—NOTES—

UNSTEADY VISCOUS FLOW IN THE VICINITY OF A STAGNATION POINT*

By NICHOLAS ROTT (*Cornell University*)

Consider a steady two-dimensional "stagnation point" flow of a viscous incompressible fluid in the upper x, y-plane; let the flow be directed towards and limited by a plate in the plane $y = 0$, with the stagnation point at $x = y = 0$. The corresponding flow pattern is well known as an exact solution of the Navier-Stokes equations.

Now, in addition, let the plate perform a harmonic motion in its own plane, i.e., in the x direction, while the flow at $y \to \infty$ remains steady. It seems to have remained unnoticed that even in this case, the exact Navier-Stokes equations yield a soluble problem of the boundary-layer type. Using for the velocity components u and v in the x- and y-directions respectively,

$$u = axf'(\eta) + be^{i\omega t}g(\eta), \tag{1}$$

$$v = -(a\nu)^{1/2}f(\eta), \tag{2}$$

and for the pressure, p,

$$p = -\frac{\rho}{2}a^2x^2 - \rho\nu aF(\eta) + p_0\,, \tag{3}$$

where

$$\eta = y\left(\frac{a}{\nu}\right)^{1/2}. \tag{4}$$

(ν = kinematic viscosity), and introducing these expressions in the Navier-Stokes equations, the following set of equations is easily obtained:

$$f'^2 - ff'' = 1 + f''', \tag{5}$$

$$ikg + gf' - fg' = g'', \tag{6}$$

$$ff' = F' - f'', \tag{7}$$

where $k = \omega/a$ is a "reduced" frequency. Equation (6) and Eq. (7) result from the equation of motion in the x-direction, by putting the terms proportional to x and independent of x respectively equal to zero. It is seen that Eq. (5) for the steady part is independent of the superimposed "g-flow." With the boundary conditions, $f(0) = f'(0) = 0$ and $f'(\infty) = 1$, Eq. (5) has the well-known Hiemenz solution. The viscous pressure term F can also be computed independently of Eq. (6).

Since the function $f(\eta)$ is known, Eq. (6) can be solved for g with the boundary conditions $g(0) = 1$, $g(\infty) = 0$. Consider first the steady motion of the plate with constant velocity b, i.e., $\omega = k = 0$. The corresponding solution g_0, say, fulfills the equation,

$$g_0'' + fg_0' - f'g_0 = 0. \tag{8}$$

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*Received Nov. 12, 1954. This research was supported by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command. The author wishes to acknowledge helpful discussions with Professor C. C. Lin during the latter's stay at Cornell University.

The exact solution is

$$g_0 = \frac{f''(\eta)}{f''(0)} = .811f'' \tag{9}$$

or, the velocity profile is proportional to the shear distribution of the Hiemenz flow (see Fig. 1). Solution (9) obviously fulfills the boundary conditions since $f''(\infty) = 0$,

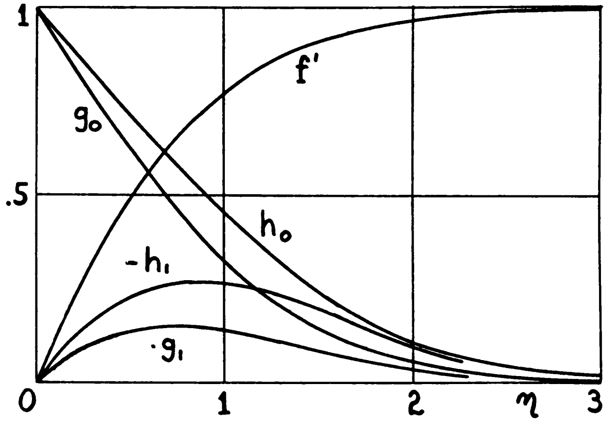

FIG. 1. Steady and quasi-steady velocity profiles.

and also the differential equation (8), because upon differentiating Eq. (5), the result is

$$f'''' + ff''' - f'f'' = 0.$$

With $f'''(0) = -1$, the shearing stress at the wall, τ_W, is proportional to

$$g_0'(0) = \frac{f'''(0)}{f''(0)} = -\frac{1}{f''(0)}, \tag{10}$$

so that the resulting τ_W from both the f- and the g-flow becomes

$$\tau_W = \rho(a\nu)^{1/2}\left\{axf''(0) - \frac{b}{f''(0)}\right\}. \tag{11}$$

Note that for $x = b/a$, the velocity outside the boundary layer is zero *relative to the wall*; the shearing stress at the wall, however, is zero at $x = .658b/a$. Evidently $\tau_W = 0$ does not necessarily mean separation for moving walls; in a system where the wall is at rest, the flow is not steady. The resulting boundary layer profiles are sketched in Fig. 2. Such development may be expected at the stagnation point of a Flettner rotor.

