force is conservative, the potential function being $mV(r, \psi)$. Then the Lagrangian function is

$$L = T - V = \tfrac{1}{2}(I + mr^2)\varphi^{\cdot 2} + mr^2\varphi^{\cdot}\psi^{\cdot} + \tfrac{1}{2}mr^2\psi^{\cdot 2} + \tfrac{1}{2}mr^{\cdot 2} - mV(r, \psi).$$

The Lagrange equations of the second kind for φ, ψ and r are respectively

$$\frac{d}{dt}\{(I + mr^2)\varphi^{\cdot} + mr^2\psi^{\cdot}\} = \lambda(I + mr^2),$$

$$4r^{\cdot}\varphi^{\cdot} + 2r\varphi^{\cdot\cdot} + 2r^{\cdot}\psi^{\cdot} + r\psi^{\cdot\cdot} - \frac{1}{r}\frac{\partial V}{\partial \psi} = \lambda r,$$

$$r^{\cdot\cdot} - r\varphi^{\cdot 2} - 2r\varphi^{\cdot}\psi^{\cdot} - r\psi^{\cdot 2} + \frac{\partial V}{\partial r} = 0.$$

From the first of these equations it follows that the multiplier λ is 0. $\varphi^{\cdot}$ and $\varphi^{\cdot\cdot}$ can be eliminated and we have two equations for the relative motion r, ψ of the particle P on the disc. They are rather complicated. Of course we can take $T + V = c$ instead of one of them. If we have the case mentioned above, where mr^2 may be neglected when compared with I, the equations are

$$\varphi^{\cdot} = -\frac{m}{I}r^2\psi^{\cdot}, \qquad r^{\cdot\cdot} - r\psi^{\cdot 2} + \frac{\partial V}{\partial r} = 0, \qquad r^2\psi^{\cdot 2} + r^{\cdot 2} - 2V = c.$$

This means that the relative motion is the same as it would be if the disc were fixed. We give a simple example: $V = k^2/2(r^2 - 2ar\cos\psi)$, the force on P being an attracting force directed to the center $C(r = a, \psi = 0)$, and proportional to the distance PC. If $x = r\cos\psi - a$, $y = r\sin\psi$, we have $x^{\cdot\cdot} = -k^2x$, $y^{\cdot\cdot} = -k^2y$, $x = C_1\cos kt + C_2\sin kt$, $y = C_3\cos kt + C_4\sin kt$, where C_i are constants of integration. Therefore

$$r^2 = (C_1^2 + C_3^2)\cos^2 kt + (C_2^2 + C_4^2)\sin^2 kt + (C_1C_2 + C_3C_4)\sin 2kt + 2a(C_1\cos kt + C_2\sin kt) + a^2,$$

$$\psi = \arctan\frac{C_3\cos kt + C_4\sin kt}{C_1\cos kt + C_2\sin kt + a},$$

$$r^2\psi^{\cdot} = ay^{\cdot} + (xy^{\cdot} - yx^{\cdot}) = ay^{\cdot} + k(C_1C_4 - C_2C_3).$$

Hence

$$\varphi = -\frac{ma}{I}(C_3\cos kt + C_4\sin kt) - \frac{mk}{I}(C_1C_4 - C_2C_3)t + C_5.$$

ON LINEAR INSTABILITY*

By AUREL WINTNER (*The Johns Hopkins University*)

1. Let the coefficient function of the linear differential equation

$$x'' + f(t)x = 0 \tag{1}$$

be real-valued and continuous for large positive t. Consider only those solutions $x(t)$ of (1) which are real-valued and distinct from the trivial solution ($\equiv 0$). Then, since

*Received August 12, 1954.

the values of x and x' at a given t determine a solution of (1) uniquely, x and x' cannot vanish simultaneously, and so $x(t)$ must change sign whenever it vanishes.

If there exists a $t = T$ beyond which the function $x(t)$ does not change sign, so that $x(t) \neq 0$ for every $t > T$, then (1) is called non-oscillatory, and otherwise oscillatory. This classification of the differential equations (1) is independent of the choice of the particular solution $x(t)$, since, in view of Sturm's separation theorem, every solution of (1) must have zeros clustering at $t = \infty$ if one solution has such zeros.

