# SOLUTIONS OF THE HYPER-BESSEL EQUATION*

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In problems of hydrodynamic stability involving axial symmetry, it is sometimes necessary to find the solutions of a differential equation of the type

$$L_1^n f = 0$$

in which $n$ is a positive integer, and (with $D \equiv d/dr$)

$$L_1 \equiv D^2 + r^{-1}D - r^{-2} - \lambda^2$$

is the Bessel operator of the first order. In this note, solutions of the equation

$$L_p^n f = 0 \tag{1}$$

in which

$$L_p \equiv D^2 + r^{-1}D - p^2 r^{-2} + k^2$$

will be given explicitly. The theorem one seeks to establish is the following: If $p$ (taken to be positive for convenience) is not an integer, the solutions of Eq. (1) are $r^m J_{\pm(p+m)}(kr)$ in which $m = 0, 1, 2, \cdots, n - 1$; otherwise they are $r^m J_{p+m}(kr)$ and $r^m N_{p+m}(kr)$, with $m$ ranging over the same integers. The symbols $J$ and $N$ stand for the Bessel function and the Neumann function, respectively.

*Proof*: It is known that the solutions of $L_p f = 0$ are $J_{\pm p}(kr)$ for $p$ not equal to an integer and $J_p(kr)$ and $N_p(kr)$ for $p$ equal to an integer. Thus it suffices to show that if $r^s Z_{p+s}(kr)$ (in which $Z$ stands for either $J$ or $N$) satisfies $L_p^{s+1} f = 0$, then $r^{s+1} Z_{p+s+1}(kr)$ satisfies $L_p^{s+2} f = 0$, since the proof for $r^m J_{-(p+m)}(kr)$ is identical with that for $r^m Z_{p+m}(kr)$. This will be accomplished if one can show that $L_p r^{s+1} Z_{p+s+1}(kr)$ is equal to a constant times $r^s Z_{p+s}(kr)$. By straightforward differentiation one has

$$\begin{aligned} L_p r^{s+1} Z_{p+s+1}(kr) &= r^{s+1} L_p Z_{p+s+1}(kr) + s(s+1) r^{s-1} Z_{p+s+1}(kr) \\ &\quad + 2(s+1) r^s D Z_{p+s+1}(kr) + (s+1) r^{s-1} Z_{p+s+1}(kr) \\ &= r^{s+1} L_p Z_{p+s+1}(kr) + (s+1)^2 r^{s-1} Z_{p+s+1}(kr) \\ &\quad + 2(s+1) r^s D Z_{p+s+1}(kr). \end{aligned}$$

But

$$L_p = L_{p+s+1} + \frac{2p(s+1) + (s+1)^2}{r^2}$$

and [1]

$$D Z_{p+s+1}(kr) = k\left[ -\frac{p+s+1}{kr} Z_{p+s+1}(kr) + Z_{p+s}(kr) \right].$$

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So

$$L_p r^{s+1} Z_{p+s+1}(kr) = r^{s+1} L_{p+s+1} Z_{p+s+1}(kr) + [2p(s+1) + (s+1)^2] r^{s-1} Z_{p+s+1}(kr)$$

$$+ (s+1)^2 r^{s-1} Z_{p+s+1}(kr) + 2(s+1) r^s k \left[ -\frac{p+s+1}{kr} Z_{p+s+1}(kr) + Z_{p+s}(kr) \right]$$

$$= 2(s+1) k r^s Z_{p+s}(kr)$$

since

$$L_{p+s+1} Z_{p+s+1}(kr) = 0$$

by definition of $Z$.

Dr. Y. C. Fung of the California Institute of Technology communicated to the writer a different proof of the present result by means of Almansi's theorem [2] on hyperharmonic functions. His proof will not be presented here.

It may be noted that since [1]

$$Z_{p-1}(kr) + Z_{p+1}(kr) = \frac{2p}{kr} Z_p(kr) \tag{2}$$

and since by the theorem just proved $rZ_{p+1}(kr)$ and $Z_p(kr)$ are solutions of

$$L_p^2 f = 0, \tag{3}$$

it follows from Eq. (2) that $rZ_{p-1}(kr)$ is also a solution of Eq. (3). In fact, by repeated use of Eq. (2) and a similar one obtained by changing $p$ to $-p$ in Eq. (2), it can be proved that if the $m$ in the subscripts of the solutions given in the theorem is changed to $-m$, the results will still be solutions of Eq. (1). These solutions are of course not independent of the ones given in the statement of the theorem.

## References

[1] E. Jahnke and F. Emde, *Table of functions*, Dover Publications, New York, 1945, pp. 144-145

[2] E. Almansi, *Sull' integrazione dell' equazione differenziale* $\Delta^{2n}u = 0$, Annali di Matimatica, (III) 2 (1899)