CANONICAL EQUATIONS FOR NON-LINEARIZED STEADY IRROTATIONAL CONICAL FLOW*

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1. Introduction. A steady flow is *conical* if it contains a *vertex*, P, such that the velocity components, pressure, density, and entropy are constant on every half line emanating from P. Numerous authors [e.g. 5, 9, 11, 12, 19] have investigated *linearized* conical flows, which can be attacked with the aid of complex function theory inside the Mach cone of the vertex. A few [2, 16, 22] have considered second order approximations or perturbed axisymmetric non-linearized conical flow [20]. So far, the study of *non-linearized* conical flow has been confined to special examples [3, 4, 10, 13, 21], or the use of relaxation methods [14] or three-dimensional characteristic treatments [15]. It is natural to ask how far the discussion of the non-linearized case can be carried. Although no attempt has been made in this note to reach the ideal conclusion, an existence and uniqueness theorem, the present inquiry does lead at least to the formulation of an interesting boundary value problem and to a basis for numerical computation.

Most of the discussion is concerned with the flow about a conical body completely surrounded by a conical shock. The change in entropy at a shock wave is of higher order than the changes in the other flow functions. For this reason the equations of supersonic flow are frequently simplified by neglecting the variation of entropy behind the shock, as will be done here. Now the flow field *may* be but is not necessarily irrotational. In this note the flow will be assumed to be irrotational. This makes it possible to reduce the partial differential equations to simple canonical forms, to determine the class of transformations under which these are invariant, and eventually to map the conical surfaces of obstacle and shock onto *known* curves. Thus the difficulties which arise from not knowing the location of the shock wave can be avoided. Finally, it should be remarked that Taylor-Maccoll flow has all of the assumed properties for the completely elliptic case. Accordingly, this special example is a fruitful source of conjectures as to the influence of the various parameters and of estimates for numerical computation.

2. Fundamental equations. The velocity potential function, ϕ, of a steady irrotational flow satisfies

$$(a^2\delta_{ij} - u_iu_j)\partial^2\phi/\partial x_i\,\partial x_j = 0, \qquad i, j = 1, 2, 3, \tag{2.1}$$

where the velocity

$$u_i = \partial\phi/\partial x_i\,, \tag{2.2}$$

the squared velocity of sound

$$a^2 = \tfrac{1}{2}(\gamma - 1)(1 - u_iu_i), \tag{2.3}$$

*Received Dec. 11, 1953.

x_i are cartesian coordinates, Kronecker's $\delta_{ij} = 1(0)$ accordingly as $i = (\neq)j$, and repetition of indices implies summation over their ranges. In a conical flow with vertex (0, 0, 0) let

$$\phi = z\Phi(X), \qquad X_\alpha = x_\alpha/z, \qquad z = x_3\,, \qquad \alpha = 1, 2. \tag{2.4}$$

Note that points in the X_1X_2 plane correspond to lines through the origin of the $x_1x_2x_3$-space, and conversely. Now (2.1) becomes

$$[a^2(\delta_{\alpha\beta} + X_\alpha X_\beta) - (u_\alpha - wX_\alpha)(u_\beta - wX_\beta)]\,\partial^2\Phi/\partial X_\alpha\,\partial X_\beta = 0, \tag{2.5}$$

where

$$u_\alpha = \partial\Phi/\partial X_\alpha\,, \tag{2.6}$$

$$w \equiv u_3 = \Phi - u_\alpha X_\alpha\,. \tag{2.7}$$

If $u_\alpha(X)$ are functionally independent, subject (2.5) to the Legendre transformation (2.7) to obtain [4]

$$[a^2(\delta_{\alpha\beta} + w_\alpha w_\beta) - (u_\alpha + ww_\alpha)(u_\beta + ww_\beta)](-1)^{\alpha+\beta}\,\partial^2 w/\partial u_{\alpha+1}\,\partial u_{\beta+1} = 0, \tag{2.8}$$

where $\alpha + 1$ and $\beta + 1$ are to be reduced mod 2, and where the velocity $u_1\,, u_2\,, w(u_1\,, u_2)$ is assigned to the point

$$X_\alpha = -\partial w/\partial u_\alpha = -w_\alpha\,. \tag{2.9}$$

Now introduce more general parameters $\mu_\alpha = \mu_\alpha(u_1\,, u_2)$. Then