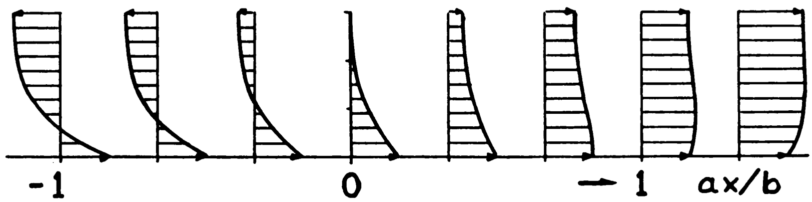

FIG. 2. Boundary layer development in the vicinity of a stagnation point with the wall moving at constant velocity b.

In the oscillating case, the two limiting cases, $k \ll 1$ and $k \gg 1$ will be considered. For $k \ll 1$, put

$$g = g_0 + ikg_1 + (ik)^2 g_2 + \cdots . \tag{12}$$

Equating powers of ik, the solution for g_0 is given by Eq. (9), obtained above; g_1 fulfills the equation

$$g_1'' + fg_1' - f'g_1 = g_0 = .811f'' \tag{13}$$

with the boundary conditions $g_1(0) = g_1(\infty) = 0$. One particular solution of the inhomogeneous equation (13) can be immediately given: $g_1 = .811f$. This function, however, does not fulfill the conditions at infinity, as $f(\infty) \to \eta$. The complete solution of Eq. (13) is obtained by adding the general homogeneous solution:

$$g_1 = \frac{f(\eta)}{f''(0)} + c_1 f'' \int_0^\eta \frac{E}{f''^2}\, d\eta + c_2 f'', \tag{14}$$

where

$$E = \exp\left(-\int_0^\eta f\, d\eta\right). \tag{15}$$

From $g_1(0) = 0$, it follows that $c_2 = 0$. The constant c_1 has to be adjusted so that $g_1(\infty) = 0$. Its value can be found with the help of the identity

$$f = f'' \int_0^\eta \frac{E}{f''^2}\, d\eta \int_0^\eta \frac{f''^2}{E}\, d\eta \tag{16}$$

which may be proved by identifying the proper inhomogeneous solution of Eq. (13) with f, or directly by using the properties of the function f as a solution of Eq. (5). It can be shown also that for $\eta \to \infty$, $f''^2/E \to 0$, and that the integrand of the inner integral in Eq. (16) is integrable between the limits 0 and ∞. Therefore, for large η, the inner integral in Eq. (16) will vary only slightly and may be replaced asymptotically by the constant value obtained after taking the limits between 0 and ∞:

$$f \to \left\{\int_0^\infty \frac{f''^2}{E}\, d\eta\right\} f'' \int_0^\eta \frac{E}{f''^2}\, d\eta. \tag{16a}$$

With this asymptotic expression, c_1 can be determined, and the final result is given by

$$g_1 = \frac{f}{f''(0)} - \left\{\frac{1}{f''(0)} \int_0^\infty \frac{f''^2}{E}\, d\eta\right\} f'' \int_0^\eta \frac{E}{f''^2}\, d\eta. \tag{17}$$

The curve $g_1(\eta)$ is also plotted in Fig. 1; its extremum is $-.152$. The shearing stress at the wall is proportional to

$$g_1'(0) = -\frac{1}{[f''(0)]^2} \int_0^\infty \frac{f''^2}{E}\, d\eta = -.496 \tag{18}$$

so that τ_W due to the g-flow is, to the first order in ik,

$$\tau_W = -\rho(a\nu)^{1/2} b(.811 + .496ik)e^{i\omega t}. \tag{19}$$

For high frequencies, $k \gg 1$, the WBK-method is appropriate. In Eq. (6), put

$$g = \exp\left(\int_0^\eta s\, d\eta\right) \tag{20}$$

then, s has to fulfill the equation

$$s' + s^2 + fs - f' = ik. \tag{21}$$

Now set

$$s = (ik)^{1/2}s_0 + s_1 + (ik)^{-1/2}s_2 + (ik)^{-1}s_3 + \cdots \tag{22}$$

which is also equal to

$$s = -\left(\frac{k}{2}\right)^{1/2}s_0 + s_1 - \left(\frac{1}{2k}\right)^{1/2}s_2 + \frac{1}{k}\left(\frac{1}{2k}\right)^{1/2}s_4 \cdots$$
$$- i\left\{\left(\frac{k}{2}\right)^{1/2}s_0 - \left(\frac{1}{2k}\right)^{1/2}s_2 + \frac{1}{k}s_3 - \frac{1}{k}\left(\frac{1}{2k}\right)^{1/2}s_4 \cdots\right\}. \tag{23}$$

