TEARING AND INTERCONNECTING AS A FORM OF TRANSFORMATION*

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1. Introduction. The mathematical problem under consideration owes its existence to the increasing complexity of engineering structures, and to a need for the piecewise formulation and solution of very complex physical systems. The author believes that he has solved his mathematical problem by experiment (in the manner of Heaviside) at least for linear systems and has also solved part of his program for certain types of non-linear dynamical systems. He is, however, an electrical engineer and not a mathematician and now wishes to submit his thesis to a mathematical audience in the hope of arousing interest for a search into the mathematical legitimacy of his reasoning. The method proposed for analyzing and solving large-scale engineering structures in easy stages actually works in practice, as many textbooks and articles by scores of independent workers—both in this country and abroad—testify. There is no doubt that arguments and discussions, leading to a more rigorous mathematical proof, cannot but help the extension of the method into the piecewise solution of large-scale non-linear problems as well.

2. The practical problem. The practical engineering problem the author attacks is as follows. Given a complex physical system containing mechanical, elastic, electrical, thermal, etc., subsystems, how can the "equations of state"—as well as the "equations of solution"—or "formulas of solutions"—be established in easy stages, without manipulating or even writing down the simultaneous equations for the entire system? The procedure may be summarized as follows:

(i) Tear apart the given physical system into a convenient number of subsystems, possessing no material contacts or other linkages with each other. (This last consideration is only a convenience, not an absolute necessity.) The two terminals of any point of tear need not be stationary but may have translations or rotations with respect to each other.

(ii) Establish and solve the equations of each subsystem separately, as though the other subsystems did not exist. (One may use any established method of solution for the subsystems; or one may further subdivide each subsystem and use the present method to find its solution.) The solutions may be in a numerical or analytical form.

(iii) Then interconnect the equations of solution (or the equations of state) of each subsystem by a routine procedure, to arrive at the solution (or at the equation of state) of the original given system.

(iv) The remaining work consists of solving for (or eliminating) the comparatively small number of constraint forces that appear at the cuts.

3. The theoretical problem. Instead of starting with one given complex system, it is possible to state the general problem in a slightly different manner.

Let it be assumed that the equations of solution of a large number of isolated systems are available. The problem is how to utilize the already available partial solutions, in order to build up the equations of solution of all possible supersystems that may be constructed by mere interconnection of the given subsystems.

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The following third formulation of the problem will be used presently for a physical proof. Assume that the equations of solution (or equations of state) of one of the supersystems have already been established. The problem is how to establish, in a routine manner, the equations of solution of any one of the other possible supersystems (built out of the same set of subsystems) by utilizing the equations already derived.

In the proof given below, each subsystem will consist of one electrical coil and each supersystem will consist of one possible interconnection of five coils. In practical problems there is, of course, no limitation on the size, physical nature, or complexity of each subsystem.

4. An invariant procedure. The proposal to tear a system apart into several subdivisions and to try to interconnect their equations has occured to many workers in these fields. The novelty of the present method lies in the proposed systematic and general procedure for accomplishing the interconnection. In particular, the proposed procedure enables one to utilize the entire apparatus of tensor analysis to organize the large variety of physical concepts and that of matrix algebra to organize the large number of constants into manageable forms.

At the very beginning of his researches into the above problem, the author realized that to accomplish his objective he must follow a hitherto untrodden path. In particular, he conceived the idea of treating the processes of tearing apart a physical system and interconnecting again the component parts, as elements of a "group of transformations" $C^{\alpha}_{\alpha'}$ (to be discussed later). Thus he proposed to formulate or solve a given complex physical system by the following tensor (invariant) reasoning:

(i) Tear apart the given complex system into a convenient number of independent subsystems.

(ii) Set up the "equations of state" (or the "equations of solution") of each subsystem in the form of a tensor equation. In the case of a dynamical system (for example a group of rotating electrical machines) the tensor equation of state of each subsystem (each rotating machine) assumes the form

$$e_{\alpha} = R_{\alpha\beta} i^{\beta} + L_{\alpha\beta} \frac{di^{\beta}}{dt} + \Gamma_{\beta\gamma.\alpha}\, i^{\beta} \frac{dx^{\gamma}}{dt}, \tag{1}$$

where $\Gamma_{\alpha\beta.\gamma}$ represents a general asymmetrical affine connection, not related to the metric tensor $L_{\alpha\beta}$. For each subsystem the geometric objects and tensors of $R_{\alpha\beta}$, $L_{\alpha\beta}$ and $\Gamma_{\beta\gamma.\alpha}$ are independently calculated.

