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THE DECAY OF MAGNETO-TURBULENCE IN THE PRESENCE OF A MAGNETIC FIELD AND CORIOLIS FORCE*

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Abstract. The final period of decay of magneto-turbulence in an external, homogeneous magnetic field is considered and it is shown that it develops pronounced axisymmetric properties, turbulence elements with finite wave numbers in the direction of the field being damped strongly under normal physical conditions. The turbulence consists of aperiodic motions as well as wave motions. An introduction of an angular velocity, inclined to the field, destroys the axisymmetry and modifies the damping effects and periodicity. The influence of the magnetic field on the damping is counteracted by the Coriolis force. A linear stationary theory on the action of the field gives results consistent with those of the theory of decay. From the results of both theories an explanation is given of the observed inhibition of turbulence in mercury by a magnetic field.

I. Introduction. From the large linear dimensions which one encounters in cosmical physics one may infer turbulence as basic for many applications in this field. In most applications the electrical conductivity is high enough for electrodynamic forces to play an essential role and for turbulence to be governed by the laws of magneto-hydrodynamics. However, it has been pointed out by Chandrasekhar[1] that even a small angular velocity of a medium of cosmical dimensions may sometimes be associated with a Coriolis force of the same importance as the electrodynamic force. In the general case a systematic magnetic field, created by constant sources, has also to be taken into account and turbulent fluctuations have to be superposed on the field.

Even if the influence of the boundaries is neglected, the general turbulent state is neither homogeneous, nor isotropic. The variation of mean velocity with position due to rotation destroys homogeneity also when the mean magnetic field is homogeneous. In this paper, however, we shall discuss turbulence in an incompressible liquid, where the centrifugal potential plays no essential role and regions are considered which are sufficiently small to justify the assumption of a homogeneous external field.

The problem of homogeneous, isotropic magneto-turbulence has been treated by Batchelor[2], Chandrasekhar[3] and Lundquist[4] among others. The second of these papers

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[1]George Darwin lecture for 1953 (in press).

[2]Proc. Roy. Soc. A. **201**, 405 (1950).

[3]Proc. Roy. Soc. A. **204**, 435 (1951); Ibid. **207**, 306 (1951).

[4]Arkiv f. fysik **5**, 338 (1952).

gives a treatment in terms of invariant theory and the third presents a decay law for the spectral tensors for large wave numbers.

The purpose of this paper is mainly to discuss how the law of decay of homogeneous magneto-turbulence is modified by the introduction of an external magnetic field of the strength **B** and of the Coriolis force due to a constant angular velocity **Ω** (MKSA-units are used in the following). Small amplitudes will be assumed and triple correlations will be neglected.

In Sec. III the law of decay in a homogeneous magnetic field is derived and it is shown that the motion becomes axisymmetric with respect to the direction of the field. In Sec. IV the analogous treatment in the presence of a constant angular velocity in a direction making an angle with the magnetic field is given. In Sec. V a brief discussion of stationary turbulence is given. The results are used in Sec. VI for an interpretation of the experimental results of Hartmann[5,6] and Lehnert[7,8] on turbulence in mercury.

II. The fundamental equations. The velocity field **v**, in an incompressible liquid with constant electrical conductivity σ, kinematic viscosity ν, absolute permeability μ and density ρ, is supposed to be nonrelativistic and the liquid is assumed to rotate with a constant angular velocity **Ω** in a homogeneous magnetic field $\mathbf{B} = \mu\mathbf{H}$, making an angle with **Ω** and produced by external sources. The conductivity is assumed to be so good and the rate of change with time so slow that the displacement current can be neglected compared with the convection current **j**, which is the source of the induced magneto-motive force **h**. We start with the equations

$$\operatorname{curl} \mathbf{h} = \mathbf{j}, \qquad \operatorname{curl} \mathbf{E} = -\mu \frac{\partial \mathbf{h}}{\partial t}, \tag{1}$$

$$\operatorname{div} \mathbf{h} = \operatorname{div} \mathbf{v} = 0, \tag{2}$$

$$\mathbf{j} = \sigma[\mathbf{E} + \mu\mathbf{v} \times (\mathbf{H} + \mathbf{h})] \tag{3}$$

and

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + 2\boldsymbol{\Omega} \times \mathbf{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})\right]$$
$$= \mu\mathbf{j} \times (\mathbf{H} + \mathbf{h}) + \nu\rho\nabla^2\mathbf{v} - \nabla(p + \rho\phi_0), \tag{4}$$

where **E** is the electric field, p the pressure, $\mathbf{x} = (x_1, x_2, x_3)$ the radius vector from an origin on the axis of rotation and ϕ_0 the gravitation potential. The reduction of the system into two equations is similar to the treatment by Walén[9] and Lundquist[4] and will not be given here in detail. If the z-axis is chosen in the direction of the angular velocity **Ω**, the equations become

$$\frac{\partial \mathbf{V}}{\partial t} = (\mathbf{W}\cdot\nabla)\mathbf{v} + \lambda\nabla^2\mathbf{V} + \operatorname{curl}(\mathbf{v} \times \mathbf{V}) \tag{5}$$

and

$$\frac{\partial \mathbf{v}}{\partial t} = (\mathbf{W}\cdot\nabla)\mathbf{V} + \nu\nabla^2\mathbf{v} + 2\mathbf{v} \times \boldsymbol{\Omega} - \nabla\phi - \mathbf{V} \times \operatorname{curl} \mathbf{V} + \mathbf{v} \times \operatorname{curl} \mathbf{v}, \tag{6}$$

[5] Kgl. Danske Vidensk. Selskab Math.-fys. Medd. **15**, No. 6 (1937).
[6] J. Hartmann and F. Lazarus, Ibid. **15**, No. 7 (1937).
[7] Arkiv f. Fysik **5**, 69 (1952).
[8] Tellus **4**, 63 (1952).
[9] Arkiv f. mat., astr. o. fysik **30A**, No. 15 (1944).

where

$$\mathbf{V} = \mathbf{h}(\mu/\rho)^{1/2}, \qquad \mathbf{W} = \mathbf{H}(\mu/\rho)^{1/2}, \qquad \lambda = 1/(\mu\sigma), \tag{7}$$

$$\phi = p/\rho + \phi_g + \phi_c + \mathbf{W}\cdot\mathbf{V} + \rho v^2/2 \tag{8}$$

and

$$\phi_c = \tfrac{1}{2}\Omega^2(x_1^2 + x_2^2) \tag{9}$$

is the centrifugal potential.

For small amplitudes the terms of second order in Eqs. (5), (6) and (8) may be neglected. Further, letting

$$\mathbf{J} = \text{curl}\,\mathbf{V}, \qquad \boldsymbol{\omega} = \text{curl}\,\mathbf{v}, \tag{10}$$

and taking the curl of Eqs. (5) and (6) we obtain

$$\frac{\partial \mathbf{J}}{\partial t} = (\mathbf{W}\cdot\nabla)\boldsymbol{\omega} + \lambda\nabla^2\mathbf{J} \tag{11}$$

and

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (\mathbf{W}\cdot\nabla)\mathbf{J} + \nu\nabla^2\boldsymbol{\omega} + 2\boldsymbol{\omega}\times\boldsymbol{\Omega} + \nabla\psi, \tag{12}$$

where, from well-known vector operations,

$$\psi = 2\mathbf{v}\cdot\boldsymbol{\Omega}. \tag{13}$$

III. The law of decay in an external magnetic field. The decay of magneto-turbulence in an external magnetic field is of special interest in connexion with the experimental investigations on mercury. In an exact theory we must introduce triple correlations which represent the basic mechanism of turbulent interaction. However, it is known that they do not play a role in the final period and in this paper we shall restrict ourselves to this stage.

1. *Discussion of the gradient term.* We start with Eqs. (5) and (6) for a liquid, contained in a finite region and suppose that outside this region the conductivity and viscosity are zero while the permeability has the same value as in the inside. Since $\mathbf{v}$ and $\mathbf{V}$ are solenoidal vectors, we obtain on taking the divergence of Eq. (6) that

$$\nabla^2\phi = 0 \tag{14}$$

if all second order terms are neglected. Outside the liquid we have no terms representing the electromagnetic and the viscous forces and the result (14) continues to hold. Further, no surface currents are allowed to exist due to the finite value of the electrical conductivity. Thus, ϕ given by the expression (8) is continuous throughout the bounding surface. In the external region $\mathbf{V}$ tends to a dipole field at large distances from the liquid and ϕ tends to a constant value, representing the balance between pressure and gravitation force. Since ϕ is constant at infinity it is also constant on the boundary and in the interior of the liquid due to Green's theorm and

$$\nabla\phi = 0 \tag{15}$$

all over the liquid.