Introduction of Eq. (22) into Eq. (21) gives upon equating powers of $(ik)^{1/2}$:

$$s_0 = 1, \qquad s_1 = -\tfrac{1}{2}f, \qquad s_2 = \tfrac{1}{8}f^2 + \tfrac{3}{4}f', \tag{24}$$
$$s_3 = -\tfrac{1}{8}ff' - \tfrac{3}{8}f'', \cdots .$$

The boundary condition $g(0) = 1$ is always fulfilled by the choice of limits in the integral in Eq. (20), and $g(\infty) = 0$ is assured by the selection of the proper sign if the double-valued quantity $(ik)^{1/2}$, namely, the one indicated in the decomposition Eq. (23). The resulting profile, using s_0 and s_1 only, is

$$g = \exp\left[-\left(\frac{k}{2}\right)^{1/2}i\eta\right]\exp\left[-\left(\frac{k}{2}\right)^{1/2}\eta\right]\exp\left(-\frac{1}{2}\int_0^\eta f\,d\eta\right). \tag{25}$$

The first two factors represent the "Stokes-solution" [1], which would be obtained by complete disregard of the f-flow. The last factor has the value of about .92 for $\eta = 1$ and .59 for $\eta = 2$. If, for large values of k, the second factor in Eq. (25) already assures a strong decay with η, the last factor remains very close to 1 in the whole domain where significant values of g may be found.

The shearing stress at the wall is

$$\tau_W = \rho(a\nu)^{1/2}bs(0)e^{i\omega t}. \tag{26}$$

Now the value of $s(0)$ differs from the Stokes value only if terms up to s_3 are included. From Eqs. (24), Eq. (23) gives

$$s(0) = -\left(\frac{k}{2}\right)^{1/2} - i\left[\left(\frac{k}{2}\right)^{1/2} - \frac{3}{8k}f''(0)\right]. \tag{27}$$

Therefore, the ratio of the out-of-phase shearing stress, τ_{W_i}, to the in-phase component, τ_{W_r}, is

$$\frac{\tau_{W_i}}{\tau_{W_r}} = 1 - .654k^{-3/2}, \tag{28}$$

whereas for low values of k, from Eq. (19), the ratio is

$$\frac{\tau_{W_i}}{\tau_{W_r}} = .612k. \tag{29}$$

In Fig. 3, the limiting cases for large and small k, Eqs. (28) and (29), are plotted, together with an estimated curve which joins the two plots smoothly. It is seen that for

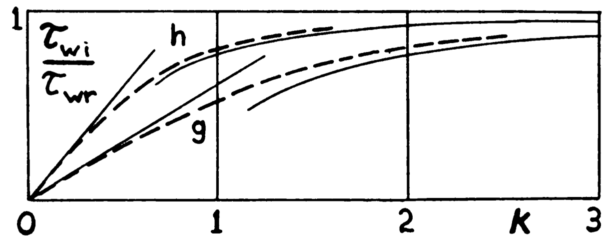

Fig. 3. The ratio of the out-of-phase and the in-phase components of the shearing stress at the wall as function of the reduced frequency k, for oscillations in x-direction (g) and z-direction (h).

a reduced frequency k larger than 2, the Stokes' value $\tau_{W_i}/\tau_{W_r} = 1$ gives errors less than 20%.

For the case of complex ω, or exponential acceleration, only one special case will be considered, for which the solution may be obtained without any computational effort. Put $k = -i$, i.e., let the time-dependence of the plate velocity be given by the factor e^{at}. The corresponding special equation for g is

$$g'' + fg' - (f' + 1)g = 0 \tag{6a}$$

which has the solution

$$g = 1 - f' \tag{30}$$

in view of Eq. (5). The complete solution is

$$u = axf'(\eta) + be^{at}[1 - f'(\eta)]$$

and the wall shearing stress is

$$\tau_W = \rho(a\nu)^{1/2}f''(0)[ax - be^{at}]. \tag{31}$$

Here it is seen that, in contrast to the case $k = 0$, Eq. (11), the zero of the shearing stress and the zero of the relative velocity between plate and fluid occur at the same value of $x[=(b/a)e^{at}]$.

