b.$$

The function $\Psi_v(\xi)$ has to satisfy the following boundary conditions:

$$\Psi_v(0) = \Psi_v(l) = \Psi_{v,\xi}(0) = \Psi_{v,\xi}(l) = 0. \tag{13}$$

Then

$$\cosh \mu_v \cos \mu_v - 1 = 0, \tag{14}$$

$$\beta_v = \frac{\cosh \mu_v - \cos \mu_v}{\sinh \mu_v - \sin \mu_v}. \tag{15}$$

The values of β_v and μ_v are presented in Table I. Since the characteristic functions are

TABLE I

Values of β_v and μ_v

v	β_v	μ_v	μ_v^4
1	0.98250	4.73004	500.564
2	1.00078	7.85320	3,803.537
3	0.99997	10.99561	14,617.630
4	1.00000	14.13717	39,943.799
5	1.00000	17.27876	89,135.407
6	1.00000	20.42035	173,881.316
$v > 6$	1.00000	$(2v + 1)\,\pi/2$	$(2v + 1)^4\,\pi^4/16$

of the form of Eq. (12), we obtain

$$X_{r,xxxx} = \frac{\mu_r^4}{a^4} X_r , \qquad Y_{s,yyyy} = \frac{\mu_s^4}{b^4} Y_s , \tag{16}$$

$$\int_0^a X_p X_r \, dx = a\delta_p^r , \text{ where } \delta_p^r = \begin{cases} 1 & \text{if} \quad p = r \\ 0 & \text{if} \quad p \neq r \end{cases} \tag{17}$$

and

$$\int_0^b Y_q Y_s \, dy = b\delta_q^s , \text{ where } \delta_q^s = \begin{cases} 1 & \text{if} \quad q = s \\ 0 & \text{if} \quad q \neq s. \end{cases} \tag{18}$$

Using the abbreviation

$$H_{pr} = a \int_0^a X_p X_{r,xx} \, dx, \qquad H_{qs} = b \int_0^b Y_q Y_{s,yy} \, dy,$$

knowing that

$$\int_0^l \Psi_i \Psi_{j,\xi\xi} \, d\xi = | \Psi_i \Psi_{j,\xi} |_0^l - \int_0^l \Psi_{i,\xi} \Psi_{j,\xi} \, d\xi$$

and using boundary condition Eq. (13), we have

$$\int_0^l \Psi_i \Psi_{j,\xi\xi} \, d\xi = -\int_0^l \Psi_{i,\xi} \Psi_{j,\xi} \, d\xi. \tag{19}$$

TABLE II

Values of $l \int_0^l \psi_{i,\xi} \Psi_{j,\xi} \, d\xi$

i \ j	1	2	3	4	5	6
1	12.30262	0	−9.73079	0	−7.61544	0
2	0	46.05012	0	−17.12892	0	−15.19457
3	−9.73079	0	98.90480	0	−24.34987	0
4	0	−17.12892	0	171.58566	0	−31.27645
5	−7.61544	0	−24.34987	0	263.99798	0
6	0	−15.19457	0	−31.27645	0	376.15008

Numerical values for the integrals, Eq. (19), are presented in Table II. Hence

$$H_{pr} = -a \int_0^a X_{p,x} X_{r,x} \, dx, \tag{20}$$

$$H_{qs} = -b \int_0^b Y_{q,y} Y_{s,y} \, dy. \tag{21}$$

Now Eq. (11) becomes

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs}\left\{\int_0^a \int_0^b [X_p X_{r,xxxx} Y_q Y_s + X_p X_r Y_q Y_{s,yyyy} + 2X_p X_{r,xx} Y_q Y_{s,yy}]\, dx\, dy - \lambda \int_0^a \int_0^b X_r X_p Y_q Y_s\, dx\, dy\right\} = 0$$

or

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs}\left\{\int_0^a \frac{\mu_r^4}{a^4} X_p X_r\, dx \int_0^b Y_q Y_s\, dy + \int_0^a X_p X_r\, dx \int_0^b \frac{\mu_s^4}{b^4} Y_q Y_s\, dy + 2\int_0^a X_p X_{r,xx}\, dx \int_0^b Y_q Y_{s,yy}\, dy - \lambda \int_0^a X_p X_r\, dx \int_0^b Y_q Y_s\, dy\right\} = 0. \tag{22}$$

After substitution of the corresponding values, and multiplication of each term by a^2 we obtain

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs}\left\{\left[\left(\frac{\mu_r^4}{a^4} + \frac{\mu_s^4}{b^4}\right)a^3 b\,\delta_{pq}^{(rs)} + 2\,\frac{a^2}{ab} H_{pr} H_{qs}\right] - \lambda a^3 b\,\delta_{pq}^{(rs)}\right\} = 0, \tag{23}$$

where

$$\delta_{pq}^{(rs)} = \int_0^a X_p X_r\, dx \int_0^b Y_q Y_s\, dy. \tag{24}$$

Equation (23) may be rewritten as

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs}\left\{\left[\left(\frac{b}{a}\,\mu_r^4 + \frac{a^3}{b^3}\,\mu_s^4\right)\delta_{pq}^{(rs)} + 2\,\frac{a}{b} H_{pr} H_{qs}\right] - \Lambda\,\delta_{pq}^{(rs)}\right\} = 0, \tag{25}$$

in which

$$\Lambda = \lambda a^3 b = \frac{a^3 b k^2}{4\rho h D}\left[1 + \left(\frac{2\rho h\omega}{k}\right)^2\right]. \tag{26}$$

