ON THE FLEXURAL VIBRATIONS OF CIRCULAR AND ELLIPTICAL PLATES*

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In this paper we express R. D. Mindlin's version of plate flexure equations, which take transverse shear and rotary inertia into account, in general orthogonal curvilinear coordinates and then we specialize these to polar and elliptical coordinates in order to find the frequency equations for the normal modes of vibration of the circular and elliptical plates respectively. In particular, we wish to discover those of the eight natural boundary conditions for which the normal modes of vibration are expressible in terms of product functions.

In rectangular coordinates, the bending moments (M_x , M_y), twisting moments ($M_{yx} = - M_{xy}$) and shear (Q_x , Q_y) are given by the equations:

$$M_x = D\left(\frac{\partial \psi_x}{\partial x} + \mu \frac{\partial \psi_y}{\partial y}\right), \qquad M_y = D\left(\frac{\partial \psi_y}{\partial y} + \mu \frac{\partial \psi_x}{\partial x}\right),$$

$$M_{yx} = -M_{xy} = \frac{1-\mu}{2} D\left(\frac{\partial \psi_y}{\partial x} + \frac{\partial \psi_x}{\partial x}\right), \tag{a}$$

$$Q_x = k^2Gh\left(\frac{\partial w}{\partial x} + \psi_x\right), \qquad Q_y = k^2Gh\left(\frac{\partial w}{\partial y} + \psi_y\right),$$

where ψ_x , ψ_y , and w are plate displacements; D, G, and μ are the plate modulus, shear and Poisson's ratio respectively; h is the plate thickness, and $k^2 = \pi^2/12$ is a constant for any plate.

In the case of free vibrations, Mindlin has shown that w, ψ_x , and ψ_y can be expressed in terms of the three functions w_1 , w_2 , and w_3 by the following equations:

$$w = w_1 + w_2 ,$$

$$\psi_x = (\sigma_1 - 1) \frac{\partial w_1}{\partial x} + (\sigma_2 - 1) \frac{\partial w_2}{\partial x} + \frac{\partial w_3}{\partial y} , \tag{b}$$

$$\psi_y = (\sigma_1 - 1) \frac{\partial w}{\partial y} + (\sigma_2 - 1) \frac{\partial w_2}{\partial y} - \frac{\partial w_3}{\partial x} ,$$

where w_1 and w_2 are components of the displacement perpendicular to the middle plane of the plate, and w_3 is the potential function which gives rise to the twist about the normal to the plane of the plate;

$$\sigma_1 = \delta_1^2(S^{-1} + R\delta_0^4)^{-1}, \qquad \sigma_2 = \delta_2^2(S^{-1} + R\delta_0^4)^{-1},$$

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where $R = \dfrac{h^2}{12}$ (coefficient of rotary inertia),

$S = \dfrac{D}{k^2Gh}$ (coefficient of transverse shear),

$\delta_0^4 = \dfrac{\rho p^2 h}{D}$ where ρ and p are the plate density and angular frequency respectively;

$$\delta_1^2 = \tfrac{1}{2}\delta_0^2\{(R + S) + [(R - S)^2 + 4\delta_0^{-4}]^{1/2}\},$$

and

$$\delta_2^2 = \tfrac{1}{2}\delta_0^2\{(R + S) - [(R - S)^2 + 4\delta_0^{-4}]^{1/2}\}.$$

Mindlin showed further that the w_i are governed by the following three separated wave equations

$$(\nabla^2 + \delta_i^2)w_i = 0, \qquad i = 1, 2, 3, \tag{c}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{and} \qquad \delta_3^2 = \frac{2(R\delta_0^4 - S^{-1})}{1 - \mu}\,.$$

The functions w_i are also linked through the following boundary conditions: One member of each of the following three products must be specified on the boundary:

$$\psi_\xi M_\xi\,, \qquad \psi_\eta M_{\xi\eta}\,, \qquad wQ_\xi\,,$$

where

$$w = w_1 + w_2\,, \qquad \psi_\xi = \psi_x \cos\theta + \psi_y \sin\theta, \qquad \psi_\eta = \psi_y \cos\theta - \psi_x \sin\theta,$$

$$Q_\xi = Q_x \cos\theta + Q_y \sin\theta, \qquad M_\xi = M_x \cos^2\theta + M_y \sin^2\theta + 2M_{yx} \sin\theta\cos\theta, \tag{d}$$

$$M_{\xi\eta} = (M_y - M_x)\sin\theta\cos\theta + M_{yx}(\cos^2\theta - \sin^2\theta),$$

θ being the angle between the normal to the boundary and the x-axis.

The classical Lagrange theory of plates is a good approximation only when the wave length is large in comparison with the thickness of the plate, and this restricts the theory generally to low frequency vibrations. The present theory permits extensions to moderately high frequency modes, essentially because it includes coupling between flexual and shear motions.