2. *The correlation tensors.* In terms of components Eqs. (5) and (6) can be written in the forms

$$\frac{\partial V_i}{\partial t} = W_k \frac{\partial v_i}{\partial x_k} + \lambda \nabla^2 V_i \tag{16}$$

and

$$\frac{\partial v_i}{\partial t} = W_k \frac{\partial V_i}{\partial x_k} + \nu \nabla^2 v_i , \tag{17}$$

where use has been made of Eq. (15). Summation over repeated indices is to be understood. Let

$$\mathbf{x}' = \mathbf{x} + \mathbf{r} \tag{18}$$

for a point at the distance $\mathbf{r}$ from $\mathbf{x}$. We shall distinguish the values of the various quantities at $\mathbf{x}'$ by a prime. Equations for the correlation tensors may now be formed in the usual manner; thus multiplying Eq. (16) by V_j' ; we get,

$$V_j' \frac{\partial V_i}{\partial t} = W_k V_j' \frac{\partial v_i}{\partial x_k} + \lambda V_j' \nabla^2 V_i \tag{19}$$

and adding to this equation that obtained from Eq. (19) by interchanging i and j and the primed and the unprimed quantities, we get

$$\frac{\partial}{\partial t}(V_i V_j') = W_k \left[\frac{\partial}{\partial x_k}(V_j' v_i) + \frac{\partial}{\partial x_k'}(V_i v_j') \right] + \lambda(\nabla^2 + \nabla'^2)(V_i V_j'). \tag{20}$$

In Eq. (20) (and similar equations in the sequel) one of the vectors $\mathbf{x}$ and $\mathbf{x}'$ is kept constant while the other is varied; then

$$\partial/\partial x_k' = \partial/\partial r_k = -\partial/\partial x_k \tag{21}$$

and Eq. (20) can be written in the form

$$\frac{\partial}{\partial t}(V_i V_j') = W_k \frac{\partial}{\partial r_k}(V_i v_j' - V_j' v_i) + 2\lambda \nabla^2 (V_i V_j'). \tag{22}$$

In the sequel all differential operators will refer to the variable $\mathbf{r}$.

Further equations are obtained in a similar manner by multiplying Eqs. (16) and (17) by v_j' . Introduce the tensors

$$M_{ij}(\mathbf{r}, t) = \tfrac{1}{2}\langle V_i(\mathbf{x}, t) V_j(\mathbf{x} + \mathbf{r}, t) \rangle, \tag{23}$$

representing the magnetic energy,

$$K_{ij}(\mathbf{r}, t) = \tfrac{1}{2}\langle v_i(\mathbf{x}, t) v_j(\mathbf{x} + \mathbf{r}, t) \rangle, \tag{24}$$

the kinetic energy, and

$$L_{ij}(\mathbf{r}, t) = \tfrac{1}{2}\langle V_i(\mathbf{x}, t) v_j(\mathbf{x} + \mathbf{r}, t) \rangle, \tag{25}$$

the interaction energy respectively. In forming these tensors the mean values are with respect to time. In homogeneous turbulence these tensors should be invariant to arbitrary displacements; this invariance leads to the following geometrical properties:

$$M_{ij}(\mathbf{r}) = \tfrac{1}{2}\langle V_i(\mathbf{x}' - \mathbf{r}) V_j(\mathbf{x}') \rangle = M_{ji}(-\mathbf{r}), \tag{26}$$

$$K_{ij}(\mathbf{r}) = K_{ji}(-\mathbf{r}) \tag{27}$$

and

$$L_{ji}(-\mathbf{r}) = \tfrac{1}{2}\langle V_j(\mathbf{x})v_i(\mathbf{x} - \mathbf{r})\rangle = \tfrac{1}{2}\langle V_j(\mathbf{x} + \mathbf{r})v_i(\mathbf{x})\rangle = L_{ji}(\mathbf{x}', \mathbf{x}). \tag{28}$$

Further, M_{ij} and K_{ij} are symmetric, whereas L_{ij} is skewsymmetric. $M_{ii}(0)$ and $K_{ii}(0)$ are the mean magnetic and kinetic energies at the point $\mathbf{x}$ and time t. The physical significance of L_{ij} is easily seen from the electrodynamic force in Eq. (17). The decrease in magnetic energy per unit mass and time due to the work of the electrodynamic force is

$$E_{ii}^{(MH)} = W_k \frac{\partial V_i}{\partial x_k} v_i , \tag{29}$$

which is a positive quantity if magnetic energy is converted into kinetic energy. From Eqs. (21) and (28) we get the expression

$$E_{ii}^{(MH)} = \lim_{r\to 0} \frac{1}{2} W_k\left(v_i' \frac{\partial V_i}{\partial x_k} + v_i \frac{\partial V_i'}{\partial x_k'}\right) = \lim_{r\to 0} \left\{-W_k \frac{\partial}{\partial r_k} [L_{ii}(\mathbf{r}) - L_{ii}(-\mathbf{r})]\right\}. \tag{30}$$

Returning to Eq. (22) and taking the mean value we get

$$\frac{\partial}{\partial t} M_{ij} = W_k \frac{\partial}{\partial r_k} [L_{ij}(\mathbf{r}) - L_{ji}(-\mathbf{r})] + 2\lambda \nabla^2 M_{ij} . \tag{31}$$

Similarly we obtain

$$\frac{\partial}{\partial t} K_{ij} = -W_k \frac{\partial}{\partial r_k} [L_{ij}(\mathbf{r}) - L_{ji}(-\mathbf{r})] + 2\nu \nabla^2 K_{ij} \tag{32}$$

and

$$\frac{\partial}{\partial t} L_{ij}(\mathbf{r}) = W_k \frac{\partial}{\partial r_k} (M_{ij} - K_{ij}) + (\lambda + \nu)\nabla^2 L_{ij}(\mathbf{r}). \tag{33}$$

Interchanging $\mathbf{r}$ and $-\mathbf{r}$ and i and j in Eq. (33) we obtain

$$\frac{\partial}{\partial t} L_{ji}(-\mathbf{r}) = -W_k \frac{\partial}{\partial r_k} (M_{ij} - K_{ij}) + (\lambda + \nu)\nabla^2 L_{ji}(-\mathbf{r}). \tag{34}$$

Now, assume that a spectral tensor, $\Lambda_{ij}(\boldsymbol{\kappa})$, exists such that

$$\Lambda_{ij}(\boldsymbol{\kappa}) = (8\pi^3)^{-1} \iiint M_{ij}(\mathbf{r}) \exp(-i\boldsymbol{\kappa}\cdot\mathbf{r})\, dr_1\, dr_2\, dr_3 \tag{35}$$

and

$$M_{ij}(\mathbf{r}) = \iiint \Lambda_{ij}(\boldsymbol{\kappa}) \exp(i\boldsymbol{\kappa}\cdot\mathbf{r})\, d\kappa_1\, d\kappa_2\, d\kappa_3 , \tag{36}$$

where $\boldsymbol{\kappa}$ is the wave number. A spectral kinetic energy tensor, $\boldsymbol{\Omega}_{ij}(\boldsymbol{\kappa})$, and a spectral interaction tensor, $\Upsilon_{ij}(\boldsymbol{\kappa})$, can be similarly defined in terms of $K_{ij}(\mathbf{r})$ and $L_{ij}(\mathbf{r})$ respectively.

Since differentiation with respect to r_k in ordinary space corresponds to multiplication with a factor $i\kappa_k$ in the wave-number space we have from Eqs. (31), (32), (33) and (34):

$$\begin{gathered}
\left(\frac{\partial}{\partial t} + 2\lambda\kappa^2\right)\Lambda_{ij} - i\kappa_k W_k[\Upsilon_{ij}(\boldsymbol{\kappa}) - \Upsilon_{ji}(-\boldsymbol{\kappa})] = 0, \\
\left(\frac{\partial}{\partial t} + 2\nu\kappa^2\right)\Omega_{ij} + i\kappa_k W_k[\Upsilon_{ij}(\boldsymbol{\kappa}) - \Upsilon_{ji}(-\boldsymbol{\kappa})] = 0, \\
\left[\frac{\partial}{\partial t} + (\lambda + \nu)\kappa^2\right][\Upsilon_{ij}(\boldsymbol{\kappa}) - \Upsilon_{ji}(-\boldsymbol{\kappa})] - 2i\kappa_k W_k(\Lambda_{ij} - \Omega_{ij}) = 0.
\end{gathered} \tag{37}$$

These equations are also valid for arbitrary amplitudes if sufficiently high wave numbers are considered.

3. *The law of decay.* The system of Eqs. (37) will be satisfied by solutions of the form exp (mt) if the determinant of the system

$$\begin{vmatrix} m + 2a & 0 & -iF \\ 0 & m + 2b & iF \\ -2iF & 2iF & m + a + b \end{vmatrix}, \tag{38}$$

where

$$a = \lambda\kappa^2; \qquad b = \nu\kappa^2; \qquad F = \kappa_k W_k , \tag{39}$$

vanishes. Hence

$$(m + a + b)[m^2 + 2(a + b)m + 4ab + 4F^2] = 0. \tag{40}$$

The roots of this equation are

$$m_{1,2} = -(a + b) \pm [(a - b)^2 - 4F^2]^{1/2}, \qquad m_3 = -(a + b) \tag{41}$$

and the solutions have the form

$$\begin{vmatrix} \Lambda_{ij}(\boldsymbol{\kappa}, t) \\ \Omega_{ij}(\boldsymbol{\kappa}, t) \\ \Pi_{ij}(\boldsymbol{\kappa}, t) \end{vmatrix} = \begin{vmatrix} \Lambda_{ij}^{(1)} & \Lambda_{ij}^{(2)} & \Lambda_{ij}^{(3)} \\ \Omega_{ij}^{(1)} & \Omega_{ij}^{(2)} & \Omega_{ij}^{(3)} \\ \Pi_{ij}^{(1)} & \Pi_{ij}^{(2)} & \Pi_{ij}^{(3)} \end{vmatrix} \cdot \begin{vmatrix} \exp(m_1 t) \\ \exp(m_2 t) \\ \exp(m_3 t) \end{vmatrix}, \tag{42}$$

where

$$\Pi_{ij}(\boldsymbol{\kappa}, t) = \Upsilon_{ij}(\boldsymbol{\kappa}, t) - \Upsilon_{ji}(-\boldsymbol{\kappa}, t) \tag{43}$$

has been introduced. When m has been chosen as a root of Eq. (40) the solution of the linear system (37) can be written in the form

$$\begin{vmatrix} \Lambda_{ij}(\boldsymbol{\kappa}, t) \\ \Pi_{ij}(\boldsymbol{\kappa}, t) \end{vmatrix} = \begin{vmatrix} [1 - (1 - \zeta^2)^{1/2}]/[1 + (1 - \zeta^2)^{1/2}] & [1 + (1 - \zeta^2)^{1/2}]/[1 - (1 - \zeta^2)^{1/2}] & 1 \\ -2i[1 - (1 - \zeta^2)^{1/2}]/\zeta & -2i[1 + (1 - \zeta^2)^{1/2}]/\zeta & -2i/\zeta \end{vmatrix} \cdot \begin{vmatrix} \Omega_{ij}^{(1)} \exp(m_1 t) \\ \Omega_{ij}^{(2)} \exp(m_2 t) \\ \Omega_{ij}^{(3)} \exp(m_3 t) \end{vmatrix}, \tag{44}$$

where we have introduced the parameter

$$\zeta = 2F/(a - b) = 2\kappa_k W_k/[\kappa^2(\lambda - \nu)] \tag{45}$$

and have written

$$m_{1,2} + 2a = a - b \pm (a - b)(1 - \zeta^2)^{1/2}, \qquad m_3 + 2a = a - b \tag{46}$$

and

$$m_{1,2} + 2b = -(a - b) \pm (a - b)(1 - \zeta^2)^{1/2}, \qquad m_3 + 2b = -(a - b). \tag{47}$$