$$u_i = u_i(\mu_1\,, \mu_2). \tag{2.10}$$

Let

$$q^2 = u_iu_i\,, \qquad q\,\partial q/\partial\mu_\alpha = u_i\,\partial u_i/\partial\mu_\alpha\,, \qquad g_{\alpha\beta} = (\partial u_i/\partial\mu_\alpha)(\partial u_i/\partial\mu_\beta),$$
$$b_{\alpha\beta} = (g_{11}g_{22} - g_{12}^2)^{-1/2}\det\,\|\,\partial^2 u_i/\partial\mu_\alpha\,\partial\mu_\beta\,, \partial u_i/\partial\mu_1\,, \partial u_i/\partial\mu_2\,\|. \tag{2.11}$$

For the special choice $\mu_\alpha = u_\alpha$ (2.8) implies

$$(a^2 g_{\alpha\beta} - q^2\,\partial q/\partial\mu_\alpha\,\partial q/\partial\mu_\beta)(-1)^{\alpha+\beta}b_{\alpha+1\,\beta+1} = 0. \tag{2.12}$$

Since $g_{\alpha\beta}\,, b_{\alpha\beta}\,,$ and $\partial q/\partial\mu_\alpha$ are covariant tensors, (2.12) is a tensor equation. Hence its form is independent of the particular choice of parameters μ_α. Now observe that (2.9) implies $x_i\partial u_i/\partial\mu_\alpha = 0$. Thus the point $u_i(\mu)$ on the hodograph surface corresponds to the half line

$$x_i = r\nu_i\,, \qquad \nu_i\,\partial u_i/\partial\mu_\alpha = 0, \tag{2.13}$$

where $\nu_i(\mu)$ is a vector normal to this surface at $u_i(\mu)$, and r is a parameter.

So far the three functions $u_i(\mu)$ have been subjected to (2.12) only. Two more equations are required to determine u_i. As suggested by [6, p. 130], they can be chosen to simplify the coefficients of (2.12) in a way which depends on the nature of the quadratic form

$$(a^2 g_{\alpha\beta} - q^2\,\partial q/\partial\mu_\alpha\,\partial q/\partial\mu_\beta)L_\alpha L_\beta\,. \tag{2.14}$$

This is *definite* (*indefinite*) if and only if the determinant of its coefficients is greater (less) than zero. For $\mu_\alpha = u_\alpha$ and by (2.9) the determinant becomes $a^2[(1 + X_\alpha X_\alpha)(a^2 -$

$q^2) + (u_\alpha X_\alpha + w)^2]$. In *subsonic* flow (2.14) is always definite. In the *supersonic* case (2.14) will be definite (indefinite) if and only if the half line (2.13) lies *inside* (*outside*) the Mach cone based on $u_i(\mu)$ and with vertex at (0, 0, 0). In the former (latter) case equations (2.5), (2.8) and (2.12) will be said to be of *elliptic* (*hyperbolic*) *type* for reasons which will become apparent in Sec. 3 and 4.

3. The hyperbolic case. This section has been included to round out the discussion, for the sake of contrast with the material in Sec. 4. It is not essential for the sequel. Examples of conical flows which contain regions where $x_i = r\nu_i$ is outside its Mach cone have been discussed in [10].

If (2.14) is indefinite, choose the parameters μ_1 and μ_2 so that

$$A \equiv a^2 u_{i1}u_{i1} - (u_i u_{i1})^2 = 0, \tag{3.1}$$

$$B \equiv a^2 u_{i2}u_{i2} - (u_i u_{i2})^2 = 0, \tag{3.2}$$

where $u_{i\alpha} = \partial u_i/\partial\mu_\alpha$. By (2.11) and (2.12)

$$u_{i12} = C_\beta u_{i\beta} \tag{3.3}$$

for some functions C_β. Eliminate u_{i12} from $A_2 = B_1 = 0$ to obtain

$$(a^2 g_{\alpha\beta} - q^2 q_\alpha q_\beta)C_\beta = qq_\alpha g_{12} - aa_{\alpha+1}g_{\alpha\alpha} \qquad (\alpha \textit{ not} \text{ summed}). \tag{3.4}$$