(iii) Represent the interconnection of the various subsystems by a "matrix of transformation" $C^{\alpha}_{\alpha'}$. This matrix is established by simply inspecting the relations of the variables i^{α} and $i^{\alpha'}$ that exist at the points of separation—before and after the tearing—as $i^{\alpha} = C^{\alpha}_{\alpha'}\, i^{\alpha'}$. The components of $C^{\alpha}_{\alpha'}$ are often plus or minus unity in simple systems, but in rotating dynamical systems they are functions of space and time.

The existence of such a non-singular C is the key to the method of tearing. The rest of the paper elaborates a physical line of reasoning to justify and construct this single concept for as large a variety of practical cases as possible.

(iv) Transform each geometric object by the aid of $C^{\alpha}_{\alpha'}$ in a routine manner, following their laws of transformations. Thus $R_{\alpha'\beta'}$, $L_{\alpha'\beta'}$ and $\Gamma_{\beta'\gamma',\alpha'}$ of the interconnected system are established in a routine manner.

(v) The equations of state (or equations of solution) of the resultant system assume in terms of geometric objects the same form as those of the component subdivisions.

For instance, the equations of state of the interconnected group of rotating electrical machines becomes analogously to Eq. (1)

$$e_{\alpha'} = R_{\alpha'\beta'} i^{\beta'} + L_{\alpha'\beta'} \frac{di^{\beta'}}{dt} + \Gamma_{\beta'\gamma',\alpha'} i^{\beta'} \frac{dx^{\gamma'}}{dt} . \tag{2}$$

5. Invariance of equations. It should be emphasized that one of the basic requirements of the method of tearing is that the form of the equations should remain invariant in passing from each member of the torn-apart system to the resultant system. (A second basic requirement, namely the invariance of power, will be treated later on.) It must also be emphasized that the form of the equations also remains invariant if the component subsystems are interconnected in any other possible manner. Moreover the invariance is maintained, even if each of the subsystems is further torn apart into several pieces and each of these pieces is interconnected into any of a large variety of supersystems.

The method of tearing assumes that between each supersystem and each set of subsystems there exists a non-singular matrix of transformation C and that their totality forms a "group". That is:

(i) Each C has an inverse.

(ii) The product of any two C's is also an element of the group.

(iii) The unit element leaves a system unchanged.

6. "Orthogonal" networks. Perhaps the most important concept in interconnecting piecewise solutions is the existence of the inverse of every matrix of transformation C. Because of this property it is possible to pass freely from the equations of state to the equations of solution and back again at any stage of the analysis. To show the existence

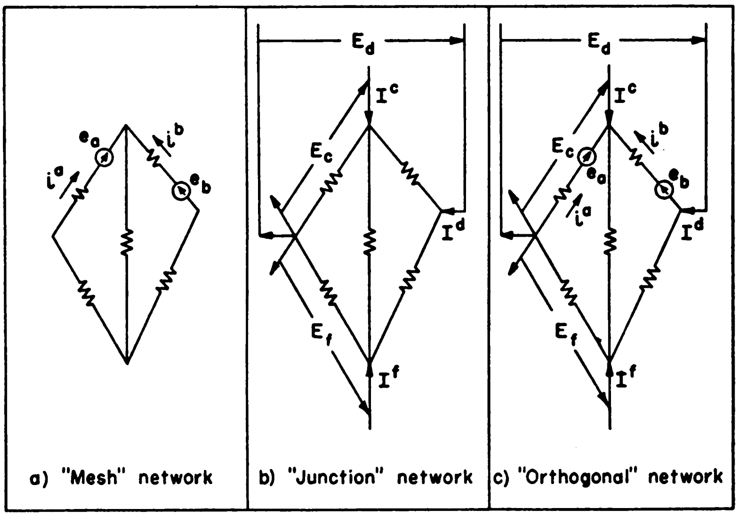

FIG. 1. Three types of representation of the same network.

of an inverse C for all engineering structures, the simple electrical network of Fig. 1a will be used as an illustration.

The network contains five coils, each coil having an impedance Z (or admittance

Y). It is well known that the equation of state (or equation of solution) of this network may be written in two different manners:

(i) By writing two mesh equations $e_\alpha = z_{\alpha\beta} i^\beta$ (Fig. 1a).