For small values of ζ the form (44) reduces to

$$\begin{vmatrix} \Lambda_{ij}(\kappa, t) \\ \\ \Pi_{ij}(\kappa, t) \end{vmatrix} = \begin{vmatrix} \zeta^2/4 & 4/\zeta^2 & 1 \\ \\ -i\zeta & -4i/\zeta & -2i/\zeta \end{vmatrix} \cdot \begin{vmatrix} \Omega_{ij}^{(1)} \exp (m_1 t) \\ \Omega_{ij}^{(2)} \exp (m_2 t) \\ \Omega_{ij}^{(3)} \exp (m_3 t) \end{vmatrix}. \tag{48}$$

All quantities have to be finite at $\zeta = 0$, and therefore, when ζ tends to zero,

$$\Lambda_{ij}^{(1)} = \mathcal{O}(\zeta^2) \to 0, \qquad \Lambda_{ij}^{(2)} = \Lambda_{ij}^{(0)} + \mathcal{O}(\zeta), \qquad \Lambda_{ij}^{(3)} = \mathcal{O}(\zeta) \to 0, \tag{49}$$

$$\Omega_{ij}^{(1)} = \Omega_{ij}^{(0)} + \mathcal{O}(\zeta), \qquad \Omega_{ij}^{(2)} = \mathcal{O}(\zeta^2) \to 0, \qquad \Omega_{ij}^{(3)} = \mathcal{O}(\zeta) \to 0, \tag{50}$$

$$\Pi_{ij}^{(1)} = \mathcal{O}(\zeta) \to 0, \qquad \Pi_{ij}^{(2)} = \mathcal{O}(\zeta) \to 0, \qquad \Pi_{ij}^{(3)} = \Pi_{ij}^{(0)} + \mathcal{O}(\zeta), \tag{51}$$

where $\Lambda_{ij}^{(0)}$, $\Omega_{ij}^{(0)}$ and $\Pi_{ij}^{(0)}$ are independent of ζ and $\mathcal{O}(\zeta^n)$ are terms at least of order n. For $\zeta = 0$ we get

$$\Lambda_{ij}(\kappa, t) = \Lambda_{ij}^{(0)}(\kappa) \exp (-2\lambda\kappa^2 t), \tag{52}$$

$$\Omega_{ij}(\kappa, t) = \Omega_{ij}^{(0)}(\kappa) \exp (-2\nu\kappa^2 t) \tag{53}$$

and

$$\Pi_{ij}(\kappa, t) = \Pi_{ij}^{(0)}(\kappa) \exp [-(\lambda + \nu)\kappa^2 t]. \tag{54}$$

These solutions represent the final period of decay of isotropic turbulence and are analogous to the case studied by Lundquist where the magneto-hydrodynamic interaction, expressed by triple correlations, was neglected and the magnetic and kinetic turbulence fields were found to decay independently of each other. The introduction of an external magnetic field, however, changes the situation even in first order and the decay is mainly governed by a coupling of the form (29).

The solutions (41) define a decay time, τ_k, given by

$$1/\tau_{1,2} = \kappa^2(\lambda + \nu) \mp \kappa^2(\lambda - \nu)(1 - \zeta^2)^{1/2}; \qquad 1/\tau_3 = \kappa^2(\lambda + \nu). \tag{55}$$

Thus, for $\zeta^2 < 1$ we have three non-periodic solutions, all with different decay times; for $\zeta^2 = 1$ all the spectral tensors decay with the same time constant

$$\tau_c = \tau_3 = 1/[\kappa^2(\lambda + \nu)] \tag{56}$$

and finally for values of $\zeta^2 > 1$ two periodic solutions are obtained, both with the same real damping, $1/\tau_3$.

4. *Physical interpretation of the law of decay.* The physical meaning underlying the results (42), (44) and (55) can be understood from a consideration of a plane state of motion in a liquid between two infinitely conducting planes at a distance L, as shown

by Fig. 1. Introduce a homogeneous magnetic field $\mathbf{B}_0$ in the z-direction, perpendicular to the planes. In a plane state of motion and for small amplitudes

$$\partial/\partial x = \partial/\partial y = 0; \qquad \mathbf{v} = (0, v, 0); \qquad \mathbf{V} = (0, V, 0) \tag{57}$$

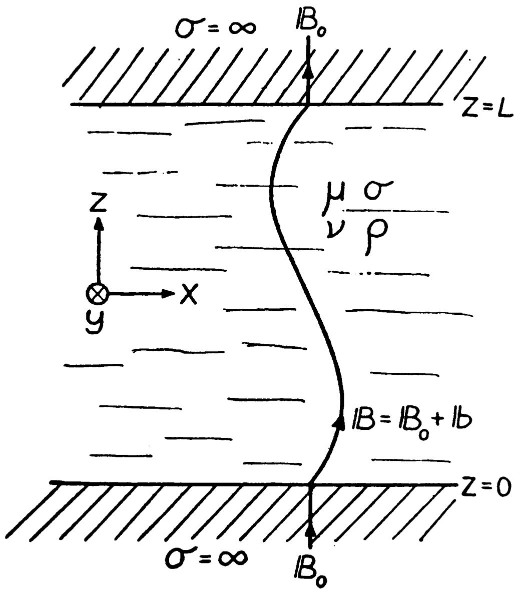

Fig. 1. Plane state of motion of an electrically conducting, viscous liquid between two infinitely conducting planes in a homogeneous, perpendicular magnetic field, $\mathbf{B}_0$.

and Eqs. (5) and (6) reduce to

$$\frac{\partial V}{\partial t} = W \frac{\partial v}{\partial z} + \lambda \frac{\partial^2 V}{\partial z^2} \tag{58}$$

and

$$\frac{\partial v}{\partial t} = W \frac{\partial V}{\partial z} + \nu \frac{\partial^2 v}{\partial z^2}. \tag{59}$$

On eliminating V we obtain the following equation:

$$\left[\frac{\partial^2}{\partial t^2} - W^2 \frac{\partial^2}{\partial z^2} - (\lambda + \nu) \frac{\partial^3}{\partial t\, \partial z^2} + \lambda\nu \frac{\partial^4}{\partial z^4}\right] v = 0. \tag{60}$$

Separating the variables in the form

$$v = Z(z)T(t) \tag{61}$$

we get

$$T''/T - W^2 Z''/Z - (\lambda + \nu)(Z''/Z)(T'/T) + \lambda\nu Z^{\mathrm{IV}}/Z = 0. \tag{62}$$

Since the magnetic field lines and the liquid are attached to the infinitely conducting walls we may assume that

$$Z = \sin\,[\pi kz/L] = \sin\,(\kappa z) \qquad (k = 1, 2, \cdots \kappa = \pi k/L) \tag{63}$$

in which case T admits a solution of the form

$$T = \exp\left(\tfrac{1}{2}mt\right), \tag{64}$$

provided

$$m^2 + 4\kappa^2W^2 + 2(\lambda + \nu)\kappa^2 m + 4\lambda\nu\kappa^4 = 0. \tag{65}$$

The roots of this equation are

$$m_{1,2} = -(\lambda + \nu)\kappa^2 \pm (\lambda - \nu)\kappa^2(1 - \zeta^2)^{1/2}, \tag{66}$$

which are the two first values given by Eq. (41). This corresponds to a motion of the magnetic lines of force, regarded as elastic strings with a tension given by the magnetic field strength and a damping due to the Joule heat and the viscous losses. For $\zeta^2 \leq 1$ the equivalent strings move aperiodically and for $\zeta^2 > 1$ damped waves travel along the strongs. Correlations may be formed by products of the solutions of Eq. (60), giving time factors of the form

$$T^{(\mathrm{I})}\cdot T^{(\mathrm{II})} = \exp\left[\tfrac{1}{2}(m_{1,2} + m_{1,2})t\right] \tag{67}$$

for a given value of κ. These factors are consistent with those given by Eqs. (41) and (42).

We shall now return to the discussion of the results (42). Whatever distribution we may start with at $t = 0$ these results will always tend to a solution, which for the kinetic tensor reduces to

$$\Omega_{ij}(\mathbf{\kappa}, t) = \Omega_{ij}^{(k)}(\mathbf{\kappa}) \exp\left(-t/\tau_k\right), \tag{68}$$

where $1/\tau_k$ is the value of the one of the decay factors (55) having the smallest real part. The solution (68) will differ for turbulence elements due to their size, the properties of the liquid and the strength of the external field.