Transforming equations (a), (b), and (c) into general orthogonal curvilinear coordinates we have:

$$w = w_1 + w_2\,,$$

$$\psi_\xi = (\sigma_1 - 1)h_1 \frac{\partial w_1}{\partial \xi} + (\sigma_2 - 1)h_1 \frac{\partial w_2}{\partial \xi} + h_2 \frac{\partial w_3}{\partial \eta}\,,$$

$$\psi_\eta = (\sigma_1 - 1)h_2 \frac{\partial w_1}{\partial \eta} + (\sigma_2 - 1)h_2 \frac{\partial w_2}{\partial \eta} - h_1 \frac{\partial w_3}{\partial \xi}\,,$$

$$Q_\xi = k^2Gh\left[\sigma_1 h_1 \frac{\partial w_1}{\partial \xi} + \sigma_2 h_1 \frac{\partial w_2}{\partial \xi} + h_2 \frac{\partial w_3}{\partial \eta}\right],$$

$$M = D\Big((\sigma_1 - 1)\Big\{h_1^2 \frac{\partial^2 w_1}{\partial_\xi^2} + \mu h_2^2 \frac{\partial^2 w_1}{\partial \eta^2} + P_{1\xi}(x, y) \frac{\partial w_1}{\partial \xi} + P_{2\eta}(x, y) \frac{\partial w_1}{\partial \eta}\Big\}$$

$$+ (\sigma_2 - 1)\Big\{h_1^2 \frac{\partial^2 w_2}{\partial_\xi^2} + \mu h_2^2 \frac{\partial^2 w_2}{\partial \eta^2} + P_1(x, y) \frac{\partial w_2}{p_\xi} + P_{2\eta}(x, y) \frac{\partial w_2}{\partial \eta}\Big\} \tag{A}$$

$$- (1 - \mu)\Big[h_1^4 h_2^2\Big\{4 \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}\Big(\frac{\partial y}{\partial \xi}\Big)^2 + \Big(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}\Big)\Big[\Big(\frac{\partial x}{\partial \xi}\Big)^2 - \Big(\frac{\partial y}{\partial \xi}\Big)^2\Big]\Big\} \frac{\partial^2 w_3}{\partial \xi\, \partial \eta}$$

$$+ R_{1\xi}(x, y) \frac{\partial w_3}{\partial \xi} + R_{2\eta}(x, y) \frac{\partial w_3}{\partial \eta}\Big]\Big),$$

$$M_{\xi i} = (1 - \mu)D\Big[(\sigma_1 - 1)\Big(h_1^4 h_2^2\Big\{4 \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}\Big(\frac{\partial y}{\partial \xi}\Big)^2 + \Big(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}\Big)\Big[\Big(\frac{\partial x}{\partial \xi}\Big)^2$$

$$- \Big(\frac{\partial y}{\partial \eta}\Big)^2\Big]\Big\} \frac{\partial^2 w_1}{\partial \xi\, \partial \eta} + R_{1\xi}(x, y) \frac{\partial w_1}{\partial \xi} + R_{2\eta}(x, y) \frac{\partial w_1}{\partial \eta}\Big)$$

$$+ (\sigma_2 - 1)\Big(h_1^4 h_2^2\Big\{4 \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}\Big(\frac{\partial y}{\partial \xi}\Big)^2 + \Big(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}\Big)\Big[\Big(\frac{\partial x}{\partial \xi}\Big)^2 - \Big(\frac{\partial y}{\partial \eta}\Big)^2\Big]\Big\} \frac{\partial^2 w_2}{\partial \xi\, \partial \eta}$$

$$+ R_{1\xi}(x, y) \frac{\partial w_2}{\partial \xi} + R_{2\eta}(x, y) \frac{\partial w_2}{\partial \eta}\Big)$$

$$- \frac{1}{2}\Big(h_1^2 \frac{\partial^2 w_3}{\partial \xi^2} - h_2^2 \frac{\partial^2 w_3}{\partial \eta^2} + S_1(x, y) \frac{\partial w_3}{\partial \xi} + S_{2\eta}(x, y) \frac{\partial w_3}{\partial \eta}\Big)\Big],$$

$$h_1^2 \frac{\partial^2 w_i}{\partial \xi^2} + h_2^2 \frac{\partial^2 w_i}{\partial \eta^2} + L_{1\xi}(x, y) \frac{\partial w_i}{\partial \xi} + L_{2\eta}(x, y) \frac{\partial w_i}{\partial \eta} = 0, \qquad i = 1, 2, 3, \tag{B}$$

where:

$$L_{ij}(x, y) \equiv \Big[h_1^2 \frac{\partial x}{\partial_\xi}\frac{\partial}{\partial_\xi}\Big(h_i^2 \frac{\partial x}{\partial j}\Big) + h_2^2 \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial x}{\partial j}\Big)\Big]$$

$$+ \Big[h_i^2 \frac{\partial y}{\partial_\xi}\frac{\partial}{\partial_\xi}\Big(h_i^2 \frac{\partial y}{\partial j}\Big) + h_2^2 \frac{\partial y}{\partial \eta}\frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial y}{\partial j}\Big)\Big],$$

$$P_{ij}(x, y) \equiv h_1^4\Big\{\frac{\partial x}{\partial_\xi}\Big[\Big(\frac{\partial x}{\partial \xi}\Big)^2 + 2\Big(\frac{\partial y}{\partial \xi}\Big)^2 - \mu\Big(\frac{\partial y}{\partial \xi}\Big)^2\Big] \frac{\partial}{\partial \xi}\Big(h_i^2 \frac{\partial x}{\partial j}\Big)$$