(ii) By writing three junction-pair equations $I^\alpha = Y^{\alpha\beta} E_\beta$ (Fig. 1b).

$$\begin{array}{cc|cc} & [i \quad\quad] & & [\quad\quad E] \\ \begin{bmatrix} e \\ \end{bmatrix} & \begin{bmatrix} z & \\ & \end{bmatrix} & \begin{bmatrix} \\ I \end{bmatrix} & \begin{bmatrix} & \\ & Y \end{bmatrix} \end{array} \tag{3}$$

For purposes of tearing it will be assumed, however, that it is always possible to write for the network as many equations of state (or equations of solution) as the sum of the number of meshes and junction-pairs (that is, as many as there are coils). The additional large number of "orthogonal" equations may assume either of the following two forms:

$$\begin{array}{ccc|cc} & & [i \quad\quad I] & & [e \quad\quad E] \\ \begin{bmatrix} e \\ E \end{bmatrix} & = & \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} & \begin{bmatrix} i \\ I \end{bmatrix} & \begin{bmatrix} Y^1 & Y^2 \\ Y^3 & Y^4 \end{bmatrix} \end{array} \tag{4}$$

(Orthogonality of the meshes and junction-pairs is proved in Ref. [1], page 974.)

That is, every network and every dynamical system may be described in a more general manner by assuming not only the active forces (e or I) but also the external

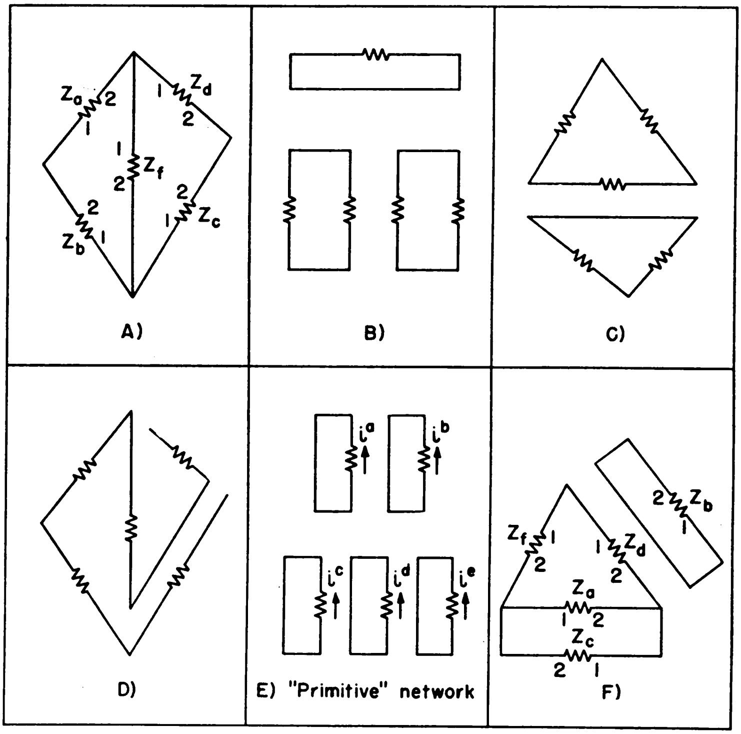

FIG. 2. Six different interconnections of five coils (stationary, linear graph).

constraints (E or i) as variables and writing for the system as many equations of state (or solution) as there are total number of variables. In conventional analysis such a course is not followed, but in the method of tearing it is necessary to have at least some of the superfluous equations available.

In order to have the extra equations and extra variables at hand, it is necessary in the method of tearing always to start the study with the original physical system or a model of it. The latter always imply all the above variables and all the above equations.

7. Example of stationary networks. Perhaps the simplest possible illustration of a group of C is given in Fig. 2, in which five stationary, one-dimensional electrical coils are interconnected in six different manners. (The number of possible interconnections of the same five coils is much larger.) The coils may have asymmetrical mutual impedances between them, and the impedances may be non-linear functions of the currents.

Let it be assumed that the orthogonal equations of state, or the orthogonal equations of solution, of one of the networks are given by Eq. (4) or by a non-linear generalization of it. The problem is to establish the analogous equations for any of the other networks by utilizing the already known equations (4). The author's thesis is that it is sufficient to establish a non-singular matrix of transformation C between the known network and the unknown network. With C established, the rest of the work is merely a routine application of tensor methods.

Selecting any two networks, for instance the two-mesh, one-piece network of Fig. 2A and the three-mesh, two-piece network of Fig. 2F, the matrix of transformation C_F^A existing between them may be established in the following way.

Reproducing the two networks in Fig. 3, assume as many current-variables in each network as there are coils, namely five. That is, assume in Fig. 3a two mesh-currents ($i^{a'}, i^{b'}$) and three junction-pair currents. (Of course, each set of variables may be selected in a variety of ways.) In Fig. 3b assume three mesh-currents ($i^{a''}, i^{b''}, i^{c''}$) and two junction-pair currents. Plot the path of each current through the network as shown. (Again the paths may be selected in a number of different ways.)