For small values of ζ and for large ratios λ/ν ($\lambda/\nu \approx 10^5$ for experiments with mercury) the decay factors (55) become

$$1/\tau_1 \approx 2\nu\kappa^2 + \tfrac{1}{2}\kappa^2(\lambda - \nu)\zeta^2 \tag{69}$$

and

$$1/\tau_2 \approx 2\lambda\kappa^2 - \tfrac{1}{2}\kappa^2(\lambda - \nu)\zeta^2, \tag{70}$$

whereas $1/\tau_3$ remains unaltered; further, $1/\tau_1$ is much smaller than $1/\tau_2$ and $1/\tau_3$, if ζ is sufficiently small. Now, Eq. (50) shows that the corresponding factor, $\Omega_{ij}^{(1)}$, has a term of zero order in ζ and consequently $\Omega_{ij}(\mathbf{\kappa}, t)$ will be represented by the asymptotic law

$$\Omega_{ij}(\mathbf{\kappa}, t) \approx \Omega_{ij}^{(1)}(\mathbf{\kappa}) \exp\left\{-2[\nu\kappa^2 + \kappa_3^2W^2/\kappa^2(\lambda - \nu)]t\right\} \tag{71}$$

during the largest part of the final period of decay, if the x_3-axis is chosen in the direction of the field and Eq. (45) is used. Analogous discussions may be carried out when $\lambda < \nu$. The results of this section may be summed up in the following statements:

During the turbulent decay of a liquid with $\lambda \gg \nu$ all periodic turbulence elements as well as the aperiodic ones with small extensions in the direction of the field (large values of κ_3) *are damped out relatively rapidly. The asymptotic state of decay is two-dimensional* with respect to the direction of the field. Only vortices with $\kappa_3 = 0$ are left in the final state;

the damping is by viscosity only and since the electric field becomes independent of x_3 no induced currents flow.

The main kinetic decay factor, $1/\tau_1$, for mercury, given by Eq. (71) is shown in Fig. 2, where we have put $\kappa_1 = \kappa_2$. The curves clearly show action of the field in suppressing elements with finite wave numbers in the x_3-direction.

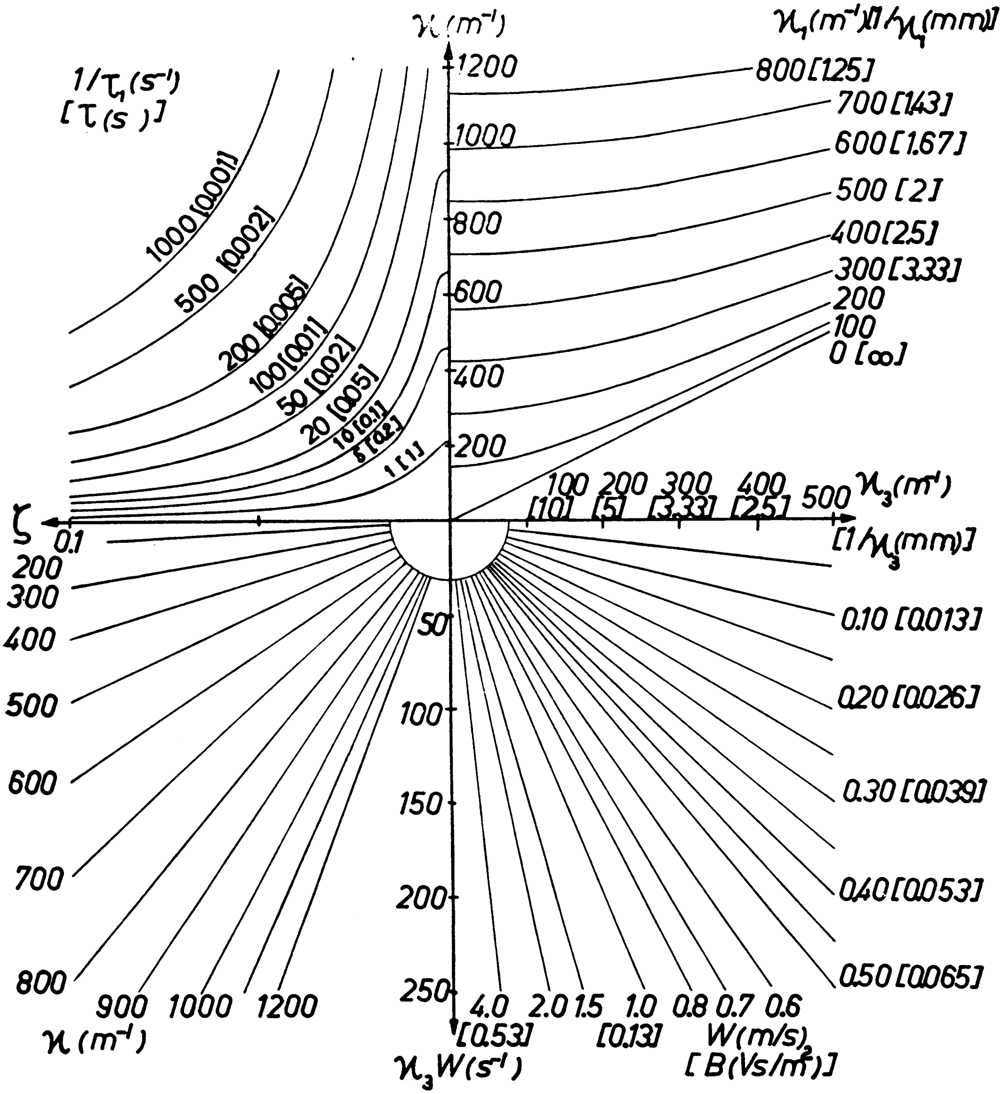

FIG. 2. The factor of decay, $1/\tau_1$, for the final period of turbulence in mercury as a function of an external magnetic field [$W = B_0(\mu\rho)^{-\frac{1}{2}}$ being the corresponding wave velocity] and the wave numbers $\kappa_1 = \kappa_2$ and κ_3. The values $\lambda = 0.763\ m^2/s$ and $\nu = 0.116 \times 10^{-4}\ m^2/s$ at 18°C have been used.

IV. The decay in a magnetic field in presence of the Coriolis force. It was pointed out at the beginning of this paper that the influence of the Coriolis force may sometimes be comparable to the electromagnetic force, both leading to first order effects which may be significant for small as well as for large turbulent intensities. The effect of these forces, when they act separately on a liquid, shows many similarities. The stationary flow of a rotating liquid becomes two-dimensional at high angular velocities[10] and the same situation is true in a conducting liquid in a strong, homogeneous magnetic field.[7,8] Similarly both have inhibiting effects on the onset of convection.[11,12] But if both forces are simultaneously present a complicated situation arises, which is not the same as the superposition of the separate effects. This is evident for example[11] from Chandrasekhar's investigations of the thermal instability of a fluid layer heated below under the joint effects of a Coriolis acceleration and a magnetic field.

1. *The correlation tensors.* We shall now extend the theory of the decay of turbulence described in the preceeding sections to allow for the effect of a stationary angular velocity in the x_3-direction: thus

$$\mathbf{\Omega} = (0, 0, \Omega) = \tfrac{1}{2}\,\mathrm{curl}\,\mathbf{U}; \qquad \mathbf{U} = \mathbf{\Omega} \times \mathbf{x}, \tag{72}$$

where $\mathbf{U}$ is the velocity of rotation of the liquid and $\mathbf{x}$ is the radius vector from the origin on the axis of rotation. We choose the coordinate system with the magnetic field vector in the x_2, x_3-plane so that

$$\mathbf{W} = (0, W_2, W_3). \tag{73}$$

As we shall see it is most convenient to discuss this problem in terms of the current density and the vorticity. We now start with Eqs. (11) and (12), which written in terms of components have the forms

$$\frac{\partial J_i}{\partial t} = W_k \frac{\partial \omega_i}{\partial x_k} + \lambda \nabla^2 J_i \tag{74}$$

and

$$\frac{\partial \omega_i}{\partial t} = -W_k \frac{\partial J_i}{\partial t} + \nu \nabla^2 \omega_i + 2\epsilon_{ilm}\omega_l \Omega_m + \frac{\partial \psi}{\partial x_i}, \tag{75}$$

where ϵ_{ilm} is the usual alternating symbol. The equations governing the various correlations can be obtained in the same manner as in Sec. III; the only difference is that now we shall have terms containing $\mathbf{\Omega}$ added to the right hand sides of the various equations; thus

$$\frac{\partial}{\partial t}(\omega_i \omega_j') = \cdots + 2\omega_j' \epsilon_{ilm} \omega_l \Omega_m + 2\omega_i \epsilon_{jlm} \omega_l' \Omega_m + \omega_j' \frac{\partial \psi}{\partial x_i} + \omega_i \frac{\partial \psi'}{\partial x_j'}, \tag{76}$$

$$\frac{\partial}{\partial t}(J_i \omega_j') = \cdots + 2J_i \epsilon_{jlm} \omega_l' \Omega_m + J_i \frac{\partial \psi'}{\partial x_j'}. \tag{77}$$