Conversely, suppose $u_i(\mu)$ is a solution of (3.3), where C_β are defined by (3.4). Then $A_2 = B_1 = 0$, so

$$A = A(\mu_1), \qquad B = B(\mu_2). \tag{3.5}$$

Equation (3.3) is a *hyperbolic* system of second order quasi-linear partial differential equations with the same principal parts, and μ_1 and μ_2 are *characteristic* variables. The existence and uniqueness, in the small, of solutions of non-characteristic and characteristic initial value problems for such systems have been discussed in [6, pp. 316-326] and there are numerous references in [1]. In conical flow the initial conditions of fluid mechanical origin, discussed in Sec. 4, appear to be more complicated than those ordinarily considered in the literature. To these must be adjoined an initial condition such as

$$A = B. \tag{3.6}$$

By (3.5) this implies $A = B =$ constant. Then it will suffice to impose $A = 0$ at *one* point, to obtain $A = B = 0$.

To simplify the formulation of initial value problems, it would be desirable to standardize initial curves in the $\mu_1\mu_2$-plane as far as possible. This raises the question as to what class of transformations $\mu_\alpha = \mu_\alpha(\mu^*)$ will preserve the forms of (3.1) and (3.2). The coefficient tensor for the transform of (2.12) is

$$a^{*2}g^*_{\alpha\beta} - q^{*2}q^*_\alpha q^*_\beta = (a^2 g_{\sigma\delta} - q^2 q_\sigma q_\delta)(\partial\mu_\sigma/\partial\mu^*_\alpha)(\partial\mu_\delta/\partial\mu^*_\beta). \tag{3.7}$$

By (3.1) and (3.2) its diagonal elements vanish if and only if $(\partial\mu_1/\partial\mu^*_1)\partial\mu_2/\partial\mu^*_1 = (\partial\mu_1/\partial\mu^*_2)\partial\mu_2/\partial\mu^*_2 = 0$. Hence either $\mu_1 = \mu_1(\mu^*_1)$, $\mu_2 = \mu_2(\mu^*_2)$, or else $\mu_1 = \mu_1(\mu^*_2)$, $\mu_2 = \mu_2(\mu^*_1)$. That is, *the most general transformation with the desired invariance consists of changes of scales along the axes, possibly combined with reflection with respect to* $\mu_2 = \mu_1$. By such transformations two initial curves can be mapped simultaneously onto straight lines.

Examples such as those considered in [10, 15] make it seem likely that every conical flow which contains regions where the equations are of hyperbolic type also contain regions of elliptic type. Hence the canonical form (3.3) cannot be used for an entire flow.

4. The elliptic case. First it should be remarked that at all points in a Taylor-Maccoll flow the ray $r\nu_i$ is inside its Mach cones. Hence (2.14) is definite throughout. It seems reasonable to conjecture that this will also be the case at least for slightly yawing or slightly distorted circular cones.

If (2.14) is definite, choose *isothermic* (relative to the tensor $a^2 g_{\alpha\beta} - q^2 q_\alpha q_\beta$) parameters μ_1 and μ_2, i.e. set

$$A \equiv a^2 u_{i1} u_{i2} - u_i u_{i1} u_j u_{j2} = 0, \tag{4.1}$$

$$2B \equiv a^2(u_{i1}u_{i1} - u_{i2}u_{i2}) - (u_i u_{i1})^2 + (u_i u_{i2})^2 = 0. \tag{4.2}$$

Then by (2.11) and (2.12)

$$u_{i\alpha\alpha} = C_\beta u_{i\beta} \tag{4.3}$$

for some functions C_β. Now form

$$\partial A/\partial\mu_1 - \partial B/\partial\mu_2 = 0, \qquad \partial A/\partial\mu_2 + \partial B/\partial\mu_1 = 0 \tag{4.4}$$

and eliminate $u_{i\alpha\alpha}$ to obtain

$$(a^2 g_{\alpha\beta} - q^2 q_\alpha q_\beta)C_\beta = \tfrac{1}{2} g_{\beta\beta}(a^2 + q^2)_\alpha - g_{\alpha\beta} a^2_\beta . \tag{4.5}$$

Conversely, suppose $u_i(\mu)$ is a solution of (4.3), where C_β are defined by (4.5). Since this implies (4.4), $A + iB$ is an analytic function of $\mu_1 + i\mu_2$.