$$+ \frac{\partial y}{\partial \xi}\Big[\Big(\frac{\partial y}{\partial \xi}\Big)^2 + \mu\Big(\frac{\partial x}{\partial \xi}\Big)^2\Big] \frac{\partial}{\partial \xi}\Big(h_i^2 \frac{\partial y}{\partial_j}\Big)\Big\}$$

$$+ h_2^4\Big\{\frac{\partial x}{\partial \eta}\Big[\mu\Big(\frac{\partial x}{\partial \eta}\Big)^2 + 2\mu\Big(\frac{\partial y}{\partial \eta}\Big)^2 - \Big(\frac{\partial y}{\partial \eta}\Big)^2\Big] \frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial x}{\partial j}\Big)$$

$$+ \frac{\partial y}{\partial \eta}\Big[\eta\Big(\frac{\partial y}{\partial \eta}\Big)^2 + \Big(\frac{\partial x}{\partial \eta}\Big)^2\Big] \frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial y}{\partial j}\Big)\Big\}. \tag{C}$$

$$R_{ij}(x, y) \equiv h_1^4\Big(\frac{\partial y}{\partial \xi}\Big)^2\Big[\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \xi}\Big(h_i^2 \frac{\partial y}{\partial j}\Big) - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \xi}\Big(h_i^2 \frac{\partial x}{\partial j}\Big)\Big]$$

$$+ h_2^4 \frac{\partial y}{\partial \eta}\Big[\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial x}{\partial j}\Big) - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial y}{\partial j}\Big)\Big],$$

$$S_{ij}(x, y) \equiv h_1^4\left\{\frac{\partial x}{\partial \xi}\left[\left(\frac{\partial x}{\partial \xi}\right)^2 + 3\left(\frac{\partial y}{\partial \xi}\right)^2\right]\frac{\partial}{\partial \xi}\left(h_i^2\frac{\partial x}{\partial j}\right) + \frac{\partial y}{\partial \xi}\left[\left(\frac{\partial x}{\partial \xi}\right)^2 - \left(\frac{\partial y}{\partial \eta}\right)^2\right]\frac{\partial}{\partial \xi}\left(h_i^2\frac{\partial y}{\partial_j}\right)\right\}$$

$$- h_2^4\left\{\frac{\partial x}{\partial \eta}\left[\left(\frac{\partial x}{\partial \eta}\right)^2 + 3\left(\frac{\partial y}{\partial \eta}\right)^2\right]\frac{\partial}{\partial \eta}\left(h_i^2\frac{\partial x}{\partial j}\right) - \frac{\partial y}{\partial \eta}\left[\left(\frac{\partial x}{\partial \eta}\right)^2 - \left(\frac{\partial y}{\partial \eta}\right)^2\right]\frac{\partial}{\partial \eta}\left(h_i^2\frac{\partial y}{\partial_j}\right)\right\},$$

$$\frac{1}{h_i^2} = \left(\frac{\partial x}{\partial j}\right)^2 + \left(\frac{\partial y}{\partial j}\right)^2, \qquad i = 1 \text{ when } j = \xi \text{ and } i = 2 \text{ when } j = \eta.$$

Specializing equations (A) and (B) into polar and elliptical coordinates we have: for polar coordinates $h_1 = 1$, $h_2 = 1/r$

$$w = w_1 + w_2\,, \qquad \psi_r = (\sigma_1 - 1)\frac{\partial w_1}{\partial r} + (\sigma_2 - 1)\frac{\partial w_2}{\partial r} + \frac{1}{r}\frac{\partial w_3}{\partial \theta}\,,$$

$$\psi_\theta = \frac{\sigma_1 - 1}{r}\frac{\partial w_1}{\partial \theta} + \frac{\sigma_2 - 1}{r}\frac{\partial w_2}{\partial \theta} - \frac{\partial w_3}{\partial r}\,,$$

$$Q_r = k^2Gh\left[\sigma_1\frac{\partial w_1}{\partial r} + \sigma_2\frac{\partial w_2}{\partial r} + \frac{1}{r}\frac{\partial w_3}{\partial \theta}\right], \tag{D}$$

$$M_r = D\left\{(\sigma_1 - 1)\left[\frac{\partial^2 w_1}{\partial r^2} + \frac{\mu}{r}\frac{\partial w_1}{\partial r} + \frac{\mu}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\right] + (\sigma_2 - 1)\left[\frac{\partial^2 w_2}{\partial r^2} + \frac{\mu}{r}\frac{\partial w_2}{\partial r} + \frac{\mu}{r^2}\frac{\partial w_2}{\partial \theta^2}\right]\right.$$

$$\left. + (1 - \mu)\left[\frac{1}{r}\frac{\partial^2 w_2}{\partial r\,\partial \theta} + \frac{1}{r^2}\frac{\partial w_3}{\partial \theta}\right]\right\},$$

$$M_{r\theta} = (1 - \mu)D\left\{(\sigma_1 - 1)\left[\frac{1}{r}\frac{\partial^2 w_1}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial w_1}{\partial \theta}\right] + (\sigma_2 - 1)\left[\frac{1}{r}\frac{\partial^2 w_2}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial w_2}{\partial \theta}\right]\right.$$

$$\left. - \frac{1}{2}\left[\frac{\partial^2 w_3}{\partial r^2} - \frac{1}{r}\frac{\partial w_3}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w_3}{\partial \theta^2}\right]\right\},$$

$$\frac{\partial^2 w_i}{\partial r^2} + \frac{1}{r}\frac{\partial w_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_i}{\partial \theta^2} + \delta_i^2 w_i = 0, \qquad i = 1, 2, 3. \tag{E}$$