Introduce the tensors

$$R_{ij}(\mathbf{x}, \mathbf{x}') = \langle J_i(\mathbf{x}) J_j(\mathbf{x} + \mathbf{r}) \rangle \equiv R_{ji}(\mathbf{x}', \mathbf{x}), \tag{78}$$

$$T_{ij}(\mathbf{x}, \mathbf{x}') = \langle \omega_i(\mathbf{x}) \omega_j(\mathbf{x} + \mathbf{r}) \rangle \equiv T_{ji}(\mathbf{x}', \mathbf{x}), \tag{79}$$

$$S_{ij}(\mathbf{x}, \mathbf{x}') = \langle J_i(\mathbf{x}) \omega_j(\mathbf{x} + \mathbf{r}) \rangle, \tag{80}$$

$$P_{ij}(\mathbf{x}, \mathbf{x}') = \left\langle \frac{\partial}{\partial r_i} [J_j(\mathbf{x} + \mathbf{r}) \psi(\mathbf{x})] \right\rangle - \left\langle \frac{\partial}{\partial r_j} [J_i(\mathbf{x}) \psi(\mathbf{x} + \mathbf{r})] \right\rangle \tag{81}$$

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[10] G. I. Taylor, Proc. Roy. Soc. **A100**, 114 (1921).
[11] S. Chandrasekhar, Phil. Mag. **43**, 501 (1952).
[12] S. Chandrasekhar Proc. Roy. Soc. **A217**, 306 (1953).

and

$$Q^*_{ij}(\mathbf{x}, \mathbf{x}') = \left\langle \frac{\partial}{\partial r_i} [\omega_j(\mathbf{x} + \mathbf{r})\psi(\mathbf{x})] \right\rangle; \qquad Q_{ij}(\mathbf{x}, \mathbf{x}') = Q^*_{ij}(\mathbf{x}, \mathbf{x}') - Q^*_{ji}(\mathbf{x}', \mathbf{x}), \tag{82}$$

where relations (18) and (21) have been used. These notations will be used for the present. They are valid also in the inhomogeneous case. The condition for homogeneity will be applied later. From Eqs. (76) and (77) we now find the equations governing the correlations:

$$\frac{\partial}{\partial t} R_{ij}(\mathbf{x}, \mathbf{x}') = W_k \frac{\partial}{\partial r_k} [S_{ij}(\mathbf{x}, \mathbf{x}') - S_{ji}(\mathbf{x}', \mathbf{x})] + 2\lambda \nabla^2 R_{ij}(\mathbf{x}, \mathbf{x}'), \tag{83}$$

$$\begin{aligned}\frac{\partial}{\partial t} T_{ij}(\mathbf{x}, \mathbf{x}') = {} & -W_k \frac{\partial}{\partial r_k} [S_{ij}(\mathbf{x}, \mathbf{x}') - S_{ji}(\mathbf{x}', \mathbf{x})] + 2\nu \nabla^2 T_{ij}(\mathbf{x}, \mathbf{x}') \\ & + 2\Omega_m[\epsilon_{ilm} R_{lj}(\mathbf{x}, \mathbf{x}') + \epsilon_{jlm} R_{il}(\mathbf{x}, \mathbf{x}')] - P_{ij}(\mathbf{x}, \mathbf{x}')\end{aligned} \tag{84}$$

and

$$\begin{aligned}\frac{\partial}{\partial t} S_{ij}(\mathbf{x}, \mathbf{x}') = {} & W_k \frac{\partial}{\partial r_k} [R_{ij}(\mathbf{x}, \mathbf{x}') - T_{ij}(\mathbf{x}, \mathbf{x}')] + (\lambda + \nu) \nabla^2 S_{ij}(\mathbf{x}, \mathbf{x}') \\ & + 2\Omega_m \epsilon_{jlm} S_{il}(\mathbf{x}, \mathbf{x}') + Q^*_{ji}(\mathbf{x}', \mathbf{x}).\end{aligned} \tag{85}$$

Interchanging i and j and $\mathbf{x}$ and $\mathbf{x}'$ in Eq. (85) and subtracting the resulting equation from Eq. (85), we obtain

$$\begin{aligned}\frac{\partial}{\partial t} [S_{ij}(\mathbf{x}, \mathbf{x}') - S_{ji}(\mathbf{x}', \mathbf{x})] & \\ = 2W_k \frac{\partial}{\partial r_k} [R_{ij}(\mathbf{x}, \mathbf{x}') - T_{ij}(\mathbf{x}, \mathbf{x}')] & + (\lambda + \nu) \nabla^2 [S_{ij}(\mathbf{x}, \mathbf{x}') - S_{ji}(\mathbf{x}', \mathbf{x})] \\ & + 2\Omega_m[\epsilon_{jlm} S_{il}(\mathbf{x}, \mathbf{x}') - \epsilon_{ilm} S_{jl}(\mathbf{x}, \mathbf{x}')] - Q_{ij}(\mathbf{x}, \mathbf{x}').\end{aligned} \tag{86}$$

2. *The spectral tensors and the law of decay.* Before the equations governing the correlations are transformed into a spectral representation we shall consider the properties of the tensors (81) and (82). It is easily seen that

$$P_{ii} = Q_{ii} = 0, \tag{87}$$

since $\mathbf{J}$ and $\boldsymbol{\omega}$ are solenoidal. If homogeneity is assumed, the tensor $Q^*_{ij}(\mathbf{x}, \mathbf{x}')$ will have the property

$$\begin{aligned}Q^*_{ij}(\mathbf{x}, \mathbf{x}') = \left\langle \frac{\partial}{\partial r_i} [\omega_j(\mathbf{x} + \mathbf{r})\psi(\mathbf{x})] \right\rangle & = \left\langle \frac{\partial}{\partial r_i} [\omega_j(\mathbf{x})\psi(\mathbf{x} - \mathbf{r})] \right\rangle \\ & = -\left\langle \frac{\partial}{\partial r_i} [\omega_j(\mathbf{x})\psi(\mathbf{x} + \mathbf{r})] \right\rangle = -Q^*_{ij}(\mathbf{x}', \mathbf{x}).\end{aligned} \tag{88}$$

From Eqs. (81) and (82) we get the result

$$Q_{ij}(\mathbf{x}, \mathbf{x}') - Q_{ji}(\mathbf{x}, \mathbf{x}') = P_{ij}(\mathbf{x}, \mathbf{x}') - P_{ji}(\mathbf{x}, \mathbf{x}') = 0. \tag{89}$$

Now introduce the Fourier transforms $\Phi_{ij}(\kappa)$, $\Psi_{ij}(\kappa)$, $\Gamma_{ij}(\kappa)$, $\chi_{ij}(\kappa)$, $\vartheta_{ij}^*(\kappa)$ and $\vartheta_{ij}(\kappa)$ of the tensors $R_{ij}(\mathbf{r})$, $T_{ij}(\mathbf{r})$, $S_{ij}(\mathbf{r})$, $P_{ij}(\mathbf{r})$, $Q_{ij}^*(\mathbf{r})$ and $Q_{ij}(\mathbf{r})$ respectively; thus

$$\Phi_{ij}(\kappa) = (8\pi^3)^{-1} \iiint R_{ij}(\mathbf{r}) \exp(-i\kappa\cdot\mathbf{r})\, dr_1\, dr_2\, dr_3\,, \qquad \text{etc.} \tag{90}$$

Also we shall write

$$\Gamma'_{ij} \equiv \Gamma_{ij}(-\kappa); \qquad \vartheta_{ij}^{*\prime} \equiv \vartheta_{ij}^*(-\kappa). \tag{91}$$

By applying the Fourier transforms to Eqs. (83), (84), (85) and (86) we obtain the following equations for the spectral tensors:

$$\left(\frac{\partial}{\partial t} + 2a\right)\Phi_{ij} - iF(\Gamma_{ij} - \Gamma'_{ji}) = 0, \tag{92}$$

$$\left(\frac{\partial}{\partial t} + 2b\right)\Psi_{ij} + iF(\Gamma_{ij} - \Gamma'_{ji}) - 2\Omega(\epsilon_{il3}\Psi_{lj} + \epsilon_{jl3}\Psi_{il}) + \chi_{ij} = 0, \tag{93}$$

$$\left(\frac{\partial}{\partial t} + a + b\right)\Gamma_{ij} - iF(\Phi_{ij} - \Psi_{ij}) - 2\Omega\epsilon_{jl3}\Gamma_{il} - \vartheta_{ji}^{*\prime} = 0, \tag{94}$$

$$\left(\frac{\partial}{\partial t} + a + b\right)\Gamma'_{ji} + iF(\Phi_{ij} - \Psi_{ij}) - 2\Omega\epsilon_{il3}\Gamma'_{jl} - \vartheta_{ij}^{*} = 0 \tag{95}$$

and

$$\left(\frac{\partial}{\partial t} + a + b\right)(\Gamma_{ij} - \Gamma'_{ji}) - 2iF(\Phi_{ij} - \Psi_{ij}) - 2\Omega(\epsilon_{jl3}\Gamma_{il} - \epsilon_{il3}\Gamma'_{jl}) + \vartheta_{ij} = 0, \tag{96}$$

where the abbreviations (39) have been introduced. Equation (95) has been obtained from Eq. (85) by interchange of i and j and $\mathbf{x}$ and $\mathbf{x}'$.