Equation (4.3) is a system of *elliptic* second order quasi-linear partial differential equations with the same principal parts. Since the coefficients C_β are rational functions of their nine arguments u_i, u_{i1}, and u_{i2}, the solutions of (4.3) are analytic [6, p. 339]. The existence and uniqueness of the solutions of Dirichlet's problem for such a system for a small enough region of the $\mu_1\mu_2$-plane have been discussed in [6, p. 287]. Unfortunately, the conical flow problem leads to the non-linear boundary conditions described below, rather than to the linear Dirichlet, Neumann, or mixed type usually considered.

As a preliminary to the formulation of a boundary value problem, first determine the class of transformations $\mu_\alpha = \mu_\alpha(\mu^*)$ that will preserve the forms of (4.1) and (4.2), and therefore of (4.3) and (4.5). The transformed coefficient tensor (3.7) becomes $a^{*2}g^*_{\alpha\beta} - q^{*2}q^*_\alpha q^*_\beta = [a^2 g_{11} - (qq_1)^2](\partial\mu_\alpha/\partial\mu^*_\alpha)(\partial\mu_\alpha/\partial\mu^*_\beta)$. Then μ^*_α will also be isothermic parameters if and only if $(\partial\mu_\alpha/\partial\mu^*_\alpha)(\partial\mu_\alpha/\partial\mu^*_\beta) = D^2\delta_{\alpha\beta}$ for some function D. Hence, *any conformal map of the* $\mu_1\mu_2$*-plane onto the* $\mu^*_1\mu^*_2$*-plane has the desired invariance.*

Suppose that the conical body (shock) is described by $f(X_1, X_2) = 0(g(X_1, X_2) = 0)$. Assume that $f = 0$ and $g = 0$ are closed curves in the X_1X_2-plane which do not intersect themselves or each other, and that $f = 0$ is inside $g = 0$. Suppose that their images $F(\mu) = 0$ and $G(\mu) = 0$ in the $\mu_1\mu_2$-plane have the same properties. All of these conditions can be satisfied by Taylor-Maccoll flow, and should also be attainable in slightly perturbed Taylor-Maccoll flows, at least. Then the region between $F = 0$ and $G = 0$ can be mapped conformally onto the annulus $1 \leqq |\mu^*_1 + i\mu^*_2| \leqq R$ for some R[7]. Assume that this has been done, that the body maps onto the unit circle, the shock onto the other boundary. Hereafter replace μ^*_α by μ_α. Now proceed to the boundary conditions for conical flow. First observe that to assure that (4.3) and (4.5) will imply the original equations, the analytic function $A + iB$ must be forced to vanish. This can be done by imposing

$$A = 0 \qquad (\text{or } B = 0) \tag{4.6}$$

on both boundaries, and

$$B = 0 \qquad (\text{or } A = 0) \tag{4.7}$$

at *one* point. Also on $| \mu_1 + i\mu_2 | = 1$ the equation of the cone

$$f(X_1, X_2) = 0 \tag{4.8}$$

and the condition of tangency

$$(u_\alpha - wX_\alpha)\, \partial f/\partial X_\alpha = 0 \tag{4.9}$$

must be satisfied, where $X_\alpha = x_\alpha/z$ must be expressed in terms of $\partial u_i/\partial \mu_\beta$ by means of (2.13). Let (0, 0, Q) be the undisturbed velocity ahead of the shock. At the (actually unknown) conical shock $g(X) = 0$, the change in velocity must be normal to the shock, i.e.

$$(\partial g/\partial X_\alpha)/u_\alpha = (X_\beta\, \partial g/\partial X_\beta)(Q - w) \qquad (\alpha \textit{ not } \text{summed}). \tag{4.10}$$

These equations imply $\partial g/\partial X_\alpha = Ku_\alpha$ for some function K, and also

$$Q = X_\alpha u_\alpha + w. \tag{4.11}$$

Then by (2.7), $\Phi = Q$, i.e. *the shock wave is an equipotential surface.* Conversely, any equipotential surface $\Phi = Q =$ constant satisfies (4.10). Finally, on $g = 0$ the equation of the shock polar

$$u_\alpha u_\alpha[\gamma - 1 + 2Q^2 - (\gamma + 1)Qw] = (Q - w)^2[(\gamma + 1)Qw - \gamma + 1] \tag{4.12}$$

must be satisfied, where X_α must be determined as above.