Now let the plate move in the z-direction perpendicular to the xy-plane, i.e., the plane of flow. If the plate motion is uniform, the case under consideration is identical with the problem of the stagnation point in yawed flow, which has been solved by Prandtl [2] and Sears [3]. For unsteady motion of the plate in z-direction, Wuest [4] has already pointed out that the problem is soluble, and has given some numerical examples. Both cases become exact solutions of the Navier-Stokes equations, if the basic steady flow is the stagnation-point flow in the half-plane, as is presently assumed. If w, the velocity component of the flow in the z-direction, is taken in the form

$$w = ce^{i\omega t}h(\eta), \tag{32}$$

then Eqs. (5) to (7) remain unaffected, and the Navier-Stokes equation in z-direction yields

$$ikh - fh' = h'' \tag{33}$$

with the boundary conditions $h(0) = 1$, $h(\infty) = 0$. For $\omega = k = 0$, Prandtl's solution is obtained:

$$h_0 = \frac{\int_\eta^\infty E\, d\eta}{\int_0^\infty E\, d\eta} \tag{34}$$

[see Eq. (15)]. A series development for small k analogous to Eq. (12) leads to the equations

$$h_0'' + fh_0' = 0, \qquad h_1'' + fh_1' = h_0\,, \qquad h_2'' + fh_2' = h_1\,, \cdots . \tag{35}$$

The solution of the second equation, adjusted to the boundary conditions $h_1(0) = h_1(\infty) = 0$, is:

$$h_1 = h_0 \int_0^\eta \frac{(1 - h_0)h_0}{h_0'}\, d\eta + (1 - h_0) \int_\eta^\infty \frac{h_0^2}{h_0'}\, d\eta. \tag{36}$$

The profiles h_0 and h_1 are plotted in Fig. 1. The shearing stress at the wall in the z-direction becomes, to a first approximation in k:

$$\tau_W = -\rho(a\nu)^{1/2}c(.571 + .685ik)e^{i\omega t}. \tag{37}$$

The investigation of the flow-behavior for $k \gg 1$ follows completely the method already employed for the g-flow. Putting

$$h = \exp\left(\int_0^\eta r\, d\eta\right). \tag{38}$$

in Eq. (33) yields

$$r' + r^2 + fr = ik \tag{39}$$

and the series development analogous to Eq. (22) gives

$$\begin{gathered} r_0 = 1, \qquad r_1 = -\tfrac{1}{2}f, \qquad r_2 = \tfrac{1}{8}f^2 + \tfrac{1}{4}f', \\ r_3 = -\tfrac{1}{8}ff' - \tfrac{1}{8}f'', \cdots . \end{gathered} \tag{40}$$

It is seen that $r_0 = s_0\,,\ r_1 = s_1\,.$

The shearing stress at the wall again differs from the Stokes value only if r_3 is included; to this approximation,

$$r(0) = -\left(\frac{k}{2}\right)^{1/2} - i\left[\left(\frac{k}{2}\right)^{1/2} - \frac{1}{8k} f''(0)\right] \tag{41}$$

and we obtain

$$\frac{\tau_{W_i}}{\tau_{W_r}} = 1 - .215k^{-3/2} \tag{42}$$

whereas for $k \ll 1$, from Eq. (37),

$$\frac{\tau_{W_i}}{\tau_{W_r}} = 1.20k. \tag{43}$$

As before, both limiting cases are plotted in Fig. 3, together with an estimated smooth transition curve. It is seen that the Stokes limit is approached faster by the h-flow than by the g-flow; for $k = 1$ the deviation for the h-flow is about 20%.

A further set of exact solutions may be obtained if the basic steady, two-dimensional, stagnation-point flow is replaced by a three-dimensional one. It is not necessary to assume rotational symmetry; the "potential" flow along the plate may be of the form $u = a_1 x$, $w = a_2 z$ with $a_1 \neq a_2$. The steady viscous solution has been obtained by Howarth [5]. If the plate moves or oscillates in any direction in the xz-plane, further exact solutions of the Navier-Stokes equation are easily obtained. No examples will be carried out for the three-dimensional stagnation point, as the cases treated before are well representative of the phenomena that may be expected.