Consider next each coil in succession and equate the currents flowing through it in both networks. For instance, considering coil Z_a, in Fig. 3a the current is $i^{a'}$ and in Fig. 3b it is $i^{a''} - i^{b''} - i^{d''} - i^{f''}$. Equating the currents in each of the five coils in succession,

$$\begin{aligned} i^{a'} \qquad\qquad\qquad &= i^{a''} - i^{b''} - i^{d''} - i^{f''} \\ i^{a'} + i^{c'} + i^{d'} + i^{f'} &= i^{c''} \\ i^{b'} - i^{d'} &= i^{a''} \\ - i^{b'} &= i^{b''} + i^{d''} \\ i^{a'} + i^{b'} + i^{c'} &= - i^{b''} \end{aligned} \tag{5}$$

(The five coils are identified in Figs. 2A and 2F.)

It should be noted that the left-hand relation may be looked upon as $C_A^E \, i^A$, transforming the network of Fig. 2E (containing isolated coils only, the so called "primitive" network) into that of Fig. 2A. Similarly the right-hand relation may be looked upon as $C_F^E \, i^F$, transforming the primitive network of Fig. 2E into that of Fig. 2F. That is, the above set of equations may be written as

$$i^E = C_A^E i^A = C_F^E i^F. \tag{6}$$

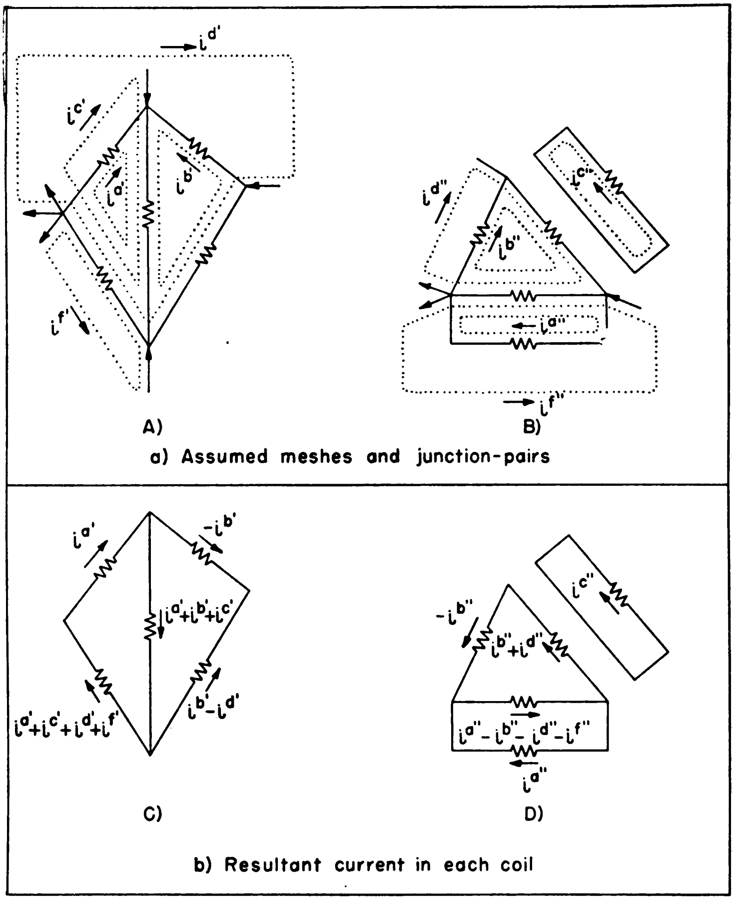

FIG. 3. Transformation between two networks

From this the relation between i^A and i^F follows as

$$i^{\cdot A} = C^A_E C^{E}_{F} i^{\cdot F}, \tag{7}$$

where $C^A_E = (C^E_A)^{-1}$. (This inverse always exists.) Hence the relation between the two network currents of Fig. 3 is

$$i^{\cdot A} = C^{A}_{F} i^{\cdot F}, \tag{8}$$

where

$$C^A_F = (C^E_A)^{-1} C^E_F, \tag{9}$$

$$C^A_F = \begin{bmatrix} 1 & -1 & & -1 & -1 \\ & -1 & & -1 & \\ -1 & 1 & & 2 & 1 \\ -1 & -1 & & -1 & \\ 1 & 1 & 1 & & \end{bmatrix}. \tag{10}$$

This is the non-singular matrix of transformation from the two-mesh, one-piece network of Fig. 2A to the three-mesh, two-piece network of Fig. 2F. With its aid (or with the aid of its inverse), any physical entity or equation (such as Eq. 4 or its non-linear generalization) related to Fig. 2A may be transformed in a routine manner to that of Fig. 2F, or vice-versa.