We shall now seek solutions of Eqs. (92) to (96) which have an exponential dependence on time of the form $\exp(mt)$. As before we shall obtain a characteristic equation for m. For the diagonal terms $(i = j)$ the four first equations give

$$(m + 2a)\Phi_{ii} - iF(\Gamma_{ii} - \Gamma'_{ii}) = 0, \tag{97}$$

$$(m + 2b)\Psi_{ii} + iF(\Gamma_{ii} - \Gamma'_{ii}) - 2\Omega\epsilon_{il3}(\Psi_{li} + \Psi_{il}) = 0, \tag{98}$$

(no summation)

$$(m + a + b)\Gamma_{ii} - iF(\Phi_{ii} - \Psi_{ii}) - 2\Omega\epsilon_{il3}\Gamma_{il} = 0, \tag{99}$$

$$(m + a + b)\Gamma'_{ii} + iF(\Phi_{ii} - \Psi_{ii}) - 2\Omega\epsilon_{il3}\Gamma'_{il} = 0, \tag{100}$$

where use has been made of the relation (87). Interchanging i and j in Eq. (92) and subtracting the resulting equation from Eq. (92) we obtain

$$(m + 2a)(\Phi_{ij} - \Phi_{ji}) - iF(\Gamma_{ij} - \Gamma'_{ji} - \Gamma_{ji} + \Gamma'_{ij}) = 0,\ (i \neq j). \tag{101}$$

Similarly for Eqs. (93) and (96) we obtain

$$(m + 2b)(\Psi_{ij} - \Psi_{ji}) + iF(\Gamma_{ij} - \Gamma'_{ji} - \Gamma_{ji} + \Gamma'_{ij}) = 0 \tag{102}$$

and

$$(m + a + b)(\Gamma_{ij} - \Gamma'_{ji} - \Gamma_{ji} + \Gamma'_{ij}) - 2iF[(\Phi_{ij} - \Phi_{ji}) - (\Psi_{ij} - \Psi_{ji})]$$
$$+ 2\Omega\epsilon'_{ij3}(\Gamma_{ii} + \Gamma'_{ii} + \Gamma_{jj} + \Gamma'_{jj}) = 0, \qquad (i \neq j;\ \text{no summation}) \tag{103}$$

where ϵ'_{ij3} refers to off diagonal terms only and $\epsilon'_{ij3} = -\epsilon'_{ji3}$. Further, from the equations of the diagonal terms we derive

$$(m + a + b)(\Gamma_{ii} + \Gamma'_{ii} + \Gamma_{jj} + \Gamma'_{jj}) - 2\Omega\epsilon'_{ij3}(\Gamma_{ij} - \Gamma'_{ji} - \Gamma_{ji} + \Gamma'_{ij}) = 0.$$

$$(i \neq j;\ \text{no summation}) \qquad (104)$$

The system of Eqs. (101), (102), (103) and (104) will be satisfied if the determinant

$$\begin{vmatrix} m + 2a & 0 & -iF & 0 \\ 0 & m + 2b & iF & 0 \\ -2iF & 2iF & m + a + b & 2\Omega\epsilon'_{ij3} \\ 0 & 0 & -2\Omega\epsilon'_{ij3} & m + a + b \end{vmatrix} \qquad (105)$$

vanishes. Hence

$$(m + a + b)^2[(m + 2a)(m + 2b) + 4F^2] + 4\Omega^2\epsilon'^2_{ij3}(m + 2a)(m + 2b) = 0. \qquad (106)$$

Letting

$$M = m + a + b \qquad \text{and} \qquad c = a - b, \qquad (107)$$

Eq. (106) can be written in the form

$$M^4 - M^2(c^2 - 4F^2 - 4\Omega^2\epsilon'^2_{ij3}) - 4\Omega^2\epsilon'^2_{ij3}c^2 = 0. \qquad (108)$$

The possible values of m are therefore

$$m_k = -(a + b) \pm (2)^{-1/2}\{c^2 - 4F^2 - 4\Omega^2\epsilon'^2_{ij3} \pm [(c^2 - 4F^2)^2 + 8\Omega^2\epsilon'^2_{ij3}(c^2 + 4F^2 + 2\Omega^2\epsilon'^2_{ij3})]^{1/2}\}^{1/2} \qquad (k = 4, 5, 6, 7). \qquad (109)$$

3. *Discussion of the law of decay.* It is seen from Eqs. (97), (98), (99), (100) and the result (109) that the time dependence of the spectral tensors with at least one index in the direction of the angular velocity vector is uninfluenced by the Coriolis force and the solutions have the same form as given by Eqs. (41) and (42). However, if i and j differ from 3 the results of Sec. III are modified, each tensor splitting up into four terms:

$$\Phi_{ij}(\boldsymbol{\kappa}, t) = \sum_k \Phi^{(k)}(\boldsymbol{\kappa}) \exp(m_k t) \qquad (k = 4, 5, 6, 7;\ i \neq 3,\ j \neq 3). \qquad (110)$$

We have corresponding solutions for the remaining tensors. The behaviour of the solutions in the limiting cases $W = 0$, $\Omega = 0$ and $W = \Omega = 0$ will not be discussed in detail here.

We shall now discuss the manner in which the angular velocity modifies the spectral tensors when $i \neq 3, j \neq 3$ and $\epsilon'^{\,2}_{ij3} = 1$. Let

$$\zeta = 2(\kappa_2 W_2 + \kappa_3 W_3)/[\kappa^2(\lambda - \nu)] \qquad (111)$$

and

$$\eta = 2\Omega/[\kappa^2(\lambda - \nu)]; \qquad (112)$$

ζ and η are associated with the forms $W_c L_c/(\lambda - \nu)$ and $U_c L_c/(\lambda - \nu)$ respectively, where W_c, L_c and U_c refer to characteristic wave velocities, lengths and rotation velocities. They may also be connected with parameters used in earlier works.[4,7,13] For small values of Ω the factors of decay become

$$m_{4,5} = -(a + b) \pm c\{1 - \zeta^2[1 - \eta^2/(1 - \zeta^2)]\}^{1/2} \tag{113}$$

and

$$m_{6,7} = -(a + b) \pm ic\eta/(1 - \zeta^2)^{1/2}, \tag{114}$$

provided $\eta^2(1 + \zeta^2)/(1 - \zeta^2) \ll 1$. From a comparison of Eqs. (109) and (41) we conclude that

$$m_{4,5} \rightarrow m_{1,2} \quad \text{and} \quad m_{6,7} \rightarrow m_3 \quad \text{when} \quad \Omega \rightarrow 0. \tag{115}$$

Thus, if the aperiodic solutions (41) are perturbed by a small angular velocity, the damping effect of the magnetic field in the factors, m_1 and m_2 will be counteracted while the term in m_3 will be split up into two periodic solutions. For sufficiently small values of Ω and F the spectral vorticity tensor will have the asymptotic behaviour

$$\Psi_{ij}(\boldsymbol{\kappa}, t) \approx \Psi_{ij}^{(4)}(\boldsymbol{\kappa}) \exp\{-2[\nu\kappa^2 + (\kappa_2 W_2 + \kappa_3 W_3)^2[1 - 4\Omega^2/\kappa^4(\lambda - \nu)^2]/\kappa^2(\lambda - \nu)]t\} \quad (i \neq 3, j \neq 3); \tag{116}$$

this corresponds to the earlier law (71).

The occurence of periodicity can be discussed in terms of M, given by Eqs. (107) and (109). We have

$$2M^2 = c^2(1 - \zeta^2 - \eta^2)\{1 \pm [1 + 4\eta^2/(1 - \zeta^2 - \eta^2)^2]^{1/2}\}. \tag{117}$$

If $\zeta^2 + \eta^2 < 1$ the solutions corresponding to the roots $m_{4,5}$ are aperiodic while the solutions corresponding to $m_{6,7}$ are periodic. On the other hand if $\zeta^2 + \eta^2 > 1$ the roots $m_{4,5}$ lead to periodic solutions while $m_{6,7}$ lead to aperiodic solutions. The critical case is given by

$$\zeta^2 + \eta^2 = 1, \tag{118}$$

when m_k has the values

$$m_{4,5}^{(c)} = -(a + b) \pm (2\Omega c)^{1/2}, \qquad m_{6,7}^{(c)} = -(a + b) \pm i(2\Omega c)^{1/2}. \tag{119}$$

The result (109) may be confirmed by a discussion similar to that in Sec. III, 4. Thus, reconsidering the situation indicated in Fig. 1 we shall now assume that the system is partaking in rotation and for simplicity suppose that the axis of rotation is in the z-direction. Further we shall assume a homogeneous state and that it is possible to introduce a local rectangular system of coordinates in which the approximations

$$\partial/\partial x \approx 0, \qquad \partial/\partial y \approx 0 \tag{120}$$

are valid. Equations (5) and (6) for the x- and y-components of small disturbances, $\mathbf{V}$ and $\mathbf{v}$, become

$$\left(\frac{\partial}{\partial t} - \lambda \frac{\partial^2}{\partial z^2}\right)(V_x, V_y) = W \frac{\partial}{\partial z}(v_x, v_y) \tag{121}$$

and

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial z^2}\right)(v_x, v_y) = W \frac{\partial}{\partial z}(V_x, V_y) + 2\Omega(v_y, -v_x). \tag{122}$$

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[13]S. Chandrasekhar, Proc. Roy. Soc. **A216**, 293 (1953).