2. With reference to a differential equation of the form (1), three types of *stability* will be considered: (i) *strict stability*, requiring

$$\limsup_{t\to\infty} \{x^2(t) + x'^2(t)\} < \infty \tag{2}$$

for every solution and its derivative; (ii) *bounded stability*, requiring only the coordinate condition

$$\limsup_{t\to\infty} | x(t) | < \infty \tag{3}$$

for every solution; (iii) *oscillatory stability*, requiring

$$\lim_{t\to\infty} N(t) = \infty \tag{4}$$

for some or for every solution, where $N(t)$ denotes the number of times the solution $x(t)$ changes sign between a fixed t_0 and a variable t.

That (i) is actually stricter than (ii) follows by choosing $f(t)$ to be any function which tends to ∞ sufficiently fast, with a sufficient degree of regularity in its growth, and possessing a continuous second derivative. For then it follows from a general asymptotic formula[1] that (3) holds for every solution but (2) does not hold for any solution.

Dynamical considerations suggest that (iii) is necessary for (ii). Actually, not only (4) but even

$$\liminf_{t\to\infty} N(t)/t > 0 \tag{5}$$

must hold[2] when (3) is satisfied by every solution. On the other hand, trivial examples show that (4) can be satisfied when (5) is not. Hence (iii) is necessary, but not sufficient, for (ii).

If an arrow is meant in the sense of logical implication, then the preceding considerations can be summarized as follows:

$$\text{(i)} \to \text{(ii)} \to \text{(iii)}, \tag{6}$$

where neither of the arrows can be reversed.

3. In what follows, f^+ will denote (as customary) the number max $(0, f)$, that is, $f^+ = 0$ if $f \leqq 0$ and $f^+ = | f |$ if $f \geqq 0$. Thus $f^+(t)$ is a continuous function, since $f(t)$ is.

The second of the implications (6) states that (1) must possess some solution satisfying

$$\limsup_{t\to\infty} | x(t) | = \infty \tag{7}$$

if (1) is non-oscillatory. But the converse is not true (see the trivial example (9) below). It is therefore natural to raise the question for a condition which will allow (1) to be

[1] Cf. A. Wintner, Phys. Rev. 72, 81-82 (1947).

[2] A. Wintner, Quart. Appl. Math. 7, 115-117 (1949).

oscillatory but will bring (1) so "close" to a non-oscillatory state that (7) will remain true for some solution. The character of such a condition on $f(t)$ is suggested by the following consideration:

Since (1) is non-oscillatory if $f(t) \equiv 0$, it follows from Sturm's comparison theorem that (1) is non-oscillatory if $f(t) \leqq 0$, that is, if $f^+(t) \equiv 0$. Hence, what suggests itself is that (1) will have some solution satisfying (7) whenever $f^+(t)$ is "small" in some sense; say in the sense that

$$\text{either} \int^{\infty} f^+(t)\,dt < \infty \qquad \text{or} \qquad f^+(t) \to 0 \tag{8}$$

as $t \to \infty$. It will turn out that either of the conditions (8), and much less, is sufficient to this end.

Needless to say, if $f^+(t)$ is "very small," then (1) can be non-oscillatory. But this will not be the case by virtue of (8). In fact, if

$$f(t) = at^{-2}, \qquad (1 \leqq t < \infty), \tag{9}$$

where a is a real constant, then, while both conditions (8) are always satisfied, (1) will be non-oscillatory (if and) only if[3]

$$-\infty < a \leqq 1/4 \; (>0). \tag{9 bis}$$

4. Put

$$F^+(t) = \int_{t_0}^{t} f^+(s)\,ds, \tag{10}$$

where t_0 is arbitrary and $t_0 \leqq t < \infty$. It will be shown that not only condition (8) but even *the condition*

$$\liminf_{t\to\infty} F^+(t)/t = 0 \tag{11}$$

is sufficient in order that (1) *have some solution satisfying* (7).