If the constant Q is prescribed, then R has to be found by first determining the flow. It is more convenient to prescribe R and to impose

$$R\, dQ/ds = (-\mu_2\, \partial Q/\partial \mu_1 + \mu_1\, \partial Q/\partial \mu_2) = 0, \tag{4.13}$$

on $| \mu_1 + i\mu_2 | = R$, where Q is defined by (4.11), and where the constant value of Q is to be found after the flow has been determined. The shock wave must likewise be found by (2.13) on this circle after a solution has been found. This is comparable to the situation in numerical computation of Taylor-Maccoll flow, where for prescribed cone angle and surface velocity the free stream velocity and shock angle are not known until the end of the computation.

To recapitulate: The problem is to find on the annulus $1 \leqq | \mu_1 + i\mu_2 | \leqq R$ for given R and $f(X)$ a solution of the elliptic system (4.3), (4.5) which satisfies (4.6), (4.8), and (4.9) on $| \mu_1 + i\mu_2 | = 1$; (4.6), (4.12), and (4.13) on $| \mu_1 + i\mu_2 | = R$; and $B = 0$ (or $A = 0$) at one point. Note that although the free stream's speed is unknown, its direction is fixed. Hence (4.8) determines both the shape of the body and its orientation relative to the undisturbed flow. It should be stressed that mere solution of this boundary value problem is not enough to guarantee the existence of irrotational conical flow about a given cone. In the units employed here, $0 \leqq q \leqq 1$; also Q must be supersonic. Thus a solution is not acceptable unless $(\gamma - 1)^{1/2}(\gamma + 1)^{-1/2} < Q < 1$. Furthermore, the map (2.13) from the hodograph to the physical space must yield a single valued velocity field.

5. Application to Taylor-Maccoll flow. To find the axisymmetric flow about the cone

$$f = (X_\alpha X_\alpha)^{1/2} - \tan \theta_0 = 0, \tag{5.1}$$

of semi-vertex angle θ_0, first seek a solution of the form

$$u_1 = U(t) \cos \phi, \qquad u_2 = U(t) \sin \phi, \qquad w = W(t), \tag{5.2}$$

$$t = (X_\alpha X_\alpha)^{1/2}, \qquad \tan \phi = X_2/X_1 . \tag{5.3}$$

For the non-isothermic parameters $\mu_1 = t$, $\mu_2 = \phi$, the definitions (2.11) and (2.13) yield

$$q^2 = U^2 + W^2, \qquad g_{11} = U'^2 + W'^2, \qquad g_{12} = 0, \qquad g_{22} = U^2,$$

$$b_{11} = (U'W'' - U''W')(U'^2 + W'^2)^{-1/2}, \qquad b_{12} = 0, \qquad b_{22} = UW'(U'^2 + W'^2)^{-1/2},$$

$$x_1 = -rW' \cos \phi, \qquad x_2 = -rW' \sin \phi, \qquad z = rU'. \tag{5.4}$$

By (5.3) and (5.4)

$$W' + tU' = 0. \tag{5.5}$$

By (2.12)

$$W' = a^2 U/[a^2(1 + t^2) - (U - tW)^2], \tag{5.6}$$

where W'' has been eliminated by means of (5.5). At the shock (4.11) and (4.13) yield

$$Q(t) = tU + W, \tag{5.7}$$

$$H(t) \equiv U^2\{2Q^2 - (1 + t^2)[(\gamma + 1)QW - \gamma + 1]\} = 0. \tag{5.8}$$

At the cone $t = \tan \theta_0 = t_0$

$$U_0 = W_0 t_0 , \tag{5.9}$$

for $U(t_0) = U_0$, $W(t_0) = W_0$.

A short digression on the boundary value problem (5.5) to (5.9) will be instructive. The problem can be replaced by an initial value problem by choosing $W_0 > 0$ such that $W_0^2(1 + t_0^2) < 1$, defining $U_0 = W_0 t_0$, and integrating (5.5) and (5.6). It can be shown that $-U(t)$, $U^2 + W^2$, $Q(t)$, and $-H(t)/U^2(t)$ are increasing functions of t, that $H(t_0) > 0$, and that $H = 0$ before a^2U or a^2U/W' can vanish. By (5.7) and (5.8)