For elliptical coordinates

$$h_1^2 = h_2^2 = \frac{1}{C^2(\cosh^2 \xi - \cos^2 \eta)} = \frac{2}{C^2(\cosh 2\xi - \cos 2\eta)}\,,$$

$$w = w_1 + w_2\,, \qquad \psi_\xi = h_1\left[(\sigma_1 - 1)\frac{\partial w_1}{\partial \xi} + (\sigma_2 - 1)\frac{\partial w_2}{\partial \xi} + \frac{\partial w_3}{\partial \eta}\right],$$

$$\psi_\eta = h_1\left[(\sigma_1 - 1)\frac{\partial w_1}{\partial \eta} + (\sigma_2 - 1)\frac{\partial w_2}{\partial \eta} - \frac{\partial w_3}{\partial \eta}\right],$$

$$Q_\xi = k^2Ghh_1\left[\sigma_1\frac{\partial w_1}{\partial \xi} + \sigma_2\frac{\partial w_2}{\partial \xi} + \frac{\partial w_3}{\partial \eta}\right],$$

$$M_\xi = Dh_1^2\left[(\sigma_1 - 1)\left\{\frac{\partial w_1^2}{\partial_\xi^2} + \mu\frac{\partial w_1^2}{\partial_\eta^2} - \frac{C^2h_1^2}{2}(1 - \mu)\left[\sinh 2\xi\frac{\partial w_1}{\partial \xi} - \sin 2\frac{\partial w_1}{\partial \eta}\right]\right\}\right.$$

$$+ (\sigma_2 - 1)\left\{\frac{\partial^2 w_2}{\partial \xi^2} + \mu\frac{\partial^2 w_2}{\partial_\eta{}^2} - \frac{C^2h_i^2}{2}(1 - \mu)\left[\sinh 2\xi\frac{\partial w_2}{\partial \xi} - \sin 2\eta\frac{\partial w_2}{\partial \xi}\right]\right\} \tag{F}$$

$$\left. + (1 - \mu)\left\{\frac{\partial^2 w_3}{\partial \xi\,\partial \eta} - \frac{C^2h_1^2}{2}\left[\sin 2\eta\frac{\partial w_3}{\partial \xi} + \sinh 2\xi\frac{\partial w_3}{\partial \eta}\right]\right\}\right],$$

$$M_{\xi\eta} = \frac{1-\mu}{2} Dh_1^2\left\{(\sigma_1 - 1)\left[2\frac{\partial^2 w_1}{\partial\xi\,\partial\eta} - C^2h_1^2\left(\sin 2\eta\,\frac{\partial w_1}{\partial\xi} + \sinh 2\xi\,\frac{\partial w_1}{\partial\eta}\right)\right]\right.$$
$$+ (\sigma_2 - 1)\left[2\frac{\partial^2 w_2}{\partial\xi\,\partial\eta} - C^2h_1^2\left(\sin 2\eta\,\frac{\partial w_2}{\partial\xi} + \sinh 2\xi\,\frac{\partial w_2}{\partial\eta}\right)\right]$$
$$\left. - \left[\frac{\partial^2 w_3}{\partial\xi^2} - \frac{\partial^2 w_3}{\partial\eta^{\ 2}} - C^2h_1^2\left(\sinh 2\xi\,\frac{\partial w_3}{\partial\xi} - \sin 2\eta\,\frac{\partial w_3}{\partial}\right)\right]\right\},$$
$$\frac{\partial^2 w_1}{\partial\xi^2} + \frac{\partial^2 w_1}{\partial\eta^2} + 2k_1^2(\cosh 2\xi - \cos 2\eta)w_i = 0, \qquad i = 1, 2, 3, \tag{G}$$

where $2k_i = \delta_i C$, C is the semi-focal length of the elliptical plate.

We will now find the frequency equations for the normal modes of vibrations of the circular and elliptical plates which satisfy the following eight boundary conditions:

CIRCULAR PLATE		ELLIPTICAL PLATE	
(1) $\psi_r = \psi_\theta = w = 0$	(clamped plate)	$\psi_\xi = \psi_\eta = w = 0$	
(2) $\psi_r = \psi_\theta = Q_r = 0$		$\psi_\xi = \psi_\eta = Q_\xi = 0$	
(3) $\psi_r = M_{r\theta} = w = 0$		$\psi_\xi = M_{\xi\eta} = w = 0$	
(4) $\psi_r = M_{r\theta} = Q_r = 0$		$\psi_\xi = M_{\xi\eta} = Q_\xi = 0$	(H)
(5) $M_r = M_{r\theta} = Q_r = 0$	(free plate)	$M_\xi = M_{\xi\eta} = Q_\xi = 0$	
(6) $M_r = M_{r\theta} = w = 0$		$M_\xi = M_{\xi\eta} = w = 0$	
(7) $M_r = \psi_\theta = Q_r = 0$		$M_\xi = \psi_\eta = Q_\xi = 0$	
(8) $M_r = \psi_\theta = w = 0$		$M_\xi = \psi_\eta = w = 0$	

when $r = r_0$ and $\xi = \xi_0$ respectively.