8. Existence of the "primitive" network. When only one network is given (say Fig. 2F) and it is necessary to establish its equation of state, or equation of solution, the author advocates that the given network first be torn apart into subdivisions for which equations of state can easily be established. These subdivisions happen to be the individual "coils" for which the equations of state may usually be obtained by mere inspection. The collection of isolated coils is called the "primitive network" shown in Fig. 2E.

The step from the equations of the primitive network, say $e_\alpha = z_{\alpha\beta} i^\beta$, to those of the given network, $e_{\alpha'} = z_{\alpha'\beta'} i^{\beta'}$, requires only the establishment of a C. Since in most practical problems the given network needs to be looked upon either as a pure mesh network or as a pure junction network, it is sufficient in such cases to establish a rectangular, thus singular C. Nevertheless the non-singular C always exists.

Any argument or proof based upon the existence of only a singular C may corroborate the formulae of the author, but are unacceptable as proofs of his method. The similarity of the singular C used in transforming from the primitive network to a given "mesh" network, to the singular C arising in "going from branches to meshes", is purely coincidental. The branches do not possess an equation of state of the form $e_\alpha = z_{\alpha\beta} i^\beta$, but something more complicated. On the other hand, the method of the author is restricted to the use of only one single invariant equation, under any type of physical or hypothetical transformation.

9. More complicated examples. A more complicated group of C exists in the presence of rotating electrical machinery, Ref. [5]. The ultimate element into which a system can be torn apart now is a two-dimensional cylindrical layer of winding (Fig. 4), in which

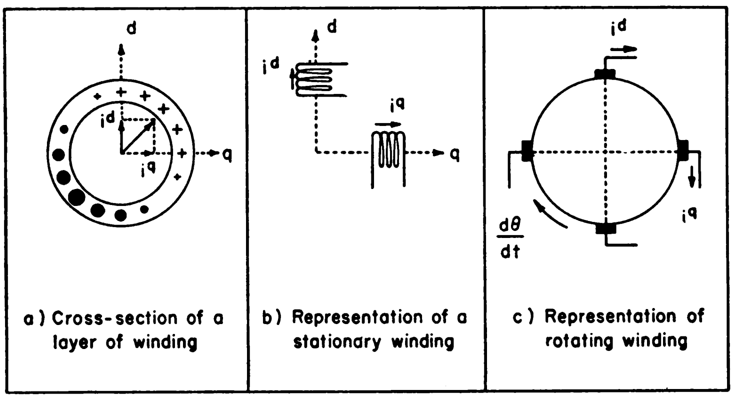

FIG. 4. Two-dimensional cylindrical windings of rotating electrical machinery.

the current-density wave is sinusoidal in space along the cross-section. The currents are removed through stationary brushes or rotating slip-rings. (One element of i^α represents now an angular velocity ω.) Fig. 5 shows three different interconnections of three such layers of windings (one stationary and two rotating layers). The simplest arrangement is shown in Fig. 5B (the "primitive" rotating machine) in which all reference axes are stationary, are at right angles in space, and all windings are electrically isolated. The author has used this machine as the starting point for the analysis and solution of all other rotating electrical machines used in industry. The equations of performance of

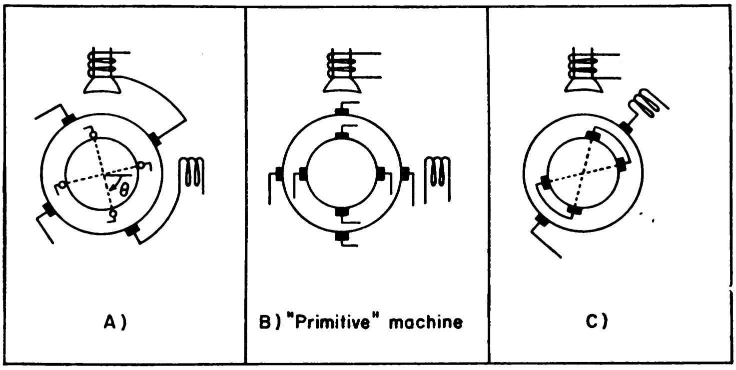

FIG. 5. Three different interconnections of three windings (rotating, two-dimensional graph).

any machine may be found from those of the primitive machine by merely establishing a C between the two machines.

In a stationary elastic beam structure (Fig. 6, see also Ref. [3]), the ultimate subdivision (one beam) has twelve degrees of freedom (six for each end). More complicated examples of engineering structures that could be torn apart into a collection of smaller structures, could be cited indefinitely.