Let

$$E_{pq}^{(rs)} = \left[\left(\frac{b}{a}\right)\mu_r^4 + \left(\frac{a}{b}\right)^3 \mu_s^4\right]\delta_{pq}^{(rs)} + 2\,\frac{a}{b} H_{pr} H_{qs}\,. \tag{27}$$

Then Eq. (25) becomes

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs}[E_{pq}^{(rs)} - \Lambda\,\delta_{pq}^{(rs)}] = 0, \tag{28}$$

where

$$\delta_{pq}^{(rs)} = \begin{cases} 1 & \text{for} \quad p = r \quad \text{and} \quad q = s \\ 0 & \text{if either} \quad p \neq r \quad \text{or} \quad q \neq s \end{cases}$$

and

$$\begin{aligned} E_{pq}^{(rs)} &= \frac{b}{a}\,\mu_r^4 + \left(\frac{a}{b}\right)^3 \mu_s^4 + 2\,\frac{a}{b} H_{rr} H_{ss} \quad \text{for} \quad p = r \quad \text{and} \quad q = s, \\ E_{pq}^{(rs)} &= 2\,\frac{a}{b} H_{pr} H_{qs} \quad \text{if either} \quad p \neq r \quad \text{or} \quad q \neq s. \end{aligned} \tag{29}$$

Equation (28) is a system of mn linear homogeneous equations in mn unknowns a_{rs} ($r = 1, 2, \cdots m$; $s = 1, 2, \cdots n$). Since a_{rs} cannot all be zero, the determinant of the system must be zero. This leads, in general, to an algebraic equation of degree mn in Λ.

From Eq. (26) it follows that

$$\omega = \left[\frac{\Lambda D}{\rho h a^3 b} - \left(\frac{k}{2\rho h}\right)^2\right]^{1/2}. \tag{30}$$

Then for any shape ratio $\sigma = b/a$, one obtains

$$\omega = \left[\frac{\Lambda D}{\rho h \sigma a^4} - \left(\frac{k}{2\rho h}\right)^2\right]^{1/2}. \tag{31}$$

If $k = 0$, then

$$\Lambda = \frac{\omega^2 \rho h \sigma a^4}{D} \tag{32}$$

which is the eigenvalue parameter for the plate with fixed boundary conditions. From Eq. (30) it can be seen that the frequency decreases with an increase in the value of the damping factor k. The zero value of ω is obtained for

$$k = \frac{2}{a^2}\left(\rho h D \frac{\Lambda}{\sigma}\right)^{1/2}. \tag{33}$$

The frequencies are calculated for the first three modes of a square plate fixed along each edge, and the results are presented in Table IV.

TABLE III

Coefficients for Vibration of Damped Square Plate

r	s	$E_{11}^{(rs)}$	$E_{12}^{(rs)}$	$E_{13}^{(rs)}$	$E_{21}^{(rs)}$	$E_{22}^{(rs)}$	$E_{23}^{(rs)}$	$E_{31}^{(rs)}$	$E_{32}^{(rs)}$	$E_{33}^{(rs)}$
1	1	1.303.84	0	−239.43	0	0	0	−239.43	0	189.38
1	2	0	5,437.18	0	0	0	0	0	−896.21	0
1	3	−239.43	0	17.551.77	0	0	0	189.38	0	−1.924.84
2	1	0	0	0	5.437.18	0	−896.21	0	0	0
2	2	0	0	0	0	11.848.30	0	0	0	0
2	3	0	0	0	−896.21	0	27,530.32	0	0	0
3	1	−239.43	0	189.38	0	0	0	17,551.77	0	−1,924.84
3	2	0	−896.21	0	0	0	0	0	27,530.32	0
3	3	189.38	0	−1,924.84	0	0	0	−1.924.84	0	48.799.58

TABLE IV

The Values of $\Lambda^{1/2}$ of Vibration of Square Plate Fixed Along Each Edge

$$\omega = \left[\frac{\Lambda D}{\rho h \sigma a^4} - \left(\frac{k}{2\rho h}\right)^2\right]^{1/2}$$

Mode	1st	2nd	3rd
$\Lambda^{1/2}$	36.11	73.73	108.85
(D. Young)	(35.99)	(73.41)	(108.27)

Table III represents the coefficients $E_{pq}^{(rs)}$ for the vibration of a clamped square plate. Using only the first two terms of the series for $E_{pq}^{(rs)}$, we obtain approximations for

$\Lambda^{1/2}$ which are compared with the D. Young solution [5] (given in parenthesis in Table IV). It can be seen that a good agreement between the results is obtained.

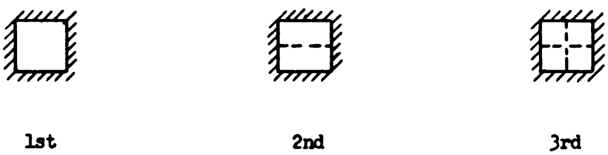

FIG. 2. Nodal Line for Square Plate

Conclusion. The problem of free vibrations of a rectangular plate fixed along each edge and having internal viscous damping is presented and solved by means of generalized Galerkin's method. By choosing the function $\Psi(\xi)$ to satisfy the boundary conditions, Galerkin's method can be applied to a plate with any type of support, even when damping forces are present.

The influence of the damping factor on the natural frequency is given by Eq. (30). It can be seen that the natural frequency decreases with increase in the k-values.

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