It should be remarked that the boundary conditions that we are assuming are particular and that other values could be assumed for the quantities involved if desired.

By assuming that the solutions of equations (E) and (G) are expressible as product solutions we obtain the following respective pairs of ordinary differential equations:

$$\left\{\begin{aligned} &\frac{d^2w_i(r)}{dr^2} + \frac{1}{r}\frac{dw_i(r)}{dr} + \left(\delta_i^2 - \frac{m^2}{r^2}\right)w_i(r) = 0 \\ &\frac{d^2w_i(\theta)}{d\theta^2} + m^2w_i(\theta) = 0 \qquad (i = 1, 2, 3) \end{aligned}\right\} \tag{I}$$

$$\left\{\begin{aligned} &\frac{d^2w_i(\eta)}{d\eta^2} + (a - 2q_i\cos 2\eta)w_i(\eta) = 0 \\ &\frac{d^2w_i(\xi)}{d\xi^2} - (a - 2q_i\cosh 2\xi)w_i(\xi) = 0 \qquad (i = 1, 2, 3) \end{aligned}\right\}, \tag{J}$$

where $q_i = k_i^2$ and a and m^2 are separation constants. The first of equations (I) is Bessel's equation. The first of equations (J) is Mathieu's equation and the second is Mathieu's modified equation. The solutions of (I) and (J) are of the respective forms:

$$w_i(r) = J_m(\delta_i\,, r), \qquad w_i(\theta) = \begin{pmatrix} \sin\, m\theta \\ \\ \cos\, m\theta \end{pmatrix} \qquad i = 1, 2, 3, \tag{K}$$

$$w_i(\xi) = \begin{pmatrix} \mathrm{Ce}_m(\xi,\, q_i),\, a = a_m \\ \\ \mathrm{Se}_m(\xi,\, q_i),\, b = b_m \end{pmatrix}, \qquad w_i(\eta) = \begin{pmatrix} \mathrm{ce}_m(\eta,\, q_i),\, a = a_m \\ \\ \mathrm{se}_m(\eta,\, q_i),\, b = b_m \end{pmatrix}, \tag{L}$$

$$i = 1, 2, 3$$

where a_m and b_m are characteristic numbers of the Mathieu functions.

Hence the solutions of (E) and (G) are products of the solutions in (K) and (L) respectively.

We shall assume the following solutions for (E) and (G) throughout in the solution of our two problems:

$$\begin{aligned} w_i^{(m)}(r,\, \theta) &= A_m^{(i)} J_m(\delta_i\,, r) \cos\, m\theta, \qquad i = 1, 2, \\ w_i^{(m)}(r,\, \theta) &= A_m^{(i)} J_m(\delta_i\,, r) \sin\, m\theta, \qquad i = 3, \end{aligned} \tag{M}$$

$$\begin{aligned} w_i^{(m)}(\xi,\, \eta) &= C_m^{(i)} \mathrm{Ce}_{2n+1}(\xi,\, q_i) \mathrm{ce}_{2n+1}(\eta,\, q_i), \qquad (a = a_m = a_{2n+1}), \\ w_i^{(m)}(\xi,\, \eta) &= C_m^{(i)} \mathrm{Ce}_{2n+1}(\xi,\, q_i) \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(q_i) \cos\,(2r + 1)\eta \qquad \text{for } i = 1, 2, \\ w_i^{(m)}(\xi,\, \eta) &= C_m^{(i)} \mathrm{Se}_{2n+1}(\xi,\, q_i) \mathrm{se}_{2n+1}(\eta,\, q_i), \qquad (b = b_m = b_{2n+1}), \\ w_i^{(m)}(\xi,\, \eta) &= C_m^{(i)} \mathrm{Se}_{2n+1}(\xi,\, q_i) \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(q_i) \sin\,(2r + 1)\eta \qquad \text{for } i = 3. \end{aligned} \tag{N}$$

By substituting the assumed solutions (M) in the equations (D), making use of the boundary conditions (H), we obtain, after some reductions, the following results for the circular plate:

PROBLEM 1: $\psi_r = \psi_\theta = w = 0$ for $r = r_0$ (clamped plated).

$$\begin{aligned} & A_m^{(1)} J_m(\delta_1\,, r_0) + A_m^{(2)} J_m(\delta_2\,, r_0) + 0 = 0, \\ & A_m^{(1)}(\sigma_1 - 1) J_m'(\delta_1\,, r_0) + A_m^{(2)}(\sigma_2 - 1) J_m'(\delta_2\,, r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3\,, r_0) = 0, \\ & A_m^{(1)}(\sigma_1 - 1) \frac{m}{r_0} J_m(\delta_1\,, r_0) + A_m^{(2)}(\sigma_2 - 1) \frac{m}{r_0} J_m(\delta_2\,, r_0) + A_m^{(3)} J_m'(\delta_3\,, r_0) = 0. \end{aligned} \tag{1}$$

PROBLEM 2: $\psi_r = \psi_\theta = Q_r = 0$ for $r = r_0$.