10. "Tearing" as a sub-group of affine transformations. Of course, simultaneously with the above C—representing actual tearing—many other transformation matrices also have to be used, in order to introduce hypothetical and physical reference frames that facilitate the tearing and the analysis. A reference frame, in which the tearing assumes a simple form, is usually not the best reference frame for the analysis itself. Practical exigencies of a problem (like knowing only real power in an electrical transmission system, instead of current or voltage) force the introduction of still other sets of reference frames.

Most of the mathematical endeavors of the author have been concentrated upon fitting many of these physical and hypothetical transformations into the framework of tensor calculus. The need for considering "tearing" and "interconnecting" also as a subgroup of the group of affine transformations is thus dictated by the appearance of so many other types of more conventional affine transformations in the analysis of complex physical systems.

11. Practical considerations. In order that the solution of supersystems be in a practicable form, it is also necessary that all partial solutions of the subsystems involve far fewer non-zero elements than do the conventional solutions. Furthermore, the method for interconnection of the partial solutions should be comparatively simple and fast. It is believed that the suggested procedure satisfies these auxiliary—nevertheless important—practical considerations.

To avoid any misunderstanding, the word "solution" refers not to a particular numerical solution of the variables, but to a general solution in which the variables still occur in a general unspecified form. For instance, $E_\alpha = Z_{\alpha\beta}I^\beta$ (a matric equation) is understood to be a general solution of $I^\alpha = Y^{\alpha\beta}E_\beta$ if the elements of $Z_{\alpha\beta}$ alone are numeric. For any particular value of I^α a corresponding set of E_α may be calculated. That is, the method of tearing is essentially an inversion procedure and the method should be evaluated accordingly.

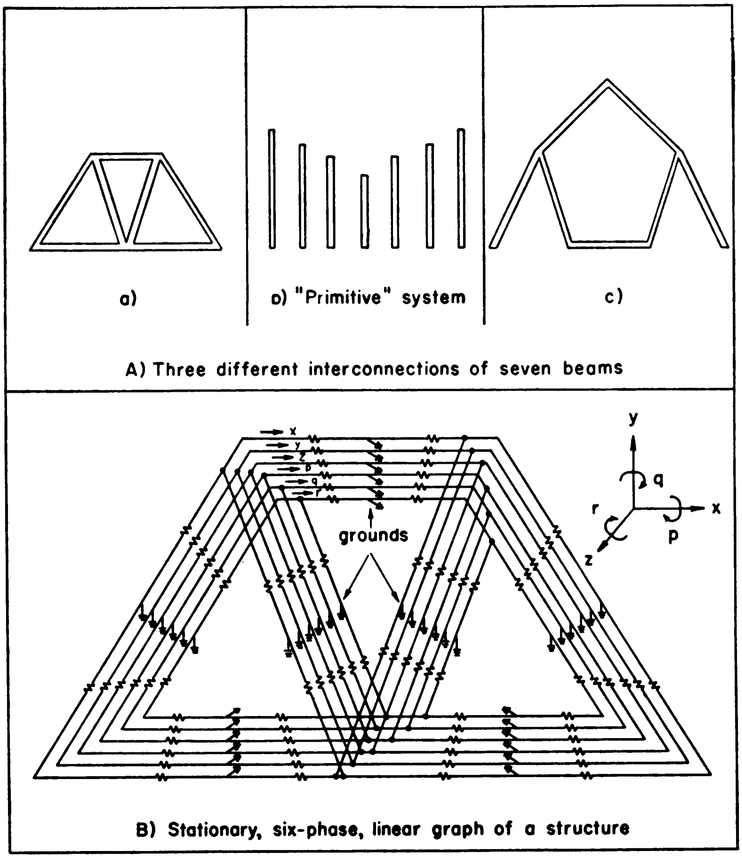

FIG. 6. Elastic structures.

It will be shown in another publication that the analytical solution of problems, with all parameters factored out, also becomes feasible by the use of the method of tearing.

In addition it must be emphasized that it is not the intention of the new method to compete with established methods of inversion (much less with methods of solution). The suggested procedure starts rather with "first-stage" solutions, already arrived at, and attempts to build up in successive stages large supersystems and their solutions. The new method intends to start where other methods end. Any overlapping in the first stage of solution is purely an educational or developmental accident. In presenting the subject this overlapping is unavoidable.

12. Physical versus geometrical model. When only a set of equations is available (say a set of partial difference equations, or the dynamical equations of Lagrange) then—in order to use the method of tearing for solving the equations—the author's first step is to replace the set of equations by a physical model and then tear the latter apart, instead of the equations themselves. The author prefers to replace the given equations by a linear or higher dimensional graph (topological model) and to use electrical-circuit terminology in tearing and analyzing the model. Of course, the geometrical model (if correctly constructed) also contains all the superfluous variables and equations that the corresponding physical system does. The electrical-circuit exists only on paper and in general cannot be (and need not be) physically realizable.