If V_x and V_y are eliminated we get

$$\left\{\left(\frac{\partial}{\partial t} - \lambda \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial z^2}\right) - W^2 \frac{\partial^2}{\partial z^2}\right\}(v_x , v_y) = 2\Omega\left(\frac{\partial}{\partial t} - \lambda \frac{\partial^2}{\partial z^2}\right)(v_y , -v_x). \quad (123)$$

Separating the variables in the manner

$$v_i = v_i^{(0)} \sin (\kappa z) \exp (nt), \qquad \kappa = \pi k/L \qquad (k = 1, 2, \cdots), \quad (124)$$

we find that with the abbreviations (39) we must satisfy the equation

$$[(n + a)(n + b) + F^2](v_x , v_y) = 2\Omega(n + a)(v_y , -v_x). \quad (125)$$

The condition that Eq. (125) allows non-zero solutions is

$$[(n + a)(n + b) + F^2]^2 + 4\Omega^2(n + a)^2 = 0. \quad (126)$$

In other words

$$n = -\tfrac{1}{2}[(a + b) \pm 2i\Omega] \pm \tfrac{1}{2}\{[(a - b) \pm 2i\Omega]^2 - 4F^2\}^{1/2}. \quad (127)$$

A product of two of these solutions gives exponents, $n^{(I)} \pm n^{(II)}$, which are easily seen to be of the form (109); thus

$$\begin{aligned} n^{(I)} \pm n^{(II)} &= -(a + b) \pm \tfrac{1}{2}\{(c + 2i\Omega)^2 - 4F^2\}^{1/2} \pm \tfrac{1}{2}\{(c - 2i\Omega)^2 - 4F^2\}^{1/2} \\ &= -(a + b) \pm N, \end{aligned} \quad (128)$$

where

$$N^2 = \tfrac{1}{2}(c^2 - 4\Omega^2 - 4F^2) \pm \tfrac{1}{2}[(c^2 - 4\Omega^2 - 4F^2)^2 + 16\Omega^2c^2]^{1/2}; \quad (129)$$

in other words the same values of M^2 as given by Eq. (117).[14]

The results of this section may be summed up as follows. If an angular velocity, $\mathbf{\Omega}$, is introduced into the turbulent state of motion of an electrically conducting liquid in a magnetic field the factors of decay of the tensor components *perpendicular to* $\mathbf{\Omega}$ *will be split up into four solutions. The passage from periodic to aperiodic motions is displaced* from $\zeta^2 = 1$ to $\zeta^2 < 1$, i.e., to weaker magnetic fields. For small values of Ω, the damping effect of the magnetic field in the aperiodic solutions is *counteracted by the angular velocity.* The remaining tensor components are uninfluenced by the angular velocity.

4. *The importance of the Coriolis force.* From the foregoing discussion it is seen that the effect of the Coriolis force will be of the same order of magnitude as the electromagnetic effects if ζ and η are of the same order, or from the expressions (111) and (112) if

$$W = B/(\mu\rho)^{1/2} \approx \Omega L/2\pi, \quad (130)$$

where L represents the linear dimensions of the disturbance. For the sun typical values are $W = 4\ m/s$ and $\Omega = 2.7 \times 10^{-6}\ s^{-1}$; it is seen that for these values the Coriolis force plays an essential role when the linear dimensions of the disturbance become larger than $L = 6 \times 10^6\ m$. Since the solar radius is about $7 \times 10^8\ m$ this will apply to a wide range of possible disturbances. It will probably be even more pronounced for stars with larger angular velocities.

In experiments on turbulence on laboratory scale, the largest turbulence elements

[14]The same solutions are also obtained in the non-dissipative state from results by Chandrasekhar, Astrophys. J. 119, 7 (1954).

will hardly exceed 10^{-2} m; and in mercury a common value of the wave velocity is about 5 m/s and of the angular velocity about 10 s^{-1}. Under these circumstances the electromagnetic effect will be about 50 times larger than that of the Coriolis force.

The influence of the Coriolis force on the critical damping in liquid sodium, as given by

$$(\kappa_k W_k)^2 + \Omega^2 = \kappa^4(\lambda - \nu)^2/4 \tag{131}$$

is shown in Fig. 3.

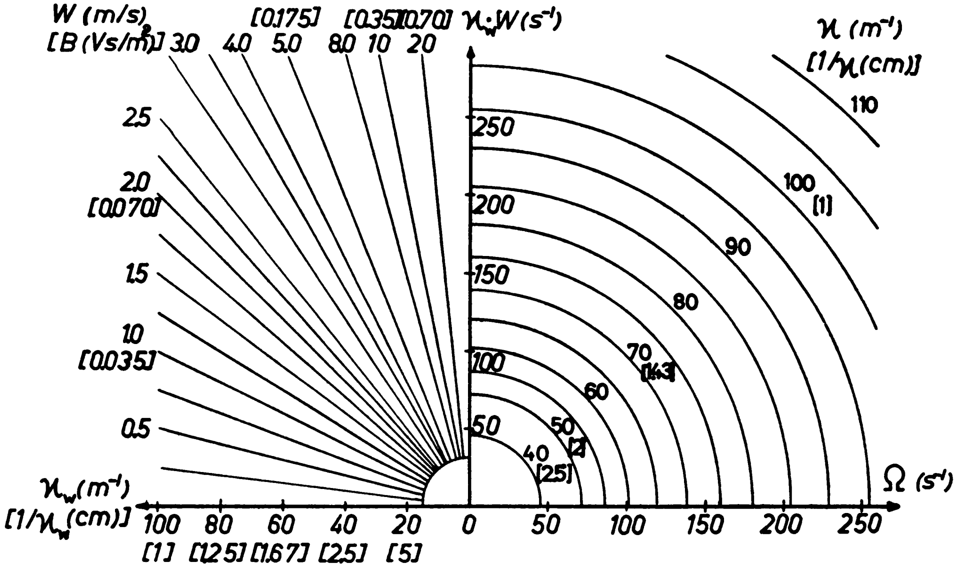

FIG. 3. Critical values of the wave velocity, W, and the angular velocity, $\mathbf{\Omega}$, in liquid sodium for turbulence elements with velocity components perpendicular to $\mathbf{\Omega}$. The total wave number is $\boldsymbol{\kappa}$ and κ_W is the component in the direction of the magnetic field, $\mathbf{B}$, where $\mathbf{W} = \mathbf{B}(\mu\rho)^{-\frac{1}{2}}$. $\lambda = 0.057$ m^2/s at 120°C.

V. The stationary state. In the previous sections we have assumed a certain spectral distribution of turbulence at an initial time, $t = 0$, and have studied the decay within a region, insulated from external sources of energy. However, in most actual cases the turbulence will be stationary and the mean intensity in every part of the spectrum will be constant with time. There is an input of energy in some parts of the spectrum and an equilibrium state is obtained, when the same amount of energy is converted into heat motion in other parts of the spectrum.

1. *The stationary equations.* We shall now investigate how far conclusions on the homogeneous, stationary state may be drawn from a linear theory that starts with the time-dependent Eqs. (31), (32), (33) and (34) for an electrically conducting liquid in a magnetic field. Introduce the tensors

$$E_{ij}^{(M)}(\mathbf{r}, t) = -\frac{\partial}{\partial t} M_{ij}(\mathbf{r}, t), \qquad E_{ij}^{(K)}(\mathbf{r}, t) = -\frac{\partial}{\partial t} K_{ij}(\mathbf{r}, t) \tag{132}$$

and

$$E_{ij}^{(MH)}(\mathbf{r}, t) = -W_k \frac{\partial}{\partial r_k} [L_{ij}(\mathbf{r}, t) - L_{ji}(-\mathbf{r}, t)] \tag{133}$$

and the corresponding Fourier transforms $\epsilon_{ij}^{(M)}(\kappa, t)$, $\epsilon_{ij}^{(K)}(\kappa, t)$ and $\epsilon_{ij}^{(MH)}(\kappa, t)$, where

$$\epsilon_{ij}^{(M)}(\kappa, t) = (8\pi^3)^{-1} \iiint E_{ij}^{(M)}(\mathbf{r}, t) \exp(-i\kappa\cdot\mathbf{r})\, dr_1\, dr_2\, dr_3 \qquad \text{etc.} \tag{134}$$

It is clear that $E_{ii}^{(M)}(0, t)$ and $E_{ii}^{(K)}(0, t)$ are the total rate of magnetic and kinetic energy losses and $E_{ii}^{(MH)}(0, t)$ is the decrease in magnetic energy per unit time and unit mass due to the magneto-hydrodynamic interaction as shown by Eqs. (29) and (30). Equations (31,) (32), (33) and (34) can now be written as

$$E_{ij}^{(M)}(\mathbf{r}, t) = E_{ij}^{(MH)}(\mathbf{r}, t) - 2\lambda\nabla^2 M_{ij}(\mathbf{r}, t), \tag{135}$$

$$E_{ij}^{(K)}(\mathbf{r}, t) = -E_{ij}^{(MH)}(\mathbf{r}, t) - 2\nu\nabla^2 K_{ij}(\mathbf{r}, t) \tag{136}$$

and

$$\left[\frac{\partial}{\partial t} - (\lambda + \nu)\nabla^2\right] E_{ij}^{(MH)}(\mathbf{r}, t) = -2W_k^2 \frac{\partial^2}{\partial r_k^2}[M_{ij}(\mathbf{r}, t) - K_{ij}(\mathbf{r}, t)]; \tag{137}$$

and the corresponding spectral equations are

$$\epsilon_{ij}^{(M)}(\kappa, t) = \epsilon_{ij}^{(MH)}(\kappa, t) + 2\lambda\kappa^2\Lambda_{ij}(\kappa, t), \tag{138}$$

$$\epsilon_{ij}^{(K)}(\kappa, t) = -\epsilon_{ij}^{(MH)}(\kappa, t) + 2\nu\kappa^2\Omega_{ij}(\kappa, t), \tag{139}$$

$$\left[\frac{\partial}{\partial t} + (\lambda + \nu)\kappa^2\right]\epsilon_{ij}^{(MH)}(\kappa, t) = 2\kappa_k^2 W_k^2[\Lambda_{ij}(\kappa, t) - \Omega_{ij}(\kappa, t)] \tag{140}$$

and

$$\epsilon_{ij}^{(MH)}(\kappa, t) = -i\kappa_k W_k[\Upsilon_{ij}(\kappa, t) - \Upsilon_{ji}(-\kappa, t)], \tag{141}$$

where the tensors of Sec. III, 2 have also been introduced.