$$\begin{aligned} & A_m^{(1)}(\sigma_1 - 1) J_m'(\delta_1\,, r_0) + A_m^{(2)}(\sigma_2 - 1) J_m'(\delta_2\,, r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3\,, r_0) = 0, \\ & A_m^{(1)}(\sigma_1 - 1) \frac{m}{r_0} J_m(\delta_1\,, r_0) + A_m^{(2)}(\sigma_2 - 1) \frac{m}{r_0} J_m(\delta_2\,, r_0) + A_m^{(3)} J_m'(\delta_3\,, r_0) = 0, \\ & A_m^{(1)} \sigma_1 J_m'(\delta_1\,, r_0) + A_m^{(2)} \sigma_2 J_m'(\delta_2\,, r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3\,, r_0) = 0. \end{aligned} \tag{2}$$

Problem 3: $\psi_r = M_{r\theta} = w = 0$ for $r = r_0$.

$$A_m^{(1)} J_m(\delta_1, r_0) + A_m^{(2)} J_m(\delta_2, r_0) + 0 = 0,$$

$$A_m^{(2)}(\sigma_1 - 1) J_m'(\delta_1, r_0) + A_m^{(2)}(\sigma_1 - 1) J_m'(\delta_2, r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3, r_0) = 0.$$

$$\begin{aligned} A_m^{(1)}(\sigma_1 - 1)\left[\frac{m}{r_0} J_m'(\delta_1, r_0) - \frac{m}{r_0^2} J_m(\delta_1, r_0)\right] & \\ + A_m^{(2)}(\sigma_2 - 1)\left[\frac{m}{r_0} J_m'(\delta_2, r_0) - \frac{m}{r_0^2} J_m(\delta_2, r_0)\right] & \\ + A_m^{(3)} \frac{1}{2}\left[J_m''(\delta_3, r_0) - \frac{1}{r_0} J_m'(\delta_3, r_0) + \frac{m^2}{r_0^2} J_m(\delta_3, r_0)\right] &= 0. \end{aligned} \tag{3}$$

Problem 4: $\psi_r = M_r = Q_{r\theta} = 0$ for $r = r_0$.

$$A_m^{(1)} \sigma_1 J_m'(\delta_1, r_0) + A_m^{(2)} \sigma_2 J_m'(\delta_2, r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3, r_0) = 0,$$

$$A_m^{(1)}(\sigma_1 - 1) J_m'(\delta_1, r_0) + A_m^{(2)}(\sigma_2 - 1) J_m'(\delta_2, r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3, r_0) = 0,$$

$$\begin{aligned} A_m^{(1)}(\sigma_1 - 1)\left[\frac{m}{r_0} J_m'(\delta_1, r_0) - \frac{m}{r_0^2} J_m(\delta_1, r_0)\right] & \\ + A_m^{(2)}(\sigma_2 - 1)\left[\frac{m}{r_0} J_m'(\delta_2, r_0) - \frac{m}{r_0^2} J_m(\delta_2, r_0)\right] & \\ + \frac{A_m^{(3)}}{2}\left[J_m''(\delta_3, r_0) - \frac{1}{r_0} J_m(\delta_3, r_0) + \frac{m^2}{r_0^2} J_m(\delta_3, r_0)\right] &= 0. \end{aligned} \tag{4}$$

Problem 5: $M_r = M_{r\theta} = Q_r = 0$ for $r = r_0$ (free plate).

$$A_m^{(1)} \sigma_1 J_m'(\delta_1, r_0) + A_m^{(2)} \sigma_2 J_m'(\delta_2, r_0) + 0 = 0,$$

$$\begin{aligned} A_m^{(1)}(\sigma_1 - 1)\left[\frac{m}{r_0} J_m'(\delta_1, r_0) - \frac{m}{r_0^2} J_m(\delta_1, r_0)\right] & \\ + A_m^{(2)}(\sigma_2 - 1)\left[\frac{m}{r_0} J_m'(\delta_2, r_0) - \frac{m}{r_0^2} J_m(\delta_2, r_0)\right] & \\ + \frac{A_m^{(3)}}{2}\left[J_m''(\delta_3, r_0) - \frac{1}{r_0} J_m'(\delta_3, r_0) + \frac{m^2}{r_0^2} J_m(\delta_3, r_0)\right] &= 0, \end{aligned} \tag{5}$$

$$\begin{aligned} A_m^{(1)}(\sigma_1 - 1)\left[J_m''(\delta_1, r_0) + \frac{\mu}{r_0} J_m'(\delta_1, r_0) - \frac{\mu m^2}{r_0^2} J_m(\delta_1, r_0)\right] & \\ + A_m^{(2)}(\sigma_2 - 1)\left[J_m''(\delta_2, r_0) + \frac{\mu}{r_0} J_m'(\delta_2, r_0) - \frac{\mu m^2}{r_0^2} J_m(\delta_2, r_0)\right] & \\ + A_m^{(3)}(1 - \mu)\left[\frac{m}{r_0} J_m'(\delta_3, r_0) - \frac{m}{r_0^2} J_m(\delta_3, r_0)\right] &= 0. \end{aligned}$$

Problem 6: $M_r = M_{r\theta} = w = 0$ for $r = r_0$.