The electrical-circuit appearance and terminology of the author's models should not mislead the mathematician into believing that he must become an electrical engineer in order to follow the author's reasoning. On the contrary, it is the author who tries to become a topologist and solve topological problems of "cells" and "chains" as best he can. The author believes that the essentials of his method could be expressed in the language of linear and higher dimensional graphs (in the manner of Weyl, Ref. [10]) without any electrical connotations.*

13. Tearing versus partitioning. It should be especially noted that it is not the set of equations that is being torn apart, but the physical system itself, or a model of it. Consequently the method of tearing has nothing whatever to do with the "partitioning" of matrices, which tears the equations themselves apart and not the physical system. The mutually exclusive nature of the two methods is clearly illuminated by the fact that the partitioning of matrices does not contribute any new information about the physical system, but the tearing apart of a model does. The method of tearing introduces additional variables,—namely the constraints (forces and velocities) appearing at the points of tearing—and their corresponding equations. (A more detailed comparison appears in Ref. [3]).

It is the appearance of apparently superfluous variables and superfluous equations, that differentiates the method of tearing from other conventional methods of solving physical problems. Surprisingly enough, these extra variables and equations do not complicate, but rather simplify, the solution of physical systems and to a far greater extent than one would expect.

14. Non-linear systems. For many years the author tried to achieve the interconnection of the "equations of state" of subsystems [5–9]. It is only during the last few years that he has also experimented systematically with interconnecting the "equations of solution" of subsystems. In interconnecting equations of state no basic difficulty existed in dealing with either linear or with certain special non-linear dynamical systems. Similarly, in interconnecting equations of solution, no basic difficulties arise with linear dynamical systems.

The author has published only one simple example for the interconnection of piecewise solution of non-linear systems [4]. This example involved the inversion of Taylor's series with several variables and its interconnection. Solutions for elastic beam structures (such as Fig. 6) with non-linear (plastic) characteristics in each beam have also been obtained by successive approximations. However, many difficulties lie ahead in inter-

*Since the writing of this article the Author's attention was called to a paper by J. P. Roth entitled "*An application of algebraic topology to numerical analysis II. The validity of Kron's method of tearing*" (to appear in the Proceedings of the National Academy of Sciences). The paper identifies the following concepts of combinatorial topology that connect with the method of tearing: "Homomorphisms of homology and cohomology sequences induced by simplicial mappings of 1-dimensional complexes."

connecting the solutions of more general types of non-linear systems. It is hoped that if competent mathematicians shed more light upon the pathway the author has been treading, the interconnection of many types of non-linear solutions will be facilitated.

15. The invariance of power. The requirement that the form of the equations of each subsystem and of each interconnected system be invariant, is not sufficient to establish the laws of transformation of the various geometric objects. An additional requirement is needed. The author has made the discovery that a linear form, namely the total power input $i^\alpha e_\alpha$ (the product of generalized forces e_α and generalized velocities i^α) also remains invariant when a given number of subsystems are interconnected into any possible super-system. Because of the existence of the relation $i^\alpha e_\alpha = i^{\alpha'} e_{\alpha'}$ between any two systems, the usual laws of transformation of all tensors and geometric objects follow automatically.

The above invariance is another key to the theory of tearing. The relation must exist, since the method of tearing does work in practice. But to prove the invariance of power in torn-apart systems in an unequivocal, scientific manner, acceptable to mathematicians, is not a simple matter.

The author has a plausible proof that appeared in Ref. [4], pp. 412–414 and 428–435, valid for stationary electrical networks only. If the proof is correct, a generalization of this proof to general physical systems should follow without difficulty. Naturally the proof is only a physical proof and not a mathematical one. Since the author is not a mathematician, he has never felt qualified to give a mathematical proof. However, all the assumptions he has made to prove his method to his own satisfaction have brought additional, useful contributions and so he feels that his proof is at least along the right track. In the following the author's physical proof will be outlined.

16. More general equations. In order to prove the invariance of power in any dynamical system, it is assumed that the most general form of the equations (of state and solution) must contain sets both of impressed and constraint forces (e, E) and velocities (i, I) along each degree of freedom. For instance, for Fig. 7, the equations given