In a certain sense, this criterion is the best possible of its type, since (11) cannot be relaxed to

$$\liminf_{t\to\infty} F^+(t)/t = \epsilon, \qquad \text{or even to} \qquad \lim_{t\to\infty} F^+(t)/t = \epsilon,$$

if $\epsilon > 0$. In fact, (10) shows that the latter condition is satisfied if $f(t)$ is a positive constant ($= \epsilon$); in which case, however, (1) is stable in every sense.

The proof of the italicized statement depends on two known facts. The first is the fact, quoted above, according to which the inequality (5) is a necessary condition for the bounded stability of (1). The second is the following fact[4]: If $N(t)$ and $F^+(t)$ are defined as above, then, without any assumption on (1), some constant multiple of the function

$$1 + \{tF^+(t)\}^{1/2} \tag{12}$$

will always exceed $N(t)$, as $t \to \infty$.

The latter estimate of $N(t)$ implies that

$$\limsup_{t\to\infty} N^2(t)/\{tF^+(t)\} < \infty.$$

[3]This classical remark of A. Kneser follows by observing that substitution of $x(t) = t^b$ into the case (9) of (1) leads, for $b = b(a)$, to a quadratic equation the roots of which are real if and only if $4a \leqq 1$.

[4]P. Hartman and A. Wintner, Amer. J. Math. 71, 206-213 (1949).

Hence, if (11) is satisfied, then

$$\liminf_{t\to\infty} N^2(t)/t^2 = 0.$$

Since this contradicts the necessary condition (5), the proof is complete.

5. In view of the first of the conditions (8), it is worth mentioning what happens to each of the three stability properties, considered above, if (1) is replaced by a differential equation

$$x'' + g(t)x = 0 \tag{13}$$

which is a *small* perturbation of (1), in the following sense:

$$\int^{\infty} | f(t) - g(t) | \, dt < \infty. \tag{14}$$

Under the assumption (14), both (1) and (13) are stable in the sense of definition (i) if either of them is, and both (1) and (13) are stable in the sense of definition (ii) if either of them is. Both of these criteria (which are quite independent, since (3) does not imply (2)) are contained, as corollaries, in known[5] asymptotic correspondences between the solutions of (1) and (13), when (14) is satisfied. The situation is changed if (i) or (ii) is replaced by (iii). In fact, if $f(t) = t^{-2}$, then (1) is oscillatory (since (9) then holds for an $a > 1/4$), and (14) is satisfied if $g(t) \equiv 0$, but (13) is then non-oscillatory.

[5]A. Wintner, Amer. J. Math. **69**, 261-265 (1947).

ON PERTURBATION METHODS INVOLVING EXPANSIONS IN TERMS OF A PARAMETER*

By RICHARD BELLMAN (*Rand Corporation*)

Summary. It is shown by means of some examples from the theories of linear algebraic equations, linear integral equations and nonlinear differential equations that the effectiveness of the method of expanding a solution in a power series in terms of a parameter may in many cases be greatly increased by expanding in terms of a suitably chosen function of the parameter. This is particularly the case when the physical setting of the problem allows only positive values of the parameter to enter.

1. Introduction. A standard tool in the theory of functional equations, of both linear and nonlinear character, is the expansion of the solution as a power series in a parameter appearing in the equation, or in the boundary conditions. Some typical examples of equations involving a parameter are

$$\begin{aligned}
&\text{(a)} && x^2 + x = \lambda, \\
&\text{(b)} && x_i + \lambda \sum_{j=1}^{N} a_{ij}x_j = c_i, \quad i = 1, 2, \cdots, N, \\
&\text{(c)} && f(x) + \lambda \int_0^1 K(x, y)f(y)\,dy = g(x), \\
&\text{(d)} && x'' + \lambda(x^2 - 1)x' + x = 0 \\
&\text{(e)} && x'' + x + \lambda x^3 = 0.
\end{aligned} \tag{1.1}$$

*Received Sept. 20, 1954.