$$
\begin{aligned}
&A_m^{(1)}J_m(\delta_1, r_0) + A_m^{(2)}J_m(\delta_2, r_0) + 0 = 0,\\
&A_m^{(1)}(\sigma_1 - 1)\left[\frac{m}{r_0}J'_m(\delta_1, r_0) - \frac{m}{r_0^2}J_m(\delta_1, r_0)\right]\\
&\qquad + A_m^{(2)}(\sigma_2 - 1)\left[\frac{m}{r_0}J'_m(\delta_2, r_0) - \frac{m}{r_0^2}J_m(\delta_2, r_0)\right]\\
&\qquad + \frac{A_m^{(3)}}{2}J''_m(\delta_3, r_0) - \frac{1}{r_0}J'_m(\delta_3, r_0) + \frac{m^2}{r_0^2}J_m(\delta_3, r_0)\Big] = 0,\\
&A_m^{(1)}(\sigma_1 - 1)\left[J''_m(\delta_1, r_0) + \frac{\mu}{r_0}J'_m(\delta_1, r_0) - \frac{\mu m^2}{r_0^2}J_m(\delta_1, r_0)\right]\\
&\qquad + A_m^{(2)}(\sigma_2 - 1)\left[J''_m(\delta_2, r_0) + \frac{\mu}{r_0}J'_m(\delta_2, r_0) - \frac{\mu m^2}{r_0^2}J_m\ \delta_2, r_0)\right]\\
&\qquad + A_m^{(3)}(1 - \mu)\left[\frac{m}{r_0}J'_m(\delta_3, r_0) - \frac{m}{r_0^2}J_m(\delta_3, r_0)\right] = 0.
\end{aligned}
\tag{6}
$$

Problem 7: $M_r = \psi_\theta = Q_r = 0$ for $r = r_0$.

$$
\begin{aligned}
&A_m^{(1)}\sigma_1J'_m(\delta_1, r_0) + A_m^{(2)}\sigma_2J'_m(\delta_2, r_0) + A_m\frac{m}{r_0}J_m(\delta_3, r_0) = 0,\\
&A_m^{(1)}(\sigma_1 - 1)\frac{m}{r_0}J_m(\delta_1, r_0) + A_m^{(2)}(\sigma_2 - 1)\frac{m}{r_0}J_m(\delta_2, r_0) + A_m^{(3)}J'_m(\delta_3, r_0) = 0,\\
&A_m^{(1)}(\sigma_1 - 1)\left[J''_m(\delta_1, r_0) + \frac{\mu}{r_0}J'_m(\delta_1, r_0) - \frac{\mu m^2}{r_0^2}J_m(\delta_1, r_0)\right]\\
&\qquad + A_m^{(2)}(\sigma_2 - 1)\left[J''_m(\delta_2, r_0) + \frac{\mu}{r_0}J'_m(\delta_2, r_0) - \frac{\mu m^2}{r_0^2}J_m(\delta_2, r_\theta)\right]\\
&\qquad + A_m^{(3)}(1 - \mu)\left[\frac{m}{r_0}J'_m(\delta_3, r_\theta) - \frac{m}{r_0^2}J_m(\delta_3, r_0)\right] = 0.
\end{aligned}
\tag{7}
$$

Problem 8: $M_r = \psi_\theta = w = 0$ for $r = r_0$.

$$
\begin{aligned}
&A_m^{(1)}J_m(\delta_1, r_0) + A_m^{(2)}J_m(\delta_2, r_0) + 0 = 0,\\
&A_m^{(1)}(\sigma_1 - 1)\frac{m}{r_0}J_m(\delta_1, r_0) + A_m^{(2)}(\sigma_2 - 1)\frac{m}{r_0}J_m(\delta_2, r_0) + A_m^{(3)}J'_m(\delta_3, r_0) = 0,\\
&A_m^{(1)}(\sigma_1 - 1)\left[J''_m(\delta_1, r_0) + \frac{\mu}{r_0}J'_m(\delta_1, r_0) - \frac{\mu m^2}{r_0^2}J_m(\delta_1, r_0)\right]\\
&\qquad + A_m^{(2)}(\sigma_1 - 1)\left[J''_m(\delta_2, r_0) + \frac{\mu}{r_0}J'_m(\delta_1, r_0) - \frac{\mu m^2}{r_0^2}J_m(\delta_2, r_0)\right]\\
&\qquad + A_m^{(3)}(1 - \mu)\left[\frac{m}{r_0}J'_m(\delta_3, r_0) - \frac{m}{r_0^2}J_m(\delta_3, r_0)\right] = 0.
\end{aligned}
\tag{8}
$$

By eliminating the constants from equations (1)-(8) we obtain the frequency equations for each of the problems.