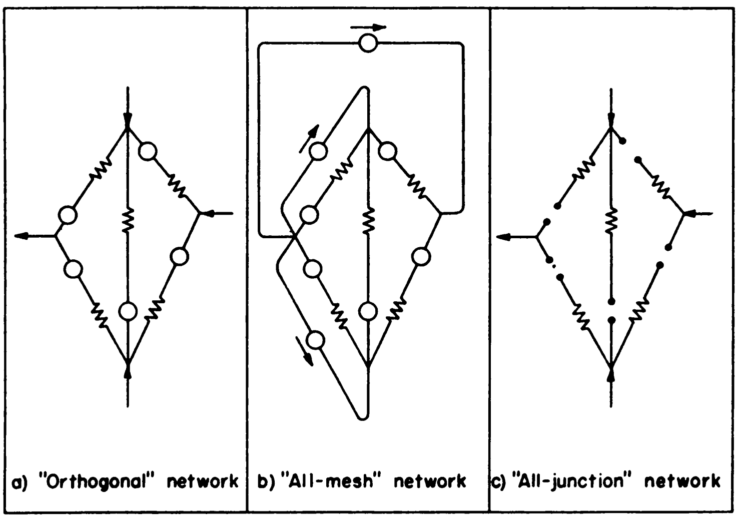

FIG. 7. Three points of view of a general network.

in Eq. (4) become generalized to:

$$\begin{array}{c|c} \begin{array}{c} \qquad\qquad [i_1 + I_1 \quad i_2 + I_2] \\ \begin{bmatrix} e_1 + E_1 \\ e_2 + E_2 \end{bmatrix} \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \end{array} & \begin{array}{c} \qquad\qquad [e_1 + E_1 \quad e_2 + E_2] \\ \begin{bmatrix} i_1 + I_1 \\ i_2 + I_2 \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \end{array} \end{array} \tag{11}$$

If the sets of mesh and junction equations are combined into one set, the above equations of state and solution may be written in the form

$$\begin{aligned} e_\alpha + E_\alpha &= z_{\alpha\beta}(i^\beta + I^\beta), \\ i^\alpha + I^\alpha &= Y^{\alpha\beta}(E_\beta + e_\beta), \end{aligned} \tag{12}$$

containing all possible forces (e_α and I^α) and constraints (i^α and E_α) along each of the topologically possible reference axes. In a similar manner, the dynamical equation (1) may assume the most general form

$$e_\alpha + E_\alpha = R_{\alpha\beta}(i^\beta + I^\beta) + L_{\alpha\beta}\frac{d(i^\beta + I^\beta)}{dt} + \Gamma_{\beta\gamma,\alpha}(i^\beta + I^\beta)(i^\gamma + I^\gamma), \tag{13}$$

where α and β denote not only the customary generalized axes of active forces, but also their dual axes of constrained forces.

Of course, it is never necessary to write down all the above equations with all the variables. Constructing a correct physical model is equivalent to having available (having written down) all the above equations in a tensor form. During the process of tearing and interconnecting the model, the engineer is enabled to write down only the absolute minimum number of equations and variables that he happens to need and to write them down only when needed.

17. Invariance of total power. In the presence of the four sets of variables, it is always possible to view every orthogonal network (Fig. 7a) in two other ways:

(i) As an "all-mesh" network (Fig. 7b) in which every impedance Z is short-circuited upon itself through a voltage, and forms a mesh.

(ii) As an "all-junction" network (Fig. 7c) in which every admittance Y is open-circuited and forms a junction-pair.

No matter what configurations are formed from the same n coils, the total current $i^\alpha + I^\alpha$ in each coil always maintains itself constant (since each coil always remains shorted through the same voltage). Simultaneously, the total voltage drop $e_\alpha + E_\alpha$ across each coil also always maintains itself constant (since each coil always remains open). Consequently the total power in each coil—and in the entire network—remains constant, no matter what networks are built out of the same coils. That is, for all possible configurations of the same n coils the following relation is satisfied

$$(e_\alpha + E_\alpha)(i^\alpha + I^\alpha) = (e_{\alpha'} + E_{\alpha'})(i^{\alpha'} + I^{\alpha'}). \tag{14}$$

It should be recalled that the non-singular matrix of transformation C between two networks was established in Eq. (5) by assuming that in each coil of both networks the total currents are equal.

This invariance of a linear form and the invariance requirements of the form of Eq. (12), enable one to establish the law of transformation of all tensors and geometric objects that arise in the method of tearing. Thus the author's thesis, that the processes of tearing and interconnecting physical systems can be viewed as a subgroup of the group of affine transformations, appears plausible.

From a practical point of view the above thesis not only opens up the possibility of the piecewise analysis and solution of very large and otherwise unmanageable engineering structures, but it also allows one to concentrate all the resources of the calculus of tensors upon such studies.

18. Acknowledgment. The author wishes to acknowledge his indebtedness to Professor Banesh Hoffmann for his constant encouragement and for many conversations, arguments and suggestions throughout the years, concerning the author's endeavor to apply highly abstract tensor concepts to the analysis and solution of practical engineering problems.

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