$$0 < Ut = 2t^2[Q^2 - (1 + t^{-2})A^2]/(\gamma + 1)(1 + t^2)Q$$

where $A^2 = 1/2(\gamma - 1)(1 - Q^2)$. Hence Q is *supersonic*. If $W_0(1 + t_0^2) \geqq 1$, $Q(t) > 1$ at $H = 0$. Thus some solutions of (5.5) and (5.6) are rejected by the restriction $Q < 1$. It is also known that there exists $q^*(t_0)$ such that for $W_0(1 + t_0^2)^{1/2} \leqq q^*(t_0)$, $(U^2 + W^2)^{1/2}$ is subsonic all the way up to the shock; for $q^*(t_0) < W_0(1 + t_0^2)^{1/2} \leqq (\gamma - 1)^{1/2}(\gamma + 1)^{-1/2}$ the flow is subsonic near the cone, supersonic near the shock; and for $(\gamma - 1)^{1/2}(\gamma + 1)^{-1/2} < W_0(1 + t_0^2)^{1/2} < 1$ the flow is completely supersonic. To obtain completely supersonic flow and $Q < 1$ one must have $t_0^2 < 2/(\gamma - 1)$. For $\gamma = 1.4$ this yields the crude estimate $\theta_0 < 66°$, by contrast with the computed bound of 57.5°. A similar crude bound for finite free stream Mach numbers can be obtained by replacing the upper bound 1 for Q by $M[M^2 + 2/(\gamma - 1)]^{-1/2}$. In general, for a given (small enough) t_0 there is a minimum free stream Mach number for Taylor-Maccoll flow. For greater Mach numbers and the same t_0 there are two solutions, one of which has a strong, the other a weak shock.

Isothermic parameters $\mu_1^* = T(t)$, $\mu_2^* = \phi$ can be defined by

$$[a^2(U'^2 + W'^2) - (UU' + WW')^2](dt/dT)^2 = a^2U^2.$$

By (5.5) and (5.6) this becomes $(dT/dt)^2 = W'/Ut^2$. Then choose

$$T(t) = 1 + \int_{t_0}^{t} (W'/Ut^2)^{1/2}\, dt.$$

At the shock let $t = \tan \theta_w = t_w$, and $T(t_w) = R$.

$$R = 1 + \int_{t_0}^{t_w} (W'/Ut^2)^{1/2}\, dt. \tag{5.10}$$

In polar coordinates T, ϕ, the cone corresponds to $T = 1$, the shock to $T = R$. Approximate values for R as functions of free stream Mach number and cone semi-angle for $\gamma = 1.4$, computed on the ENIAC and ORDVAC, are shown in Fig. 1. The curve for $\theta = 40°$ (not shown) practically coincides with that for $\theta = 35°$ above $M = 3$.

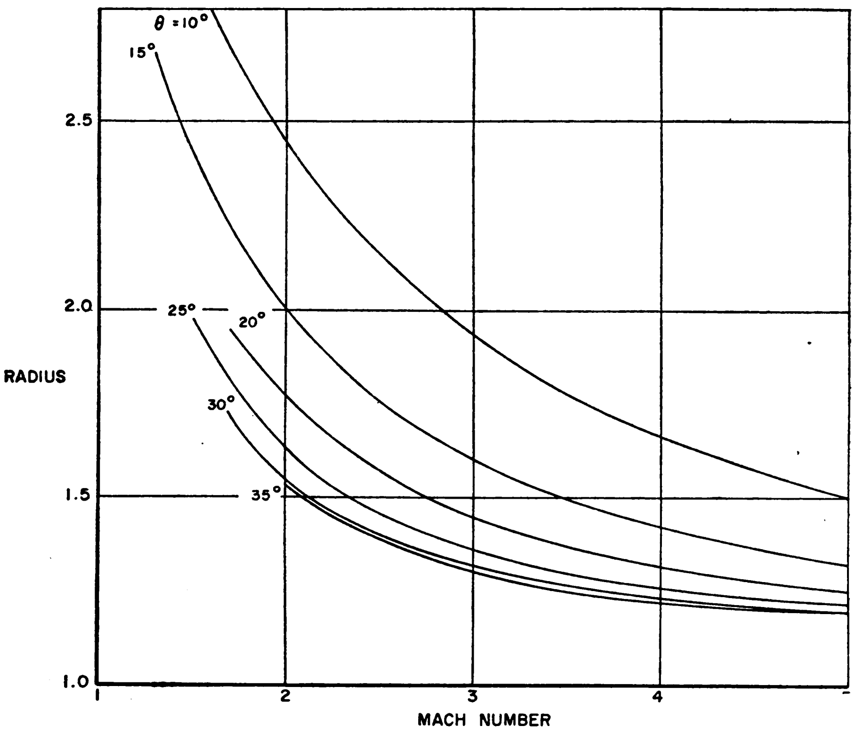

FIG. 1. Outer radius for Taylor-Maccoll flow.