By substituting the assumed solution (N) into equations (F), making use of the boundary conditions (H) and simplifying we obtain the following results for the elliptical plate:

PROBLEM I: $\psi_\xi = \psi_\eta = w = 0$ for $\xi = \xi_0$ (clamped plate).

$$
\begin{aligned}
&A_r^{(1)}\mathrm{Ce}_{2n+1}(\xi_0\,,\,q_1) + A_r^{(2)}\mathrm{Ce}_{2n+1}(\xi_0\,,\,q_2) + 0 = 0,\\
&A_r^{(1)}(\sigma_1 - 1)\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_1) + A_r^{(2)}(\sigma_2 - 1)\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_2)\\
&\qquad\qquad + A_r^{(3)}(2r+1)\mathrm{Se}_{2n+1}(\xi_0\,,\,q_3) = 0 \qquad\qquad \text{(I)}\\
&A_r^{(1)}(\sigma_1 - 1)(2r+1)\mathrm{Ce}_{2n+1}(\xi_0\,,\,q_1) + A_r^{(2)}(\sigma_2 - 1)(2r+1)\mathrm{Ce}_{2n+1}(\xi_0\,,\,q_2)\\
&\qquad\qquad + A_r^{(3)}\mathrm{Se}'_{2n+1}(\xi_0\,,\,q_3) = 0
\end{aligned}
$$

for every value of r.

PROBLEM II: $\psi_\xi = \psi_\eta = Q_\xi = 0$ for $\xi = \xi_0$.

$$
\begin{aligned}
&A_r^{(1)}(\sigma_1 - 1)\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_1) + A_r^{(2)}(\sigma_2 - 1)\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_2) + A_r^{(3)}\mathrm{Se}_{2n+1}(\xi_0\,,\,q_3) = 0,\\
&A_r^{(1)}(\sigma_1 - 1)\mathrm{Ce}_{2n+1}(\xi_0\,,\,q_1) + A_r^{(2)}(\sigma_2 - 1)\mathrm{Ce}_{2n+1}(\xi_0\,,\,q_2) \qquad\qquad \text{(II)}\\
&\qquad\qquad + A_r^{(3)}(2r+1)\mathrm{Se}_{2n+1}(\xi_0\,,\,q_3) = 0,\\
&A_r^{(1)}\sigma_1\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_1) + A_r^{(2)}\sigma_2\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_2) + A_r^{(3)}(2r+1)\mathrm{Se}_{2n+1}(\xi_0\,,\,q_3) = 0.
\end{aligned}
$$

for every value of r.

By eliminating the constants in (I) and (II) we obtain the frequency equations for problems I and II respectively.

PROBLEM III: $\psi_\xi = M_{\xi\eta} = w = 0$ for $\xi = \xi_0$.

$$
\begin{aligned}
&A_r^{(1)}\mathrm{Ce}_{2n+1}(\xi_0\,,\,q_1) + A_r^{(2)}\mathrm{Ce}_{2n+1}(\xi_0\,,\,q_2) + 0 = 0,\\
&A_r^{(1)}(\sigma_1 - 1)\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_1) + A_r^{(2)}(\sigma_2 - 1)\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_2)\\
&\qquad\qquad + A_r^{(3)}(2r+1)\mathrm{Se}_{2n+1}(\xi_0\,,\,q_3) = 0,\\
&A_r^{(1)}(\sigma_1 - 1)\{-2(2r+1)[\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_1) - G\,\mathrm{Ce}_{2n+1}(\xi_0\,,\,q_1)]\sin(2r+1)_\eta\\
&\qquad\qquad + 2F\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_1)\cos(2r+1)_\eta\}\\
&\quad + A_r^{(2)}(\sigma_2 - 1)\{-2(2r+1)\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_2) - G\,\mathrm{Ce}_{2n+1}(\xi_0\,,\,q_2)]\sin(2r+1)_\eta \qquad \text{(III)}\\
&\qquad\qquad + 2F\mathrm{Ce}'_{2n+1}(\xi_0\,,\,q_2)\cos(2r+1)_\eta\}\\
&\quad - A_r^{(3)}\{[\mathrm{Se}''_{2n+1}(\xi_0\,,\,q_3) + (2r+1)^2\mathrm{Se}_{2n+1}(\xi_0\,,\,q_3) - 2G\mathrm{Se}'_{2n+1}(\xi_0\,,\,q_2)]\sin(2r+1)_\eta\\
&\qquad\qquad - 2(2r+1)F\mathrm{Se}_{2n+1}(\xi_0\,,\,q_3)\cos(2r+1)_\eta\} = 0,
\end{aligned}
$$

where

$$
F \equiv -\frac{C^2h_1^2}{2}\sin 2\eta, \qquad G \equiv \frac{C^2h_1^2}{2}\sinh 2\xi_0\,, \qquad \text{and}
$$

$$
h_1^2 = h_2^2 = \frac{1}{C^2(\cosh^2\xi - \cos^2\eta)} = \frac{2}{C^2(\cosh 2\xi - \cos 2\eta)}\,.
$$

But since the above equations must be independent of η we cannot solve this problem. It is found that the same thing is true for the remaining five problems of the elliptical plate.

We can thus sum up our conclusions as follows:

Conclusion I: The problem of finding the frequency equations for the normal modes of vibration for a circular plate under the boundary conditions (H) can be solved in closed form and expressed in terms of Bessel functions by assuming product solutions for Mindlin's equations (E).

Conclusion II: The problem of finding the frequency equations for the normal modes of vibration of the elliptical plate, under the boundary conditions (H) can be solved in closed form and expressed in terms of Mathieu functions by assuming product solutions for Mindlin's equations (G), *ONLY* in the cases when the boundary conditions (H) are independent of the bending and twisting moments M_ξ and $M_{\xi\eta}$ respectively. Thus, the normal modes for the important case of the free elliptical edge do not appear to be expressible as product functions in elliptical coordinates.