It is interesting to note that the heat transfer to the plate remains unaffected by the motion of the plate in its plane, if the plate temperature is constant. The temperature field is obtained as a solution of the equation (in two dimensions)

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\nu}{\sigma} \frac{\partial^2 T}{\partial y^2}, \tag{44}$$

where T is the temperature and σ is the Prandtl number; dissipation is neglected, as is permissible for high temperature differences between the fluid and the plate. For constant plate temperature (and constant temperature difference between the plate and the fluid), the solution with the property $\partial T/\partial x = 0$ is appropriate. But the plate motion under consideration only affects u and leaves v unchanged, so that for $\partial T/\partial x = 0$ no effect on the solution of Eq. (44) will be felt.

If unsteady heat transfer is enforced by a variable temperature on the plate, which becomes a function of time but not of x, the resultant equation becomes completely analogous to that of the h-flow, discussed before. For $\sigma = 1$, the same solution can be used as before; modifications for $\sigma \neq 1$ are easily obtained.

A case which has not been considered so far in this paper is the problem associated with a plate oscillating perpendicular to its plane, or the equivalent case of the oscillating basic stagnation-point flow. If the f-flow is unsteady, a $\partial f'/\partial t$ — term appears in the non-linear equation (5), so that solutions with a harmonic time-dependence can be obtained only to a linearized approximation. Ultimately, however, the unsteady part of the flow will influence the development of the steady part of the f-flow, due to the non-linearity of Eq. (5).

The linearized approximation, or the superposition of a small oscillating part to a basically steady f-flow has been investigated by Lighthill [6] as a special case of fluctuating flow problems with arbitrary velocity distributions. The linearized solution exhibits essentially the same features as found in the cases discussed before, inasmuch that a quasi-steady type for low frequencies and a high-frequency type approaching Stokes' solution can be distinguished. Lighthill finds as a "limit" between these cases, the value of $k = 5.6$ for the reduced frequency, i.e., higher than the limits which have been found above for the g- and h-flows. Lighthill also discussed the effects of the time-dependent f-flow on heat transfer, which does not vanish, as the v-component of the flow is affected.

Considering the flow at the stagnation-point of an oscillating airfoil with nose-radius R, the value of a is about U/R. For all reduced frequencies of practical interest in airfoil flutter, the high frequency approximation is appropriate for all unsteady boundary layer phenomena.

It may be noted that unsteady rigid rotary motion of the plate in its own plane leads to problems related to the well-known Kármán-Cochran case and its generalizations; again, however, solutions with harmonic time-dependence can be found only to a linearized approximation.

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STABILITY OF SPHERICAL BUBBLES*

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The perturbation equations for a spherical bubble of radius $b(t)$ are [1, p. 306]

$$b(t)b_h'' + 3b'(t)b_h' - (h - 1)b''(t)b_h = 0. \tag{1}$$

It is the purpose of this note to give a general stability criterion for the stability of (1). Rewriting (1) in the form

$$x'' + p(t)x' + q(t)x = 0, \qquad p = 3b'/b, \qquad q = -(h - 1)b''/b, \tag{2}$$

we consider first the formal identity

$$\frac{d}{dt}\{x^2 + qx'^2\} = -\frac{x'^2}{q^2}[2pq + q'], \tag{3}$$

which is an easy consequence of (2).

Theorem 1. If $q < 0$, or if $q > 0$ and $2pq + q' < 0$, then (2) is *unstable*. If $q > 0$ and $2pq + q' > 0$, then (2) is *stable*.

Proof. If $q(t) < 0$, and $x(t)$, $x'(t)$ have the same sign for $t = t_0$, then they have the same sign for all $t > t_0$. This is evident since $x'(t_1) = 0$ offers the only possibility for the first sign change, and it implies $x''(t_1) = -qx(t_1)$, whence $x'(t_1 + dt) = -q(t_1)x(t_1)dt$ has the same sign as $x(t_1 + dt)$. Hence $x(t)$ grows forever in magnitude; this is of course the *non-oscillatory case*.

If $q(t) > 0$, then we are in the *oscillatory* case. To see this, replace (2) by the self-adjoint

$$d(Px')/dt + Qx = 0, \qquad P = \exp\left(\int p\,dt\right), \qquad Q = q\exp\left(\int p\,dt\right). \tag{4}$$

Then we use the Bocher-Prüfer variable Θ, defined by $\tan\Theta = -Px'/x$. Differentiating $\tan\Theta$, using (4), and simplifying, we get

$$d\Theta/dt = Q(t)\cos^2\Theta + \frac{1}{P(t)}\sin^2\Theta > 0. \tag{5}$$

*Received Dec. 21, 1954; revised manuscript received March 17, 1955.