6. Generalities and speculations. A. Numerical solutions of the boundary value problem can be attempted by relaxation methods. Estimates of the outer radius, R, of the annulus for prescribed free stream Mach number can be based on Fig. 1. The treatment of boundary conditions and the choice of coordinate lattice will be greatly simplified by the transformation $\mu_1^* + i\mu_2^* = \log (\mu_1 + i\mu_2)$ which maps the annulus $1 \leqq | \mu_1 + i\mu_2 | \leqq R$ onto the rectangle $0 \leqq \mu_1^* \leqq \log R$, $0 \leqq \mu_2^* \leqq 2\pi$. Computation could be directed toward a great variety of goals, such as (i) comparisons with Kopal's yawing cone calculations, based on A. H. Stone's theory [20]; (ii) checks of Moeckel's three-dimensional characteristic method for calculating the same types of flows [15]; (iii) use of computed results to estimate the importance of Ferri's singularity in the entropy [8]; (iv) determinations of limitations on shape, such as (a) maximum inclination between a surface element and the incident flow; (b) maximum eccentricity for elliptic cones, i.e., can one pass to the limiting case of the delta wing at an angle of attack; (c) maximum yaw for a circular cone, i.e. can the angle of attack exceed the semicone angle?

B. Since Taylor-Maccoll flow is a solution of this type of boundary value problem, it might be possible to apply the iterative method described in [6, p. 287], Green's function for an annulus being known, to show that flows can be constructed about slightly yawing almost circular cones $0 = f(X_1, X_2) = X_\alpha X_\alpha - \tan^2\theta_0 + \epsilon g(X_1, X_2)$ for reasonable g (not a function of $X_\alpha X_\alpha$ only) and small enough ϵ.

C. M. Shiffman [17] has recently developed a proof of the existence and uniqueness of subsonic plane flows about fairly general shapes. This involves reformulating of the plane flow problem in variational form and then applying direct methods in the calculus of variations. It can be shown that this is impossible for the system (4.3), (4.5), i.e. that these are not the Euler equations for any integral

$$I = \iint F(\mu_1, \mu_2, u, \partial u/\partial\mu_1, \partial u/\partial\mu_2)\, d\mu_1\, d\mu_2 . \tag{6.1}$$

D. Can a new linearization of the conical flow equations be based on (4.3), e.g. by suppressing the right members? Instead of defining in advance the transformation from the X_1X_2-plane to the $\mu_1\mu_2$-plane, as is done in the usual linearization, use (2.13) for this purpose. If possible, retain all of the boundary conditions in Sec. 4.

7. Remarks on plane flow. Shiffman [17] and Bers [unpublished results] have recently established existence and uniqueness theorems for completely subsonic flows about plane profiles. Accordingly, it may be significant that their boundary value problem can be restated, as shown below, in a form intimately related to the problem of Sec. 4.

Since $u_\alpha(x_1, x_2)$ are functionally independent in subsonic plane flow, apply the Legendre transformation

$$k(u_1, u_2) = \phi - x_\alpha u_\alpha , \tag{7.1}$$

to (2.1) to obtain

$$(a^2\delta_{\alpha\beta} - u_\alpha u_\beta)(-1)^{\alpha+\beta}\, \partial^2 k/\partial u_{\alpha+1}\, \partial u_{\beta+1} = 0, \tag{7.2}$$

where $\alpha + 1$ and $\beta + 1$ are to be reduced mod 2, and where the velocity u_α is assigned to the point

$$x_\alpha = -\partial k/\partial u_\alpha . \tag{7.3}$$

Now introduce the more general parameters $\mu_\alpha = \mu_\alpha(u_1, u_2)$. Then