If no external energy sources are present, the right hand sides of the contracted equations (138) and (139) represent the energies lost per unit time in the range $d\kappa_1\, d\kappa_2\, d\kappa_3$ due to interaction and dissipation. In a stationary state this loss is balanced by an equal input of energy from external sources. We now make the assumption that these considerations are valid not only for the contracted expressions at $\mathbf{r} = 0$ but can be generalized to all components and all values of $\mathbf{r}$, i.e., the stationary problem corresponds to the case when there is no dependence on t in Eqs. (133), (135), (136), (137) and their spectral equivalents. The quantities $E_{ij}^{(M)}(\mathbf{r})$ and $E_{ij}^{(K)}(\mathbf{r})$ now represent the external magnetic and kinetic "inputs of energy" per unit time.

There is no formal difficulty in extending the foregoing results to the general, non-linear case. The various correlations can be formed in the same straightforward manner as in Sec. III and the derivations need not be given here. The stationary equivalents of Eqs. (138) and (139) become

$$\epsilon_{ij}^{(M)}(\kappa) = \epsilon_{ij}^{(MH)}(\kappa) + 2\lambda\kappa^2\Lambda_{ij}(\kappa) + \alpha_{ij}(\kappa) \tag{142}$$

and

$$\epsilon_{ij}^{(K)}(\kappa) = -\epsilon_{ij}^{(MH)}(\kappa) + 2\nu\kappa^2\Omega_{ij}(\kappa) + \beta_{ij}(\kappa), \tag{143}$$

where α_{ij} and β_{ij} are the non-linear terms. The sum of these equations is

$$\epsilon_{ij}(\kappa) = 2\lambda\kappa^2\Lambda_{ij}(\kappa) + 2\nu\kappa^2\Omega_{ij}(\kappa) + \alpha_{ij}(\kappa) + \beta_{ij}(\kappa). \tag{144}$$

2. *The terms of interaction.* The equations in the preceding section form a starting point for a discussion of the physical consequences of different approximations. The simplest situation occurs for small amplitudes and without external fields when there are no interaction terms in Eqs. (142) and (143). If the external sources are taken away the turbulent fields of the magnetic and the kinetic energies decay independently of each other as is evident from Eqs. (52) and (53). In the stationary state every spectral range requires a separate magnetic and kinetic energy source; further there will be no exchange also of the energy in the different wave numbers and in different directions[15].

However, if an external magnetic field is introduced the expressions (142) and (143) show that there is now a coupling between the magnetic and the kinetic fields in every spectral range, also in first order. With an input $\epsilon_{ii}^{(M)}(\boldsymbol{\kappa}_0) = \epsilon_0^{(M)}$ of magnetic energy per unit time and unit range of wave number space and an input $\epsilon_0^{(K)}$ of kinetic energy at the wave number $\boldsymbol{\kappa}_0$ the equations (142), (143) and (140) become:

$$\epsilon_0^{(M)} = \epsilon_{ii}^{(MH)}(\boldsymbol{\kappa}_0) + 2\lambda\kappa_0^2\Lambda_{ii}(\boldsymbol{\kappa}_0), \tag{145}$$

$$\epsilon_0^{(K)} = -\epsilon_{ii}^{(MH)}(\boldsymbol{\kappa}_0) + 2\nu\kappa_0^2\Omega_{ii}(\boldsymbol{\kappa}_0) \tag{146}$$

and

$$(\lambda + \nu)\kappa_0^2\epsilon_{ii}^{(MH)}(\boldsymbol{\kappa}_0) = 2\kappa_{k0}^2W_k^2[\Lambda_{ii}(\boldsymbol{\kappa}_0) - \Omega_{ii}(\boldsymbol{\kappa}_0)]. \tag{147}$$

The solutions of these equations are

$$\Lambda_{ii}(\boldsymbol{\kappa}_0) = \{[\nu(\lambda + \nu)\kappa_0^4 + F_0^2]\epsilon_0^{(M)} + F_0^2\epsilon_0^{(K)}\}/\{2\kappa_0^2(\lambda + \nu)(\lambda\nu\kappa_0^4 + F_0^2)\}, \; F_0 = \kappa_{k0}W_k \tag{148}$$

and

$$\Omega_{ii}(\boldsymbol{\kappa}_0) = \{[\lambda(\lambda + \nu)\kappa_0^4 + F_0^2]\epsilon_0^{(K)} + F_0^2\epsilon_0^{(M)}\}/\{2\kappa_0^2(\lambda + \nu)(\lambda\nu\kappa_0^4 + F_0^2)\}. \tag{149}$$

In particular for $W \rightarrow \infty$ we have

$$\Lambda_{ii}/\Omega_{ii} = 1. \tag{150}$$

This is to be expected, since the losses become less important for large values of the external field and the magneto-hydrodynamic waves approach the "ideal" state, where

$$\mathbf{V} = \mathbf{h}(\mu/\rho)^{1/2} = \mathbf{v} \tag{151}$$

and there is equipartition between the magnetic and the kinetic energies. Especially for zero input of magnetic energy, the kinetic energy given by Eq. (149) decreases with a ratio

$$\Omega_{ii}(\boldsymbol{\kappa}_0\, , F_0 = \infty)/\Omega_{ii}(\boldsymbol{\kappa}_0\, , F_0 = 0) = \nu/(\lambda + \nu) \tag{152}$$

when the quantity F_0 increases from zero to infinity. This conversion of three-dimensional turbulence into two-dimensional turbulence is consistent with the decrease in the time of decay given by Eq. (69).

Finally, we may conclude as in earlier discussions of turbulence that an energy transport through the whole spectrum is only possible in the presence of the non-linear terms. The sum of these terms in Eq. (144) generally differs from zero and there is an energy flow through the spectral range in question.

[15]A discussion of the properties of the interaction terms in hydrodynamics may be found in G. K. Batchelor, *The theory of homogeneous turbulence*, Cambridge Univ. Press, Chap. V, 1953.

VI. Applications to experiments. Experiments on turbulent flow of mercury in a homogeneous magnetic field have been carried out by Hartmann[5,6] and Lehnert[7,8] with a maximum field strength of about 1 Vs/m^2 (10^4 gauss). Hartmann has shown that the necessary pressure difference to keep constant volume flow between the ends of a rectangular channel decreases in the turbulent state when the magnetic field increases (similar investigations have been carried out by Murgatroyd[16]). In the latter experiment the torque between an inner, rotating and an outer, fixed cylinder decreases for a given angular velocity if a homogeneous, axial magnetic field is introduced. Thus, in both experiments the field seems to have the effect of suppressing the intensity of turbulence.

These results may be given a qualitative explanation in terms of the linear theories of Secs. III and V. The energy sinks formed by the dominating dissipation for large wave numbers will cause a mean transfer of energy from the largest whirls, which receive the greatest part of the external input of energy. It is reasonable to assume that a great part of this energy will reach the energy consuming region for large wave numbers before being dissipated. In this region we are allowed to apply the linear theory and obtain a connexion between input power and turbulent intensity of the small scale motion. Since it is unlikely that the external magnetic field will have a reverse effect on the large scale motion we may expect that the linear theory would not depart too far from the real physical situation.

From Eq. (71) and Fig. 2, giving the result of the theory of decay of turbulence, the damping effect on vortices having finite wave numbers in the direction of the magnetic field the reduction of the total isotropic intensity by a factor of about 2/3 is apparent. This implies that a given intensity of the kinetic energy Ω_{ii} requires a larger input of energy in the presence of the field than if the field were absent. Further, the tangential force due to turbulent exchange of momentum to the walls of the experimental arrangements increases with the intensity of the kinetic energy. Thus, a given pressure difference or a given torque requires an increasing input of energy for an increasing field strength, i.e. an increasing volume flow and an increasing angular velocity respectively. Consequently, if a magnetic field is present, the curves for the pressure difference and the torque as functions of the volume flow and the angular velocity respectively must lie below the corresponding curves in the absence of the magnetic field.

These same conclusions may be drawn from Eqs. (149) and (152), representing stationary turbulence; these equations show how the intensity of the kinetic energy decreases with increasing field and constant input of energy. Even if we assume an input of both kinds of energy the ratio

$$\Omega_{ii}(\kappa_0\ , F_0 = \infty)/\Omega_{ii}(\kappa_0\ , F_0 = 0) = \nu\epsilon_0^{(K)}/[(\lambda + \nu)(\epsilon_0^{(K)} + \epsilon_0^{(M)})] \tag{153}$$

will be very small, since $\lambda \gg \nu$ for mercury and $E^{(K)}$ and $E^{(M)}$ will probably be of the same order during the transport of energy towards high wave numbers. Consequently the conclusions both from the theory of decay and the stationary theory of turbulence are in agreement with experimental observations.

Finally, it also follows from the results of Sec. IV, 4 that the Coriolis force does not play any essential role in the experiment with rotating cylinders.

VII. Concluding remarks. A linear theory gives some information about the physical nature of turbulence in magneto-hydrodynamics, especially in the case of linear inter-

[16]Phil. Mag. 44, 1348 (1953).

action forces caused by external magnetic fields and Coriolis fields. However, this does not show the feature of the general turbulent interaction and the process of distribution of energy. One way to solve this difficult problem is to continue the work on the "quasi-statistical" theory of correlations. Another way, which at the present stage also seems to be associated with great difficulties, is to make a purely statistical attack, somewhat along the lines of Burgers[17] and Onsager[18]. But such a theory, even in the absence of external fields, is probably not available.

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[17]Proc. Acad. Sci. Amsterdam **32**, 414 (1929).

[18]Nuovo Cim., Supplement **6**, No. 2, 279 (1949).