$$u_\alpha = u_\alpha(\mu_1, \mu_2). \tag{7.4}$$

Let

$$Q^2 = u_\alpha u_\alpha , \qquad Q\, \partial Q/\partial\mu_\beta = u_\alpha\, \partial u_\alpha/\partial\mu_\beta , \qquad G_{\alpha\beta} = (\partial u_\gamma/\partial\mu_\alpha)(\partial u_\gamma/\partial\mu_\beta) \tag{7.5}$$

and let the third order determinants

$$\det \begin{vmatrix} k_{,\alpha\beta} & \partial k/\partial\mu_\delta \\ u_{\gamma,\alpha\beta} & \partial u_\gamma/\partial\mu_\delta \end{vmatrix} = B_{\alpha\beta} , \tag{7.6}$$

where $k_{,\alpha\beta}$ and $u_{\gamma,\alpha\beta}$ are the second covariant derivatives of the scalars k and u_γ with respect to μ_α and μ_β based on $G_{\alpha\beta}$. For the special choice $\mu_\alpha = u_\alpha$ (7.2) implies

$$(a^2G_{\alpha\beta} - Q^2\, \partial Q/\partial\mu_\alpha\, \partial Q/\partial\mu_\beta)(-1)^{\alpha+\beta}B_{\alpha+1\,\beta+1} = 0. \tag{7.7}$$

Since $G_{\alpha\beta}$ and $\partial Q/\partial\mu_\beta$ are tensors, while $B_{\alpha\beta}$ is a relative tensor of weight one, the form of (7.7) is independent of the particular choice of parameters μ_α . Also observe that (7.3) implies

$$x_\alpha\, \partial u_\alpha/\partial\mu_\beta + \partial k/\partial\mu_\beta = 0. \tag{7.8}$$

For subsonic flow choose *isothermic parameters* defined by

$$A \equiv a^2u_{\alpha1}u_{\alpha2} - u_\alpha u_{\alpha1}u_\beta u_{\beta2} = 0, \tag{7.9}$$

$$2B \equiv a^2(u_{\alpha1}u_{\alpha1} - u_{\alpha2}u_{\alpha2}) - (u_\alpha u_{\alpha1})^2 + (u_\alpha u_{\alpha2})^2 = 0. \tag{7.10}$$

Now by (7.6) and (7.7)

$$\partial^2k/\partial\mu_\alpha\, \partial\mu_\alpha = D_\beta\, \partial k/\partial\mu_\beta , \qquad \partial^2u_\gamma/\partial\mu_\alpha\, \partial\mu_\alpha = D_\beta\, \partial u_\gamma/\partial\mu_\beta . \tag{7.11}$$

Since (7.5), (7.9), and (7.10) can be obtained by setting $w \equiv 0$ in (2.11), (4.1), and (4.2), then by analogy with (4.5)

$$(a^2G_{\alpha\beta} - Q^2\, \partial Q/\partial\mu_\alpha\, \partial Q/\partial\mu_\beta)D_\beta = 0.5G_{\beta\beta}\, \partial(a^2 + Q^2)/\partial\mu_\alpha - G_{\alpha\beta}\, \partial a^2/\partial\mu_\beta . \tag{7.12}$$

In the present case D_β are rational functions of the six arguments u_α , $\partial u_\alpha/\partial\mu_\beta$. As in Section 4, if $k(\mu)$ and $u_\gamma(\mu)$ are solutions of (7.11) where D_β are defined by (7.12), then $A + iB$ is an analytic function of $\mu_1 + i\mu_2$.

Suppose that the airfoil is described by $f(x_1 , x_2) = 0$. Assume that $f = 0$ is transformed into a closed curve $F(\mu_1 , \mu_2) = 0$, and that the flow is mapped onto the exterior of $F = 0$. Since (7.9) and (7.10) are invariant under conformal transformation, it may be assumed with no loss of generality that $F = 0$ is $| \mu_1 + i\mu_2 | = 1$, and that the point at infinity in the $x_1 + ix_2$ plane maps onto the point at infinity in the $\mu_1 + i\mu_2$ plane. On the unit circle impose the boundary conditions A (or B) $= 0$; $f(x_1 , x_2) = 0$, where x_α are defined by (7.8); and $u_\alpha\partial f/\partial x_\alpha = 0$. At infinity $u_\alpha = U_\alpha =$ constants; and $x_1 + ix_2 = \infty$. Finally, at some point impose B (or A) $= 0$.

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